In this paper, we address the pursuit-evasion problem of tracking an Omnidirectional Agent (OA) at a bounded variable distance using a Differential Drive Robot (DDR), in an Euclidean plane without obstacles. We assume that both players have bounded speeds, and that the DDR is faster than the evader, but due to its nonholonomic constraints it cannot change its motion direction instantaneously. Only a purely kinematic problem is considered, and any effect due to dynamic constraints (e.g., acceleration bounds) is neglected. We provide a criterion for partitioning the configuration space of the problem into two regions, so that in one of them the DDR is able to control the system, in the sense that, by applying a specific strategy (also provided), the DDR can achieve any inter-agent distance (within an error bound), regardless of the actions taken by the OA. Particular applications of these results include the capture of the OA by the DDR and maintaining surveillance of the OA at a bounded variable distance.

Keywords: pursuit-evasion, tracking, capturing, differential drive robot.

1. Introduction

In this paper, we address the pursuit-evasion problem of tracking an Omnidirectional Agent (OA) at a bounded variable distance using a Differential Drive Robot (DDR), on a Euclidean plane without obstacles. We assume that both players have bounded speeds, and that the DDR is faster than the evader, but due to its nonholonomic constraints it cannot change its motion direction instantaneously. Only a purely kinematic problem is considered, and any effect due to dynamic constraints (e.g., acceleration bounds) is neglected. We provide a criterion for partitioning the configuration space of the problem into two regions, so that in one of them the DDR is able to control the system, in the sense that, by applying a specific strategy (also provided), the DDR can achieve any inter-agent distance (within an error bound), regardless of the actions taken by the OA. Particular applications of these results include the capture of the OA by the DDR and maintaining surveillance of the OA at a bounded variable distance.

In our previous work (Murrieta-Cid et al., 2011), we presented a solution for the problem of tracking an omnidirectional mobile evader at a constant distance with a differential drive robot. In that paper, we obtained motion strategies for both the players and a long term solution for the game. The current work represents a generalization of the research presented there. In this paper, we provide conditions that establish whether or not it is possible for a DDR to track an OA at a bounded variable distance, and the DDR’s motion strategies to perform the task. The pursuer’s control objective is to reach an inter-player distance within an interval \([L_G - \epsilon, L_G + \epsilon]\), where \(L_G\) is the goal distance for the pursuer, and \(\epsilon\) is a prescribed small tolerance (determined by a positive real number); \(\epsilon\) represents an overshoot or undershoot, due to the assumption that while one of the players changes the inter-player distance, the motion direction of the other player is unknown.

The main difference between the current work and that presented by Murrieta-Cid et al. (2011) is that previously we only provided a pursuer motion strategy that guarantees to maintain a constant distance between the players, while in this work we provide a pursuer motion policy that guarantees that it will be able to reach an inter-player distance within an interval \([L_G - \epsilon, L_G + \epsilon]\). This \(L_G\) distance can be smaller or larger than the initial distance \(L_I\) between the players.

We model the pursuer-evader system as a hybrid control problem combining two motion strategies. The
first part of the pursuer’s strategy is modeled continuously, and the goal is to maintain a constant distance between the players while the DDR aligns its heading with the evader’s location (cf. Murrieta-Cid et al., 2011). The second part of the pursuer’s strategy is modeled using a discrete system, in which the pursuer increases or decreases the inter-player distance. The DDR switches between both strategies until the desired condition (distance) is achieved.

In this work, we define a manifold over the space of parameters for the game, such that it induces a partition of this space into two disjoint regions. The DDR will be able to control the system whenever it is in one of those regions. By controlling we mean that the DDR can vary the inter-player distance freely. When the system is exactly on the manifold, no player controls it and this can be interpreted as a tied game. The motion strategies presented in this paper are applicable to several problems related to surveillance or capture:

- They allow a DDR to maintain an omnidirectional evader within a limited sensing range defined by a maximal \( L_{\text{max}} \) and a minimal \( L_{\text{min}} \) sensing distances.

- They allow a DDR to reduce the distance to the evader.

In the remainder of the paper, we describe the conditions that make the tasks listed above possible.

2. Previous work

A lot of work has been done in the area of pursuit-evasion games (Hájek, 1965; Isaacs, 1965; Baṣar and Olsder, 1982), and three main problems have received a lot of attention. They include finding, tracking and capturing a mobile evader with one or several pursuers.

In the finding problem (Isler et al., 2005; Hollinger et al., 2009), the objective is to establish some sort of visibility between the pursuer and the evader. In this case, the pursuer must sweep the environment so that the evader is not able to eventually sneak into an area that has already been explored. Deterministic (Suzuki and Yamashita, 1992; Guibas et al., 1999; Sachs et al., 2004; Tovar and LaValle, 2008) and probabilistic (Vidal et al., 2002; Hespanha et al., 2000; Chung, 2008) algorithms have been proposed to solve this problem.

In the capturing problem, the pursuer tries to get closer than a given distance \( l \) to the evader. The goal of the evader is to keep the pursuer at all times farther from it than this capture distance. A classical problem is that of the homicidal chauffeur (Isaacs, 1965; Merz, 1971). In that game a faster pursuer (with respect to the evader) has as its objective to get closer than a given distance (the capture condition) to a slower but more agile evader, in order to run it over. The pursuer is a vehicle with a minimal turning radius. The game takes place in the Euclidean plane without obstacles, and the evader aims to avoid the capture condition. The problem tackled in this paper and its proposed solution are different to the homicidal chauffeur issue. Note that the change in the mechanical model for the pursuer (if this role is taken by the DDR, which can rotate in place) has as distinctive consequences that both the condition defining the winner and the motion strategies of the two players also change with respect to the homicidal chauffeur solution. Ruiz and Murrieta-Cid (2012) as well as Ruiz et al. (2013) presented time-optimal strategies for the game of capturing an omnidirectional evader with a differential drive robot. The results presented here, although not time-optimal, have the advantage over those of Ruiz and Murrieta-Cid (2012) as well as Ruiz et al. (2013) of allowing solution of two problems: capturing an OA evader with a DDR pursuer and maintaining surveillance at a bounded variable distance of an OA with a DDR.

In the tracking problem, the goal is maintaining visibility of the evader at all times, usually in an environment with obstacles (LaValle et al., 1997; González et al., 2002; Jung and Sukhatme, 2002; Bandyopadhyay et al., 2006; Bhattacharya and Hutchinson, 2010).

In recent years there has been a growing interest in related problems within the community of autonomous robots (Jung and Sukhatme, 2002; Kowalczuk and Czubenko, 2011), and specifically in robot motion planning (LaValle et al., 1997; González et al., 2002; Murrieta-Cid et al., 2007). LaValle et al. (1997) proposed game theory as a framework to formulate the tracking problem. Becker et al. (1995) presented an algorithm that operates by maximizing the probability of future visibility of the evader. This algorithm is also studied more formally by LaValle et al. (1997). Fabiani et al. (2002) present an approach that takes into account the positioning uncertainty of the robot pursuer.

The approach presented by Murrieta-Cid et al. (2005) computes a motion strategy by maximizing the shortest distance to escape, i.e., the shortest distance the evader needs to move in order to escape the pursuer’s visibility region. González et al. (2002) propose a technique to track an evader without the need for a global map. Instead, a range sensor is used to construct a local map of the environment, and a combinatorial algorithm is then employed to compute a motion for the pursuer at each iteration. In our previous work (Murrieta-Cid et al., 2007) we specifically considered the case in which both the pursuer and the evader are omnidirectional; that led to a sufficient escape condition for the evader. Then (Murrieta-Cid et al., 2008) we again considered both players as omnidirectional systems moving in an
environment containing obstacles. Further (Murrieta-Cid et al., 2008), we specifically addressed the combinatorial problem inherent to any strategy that considers visiting several locations in an environment with obstacles, and we provided a complexity result for this problem.

Bandyopadhyay et al. (2006) used a greedy approach for the problem of evading surveillance. To drive the greedy motion planning algorithm, a local minimum risk function is applied, called the vantage time. In the work of O’Kane (2008), a robot has to track an unpredictable target with bounded speed. The robot’s sensors are manipulated to record general information about the target’s movements and avoid the need for detailed, potentially damaging information about the target’s position being available if the robot’s sensors are accessed by other agent.

An interesting version of the problem involves multiple participants (several pursuers and evaders). Parker (2002) developed a method which attempts to minimize the total time in which the evaders escape surveillance. Jung and Sukhatme (2002) combined the application of mobile and static sensors. The authors used a metric for measuring the degree of occlusion, based on the average mean free path of a random line segment.

Pursuit-evasion can be used in a variety of applications. For example, Tekdas and Yang (2010) noticed the similarity between pursuit-evasion games and mobile routing for networking. Applying this similarity, they proposed motion planning algorithms for robotic routers to maintain connectivity between a mobile user and a base station. That work also includes a proof-of-concept implementation.

Our problem consists in determining motion strategies to always maintain surveillance of the evader (assuming surveillance at the beginning of the game). The evader is under pursuer surveillance whenever it is at a bounded variable distance to the pursuer. It is interesting to analyze this case because commercially available sensors have upper and lower range limits. In particular, even in the absence of obstacles, if the evader is farther or closer than the sensor range, then its location is unknown, and the surveillance is broken. Our results are applicable to problems in which the pursuer is a wheeled mobile robot tracking a human evader. An example of those problems is monitoring children with mobile robots to prevent them straying out-with a prescribed area set by their guardian. The results can also be applied for capturing an evader, i.e., moving the pursuer closer than a given distance to the evader.

3. Preliminaries

3.1. System model. Figure 1 shows the geometric description of the system. In an Euclidean plane, the OA’s position is represented by \((x_e, y_e)\) and the DDR’s position by \((x_p, y_p)\). We will refer to the line segment connecting these positions as the \(p\) rod, using an analogy with the problem presented by Schwartz and Sharir (1983). The length \(L\) of this rod corresponds to the distance between both players (it can be interpreted as the measure of a range sensor). The angle \(\theta_p\) denotes the angle of the pursuer’s wheels with respect to the \(x\)-axis, and \(\phi\) represents the angle of the rod (sensor’s orientation) with respect to the \(x\)-axis. \(\psi\) corresponds to the motion direction of the evader.

\[
\begin{align*}
\dot{x}_e &= u_1 \cos u_2, \\
\dot{y}_e &= u_1 \sin u_2, \\
\dot{x}_p &= u_3 \cos \theta_p, \\
\dot{y}_p &= u_3 \sin \theta_p, \\
\dot{\theta}_p &= u_4,
\end{align*}
\]

where \(u_1 \in [0, V_{\text{max}}]\) and \(u_2 \in [0, 2\pi]\) are the OA’s controls, and they represent its speed and motion direction, respectively. Note that we assume that \(u_1\) takes only positive values, but since \(u_2\) takes any value between \([0, 2\pi]\) after a time \(t\) the evader can reach any position inside or in the boundary of a circle with radius \(u_1 t\) centered at the evader’s initial position. For the DDR pursuer, we use the usual assignment of control inputs (Balkcom and Mason, 2002). The DDR controls its linear velocity \(u_3 \in [-V_{\text{p}}^{\text{max}}, V_{\text{p}}^{\text{max}}]\) and the rate of change of its motion direction \(u_4\).

3.2. Previous results. Previously (Murrieta-Cid et al., 2011) we presented the conditions under which it is possible for a differential drive robot (the pursuer) to track...
an omnidirectional mobile evader at a constant distance.

We have that the kinematic equations for a DDR (see LaValle, 2006) are given by

\[ u_3 = V_p = \frac{(w_r(t) + w_l(t))R}{2}, \quad (2) \]

\[ u_4 = \dot{\theta}_p = \frac{(w_r(t) - w_l(t))R}{2b}, \quad (3) \]

where \( u_3 \) is the linear velocity, \( u_4 \) is its angular velocity, \( w_r \) is the angular velocity of the \( i \)-th wheel, \( R \) is the radius of the wheels, and \( b \) is the distance between the center of the robot and the wheel location.

Without loss of generality, in what follows we will assume \( R = 1 \). Adding and subtracting Eqns. (2) and (3) and solving for \( u_4 \), one obtains

\[ u_4 = \frac{1}{b}(w_r - u_3) \quad (4) \]

and

\[ u_4 = \frac{1}{b}(-w_l + u_3). \quad (5) \]

These equations mean that for a given value of \( w_r \) (resp. \( w_l \)) there is a linear relation between the controls \( u_3 \) and \( u_4 \).

Recalling that \( R = 1 \), the absolute value of the angular velocities \( w_r, w_l \) is bounded by \( |V_p^{\text{max}}| \), the maximum attainable linear speed. The maximum counterclockwise turning speed \( u_4^{\text{max}} \) is obtained when either \( w_r = V_p^{\text{max}} \) or \( w_l = -V_p^{\text{max}} \). For these values, one may obtain from Eqn. (4) and (5) respectively

\[ 0 \leq u_4^{\text{max}} = \frac{1}{b}(V_p^{\text{max}} - u_3) \quad (6) \]

with

\[ u_3 = \frac{1}{2}(V_p^{\text{max}} + w_l) \geq 0, \quad (7) \]

and

\[ 0 \leq u_4^{\text{max}} = \frac{1}{b}(V_p^{\text{max}} + u_3) \quad (8) \]

with

\[ u_3 = \frac{1}{2}(w_r - V_p^{\text{max}}) \leq 0. \quad (9) \]

Similarly, the maximum clockwise turning speed \( u_4^{\text{min}} \) is obtained when either \( w_r = -V_p^{\text{max}} \) or \( w_l = V_p^{\text{max}} \), and one has from Eqns. (4) and (5)

\[ 0 \geq u_4^{\text{min}} = \frac{1}{b}(-V_p^{\text{max}} - u_3) \quad (10) \]

with

\[ u_3 = \frac{1}{2}(-V_p^{\text{max}} + w_l) \leq 0, \quad (11) \]

and

\[ 0 \geq u_4^{\text{min}} = \frac{1}{b}(-V_p^{\text{max}} + u_3) \]

with

\[ u_3 = \frac{1}{2}(-V_p^{\text{max}} + w_l) \geq 0. \quad (12) \]

with

\[ u_3 = \frac{1}{2}(w_r + V_p^{\text{max}}) \geq 0. \quad (13) \]

Equations (6), (8), (10) and (12) may be combined in the inequality

\[ |u_4| = |\dot{\theta}_p| \leq \frac{1}{b}(V_p^{\text{max}} - |u_3|). \quad (14) \]

This inequality characterizes the space of valid controls (control space \((u_3, u_4)\)) for the DDR, which corresponds to the boundary and interior of the rhombus depicted in Fig. 2.

![Fig. 2. Control space \((u_3, u_4)\).](image)

The relation between the evader and the pursuer positions is given by

\[ \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} x_e \\ y_e \end{pmatrix} + L \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}. \quad (15) \]

Computing the time derivative of Eqn. (15), and recalling that \( L = 0 \) (Murrieta-Cid et al., 2011), we proved that the linear speed \( u_3^* \) of the pursuer required to maintain a constant distance \( L \) to the evader is in fact fixed: for given values of \( u_1, u_2, \dot{\theta}_p \) and \( \phi, u_3^* \) is given by

\[ u_3^*(\phi, \dot{\theta}_p, u_1, u_2) = \frac{u_1 \cos(u_2 - \phi)}{\cos(\dot{\theta}_p - \phi)}. \quad (16) \]

The authors also proved that when the pursuer successfully tracks the evader to a constant distance (i.e., when \( u_3 = u_3^* \)), the angular velocity of the rod \( \phi \) is

\[ \dot{\phi}(\phi, \dot{\theta}_p, u_1, u_2) = \frac{u_1 \sin(\dot{\theta}_p - u_2)}{L \cos(\dot{\theta}_p - \phi)}. \quad (17) \]

Evaluating the expression (13) for \( u_3 = u_3^* \), we get

\[ \max |\dot{\theta}_p| = \max |u_4| |u_3^*| = \frac{1}{b}(V_p^{\text{max}} - |u_3^*|). \quad (18) \]
The difference \( u \) where \( \theta \) lead the system to \( u \) of the players' controls is given by the velocity of the pursuer reducing the maximum feasible \( u \) whenever the evader applies controls \( u \) the pursuer's maximal angular velocity that it can use evader applies controls \( u \) the pursuer's linear velocity that maintains the constant speed must avoid the situation when \( \theta \) the one that maximizes the difference \( |\phi|\). To this end, both players manipulate the change in \( \theta \) given by \( \phi - \theta_p \).

Note that the evader can control \( \phi \) directly using its controls \( u_1 \) and \( u_2 \) (Eqn. (17)), but also \( \theta_p \), indirectly, through \( u_3 \), since it can maximize the required linear velocity of the pursuer reducing the maximum feasible value for \( u_1 \) (see Eqn. (16)).

Hence, the evader’s optimal motion direction \( u_2^* \) is the one that maximizes the difference \( |\phi| - \max |\theta_p| \). The equation below establishes \( u_2^* \):

\[
\begin{align*}
\psi_4 &= \arctan\left(-\frac{\sin \phi + \frac{1}{L} \cos \theta_p}{-\cos \phi - \frac{1}{L} \sin \theta_p}\right) & \text{if} \quad (\theta_p - \phi) \in [0, \pi], \\
\psi_3 &= \arctan\left(-\frac{\sin \phi - \frac{1}{L} \cos \theta_p}{-\cos \phi + \frac{1}{L} \sin \theta_p}\right) & \text{if} \quad (\theta_p - \phi) \in (\pi, 2\pi).
\end{align*}
\]

The difference \( |\phi| - \max |\theta_p| = 0 \) expressed as a function of the players’ controls is given by

\[
\frac{|u_3^* \sin(\theta_p - u_2^*)|}{|L \cos(\theta_p - \phi)|} - \frac{1}{b} \left( V_{p}^{\max} - \frac{u_1^* \sin(\theta_p - u_2^*)}{\cos(\theta_p - \phi)} \right) = 0,
\]

where \( u_1^* \) is the maximal linear velocity of the evader, that is, \( u_1^* = V_{e}^{\max} \), \( u_2^* \) is defined by Eqn. (19), \( u_3^* \) denotes the pursuer’s linear velocity that maintains the constant inter-player distance between the players whenever the evader applies controls \( u_1^* \) and \( u_2^* \), and \( u_4^* \) denotes the pursuer’s maximal angular velocity that it can use whenever the evader applies controls \( u_1^* \) and \( u_2^* \).

Doing some algebraic manipulation of Eqn. (20), Murrieta-Cid et al. (2011) proved that the condition \( |\phi| - \max |\theta_p| = 0 \) can be written as

\[
V_{p}^{\max} (|\cos(\theta_p - \phi)|) = |u_1^*|g(\phi, \theta_p, u_2^*),
\]

where

\[
g(\phi, \theta_p, u_2^*) = \begin{cases} 
-\cos(\phi - u_2^*) - \frac{1}{L} \sin(\theta_p - u_2^*) & \text{if} \quad (\theta_p - \phi) \in [0, \pi], \\
-\cos(\phi - u_2^*) + \frac{1}{L} \sin(\theta_p - u_2^*) & \text{if} \quad (\theta_p - \phi) \in (\pi, 2\pi).
\end{cases}
\]

We stress the fact that the evader’s motion is not given a priori. In our previous work (Murrieta-Cid et al., 2011), we presented a worst-case analysis. We proved that if the evader uses at all times the controls that maximize the difference \( |\phi(t)| - \max |\theta_p(t)| \), that is, \( u_1^* = V_{e}^{\max} \) and \( u_2^* \) defined by Eqn. (19), and this difference is still negative, then no evader’s control will make the difference equal to or greater than zero. Hence, whenever this condition holds, the pursuer can maintain a constant distance to the evader and it can align its heading with the evader location in finite time regardless of the evader motion.

This main result was presented by Murrieta-Cid et al. (2011) defining the manifold

\[
M(V_{e}^{\max}, V_{p}^{\max}, \theta_p, \phi) = |\phi(u_1^*, u_2^*)| - \frac{1}{b} (V_{p}^{\max} - |u_3^*|),
\]

exactly equivalent to

\[
M(V_{e}^{\max}, V_{p}^{\max}, L, \theta_p, \phi) = V_{p}^{\max} (|\cos(\theta_p - \phi)|) - |u_1^*|g(\phi, \theta_p, u_2^*) = 0.
\]

The manifold \( M(V_{e}^{\max}, V_{p}^{\max}, \theta_p, \phi) = 0 \) is thus crucial for determining the behavior of the system, since it divides the state space into two regions: one \( (M > 0) \) in which the evader can break the constant distance to the pursuer, and the other \( (M < 0) \) in which the pursuer can maintain the constant distance to the evader and align its heading with the evader’s location. Note that at the moment the pursuer’s heading reaches parallelism with the rod (the pursuer’s heading is pointing to the evader’s location): it is possible for the pursuer to keep this parallelism by applying \( u_4 = \theta_p = \phi \).

4. Pursuit-evasion at a bounded variable distance

In this section, we present the conditions and motion strategies for both players that allow one to relax the constant distance constraint as long as the configuration of the system remains in the initial region.
For simplicity, instead of working with the variables $(V_e^{\text{max}}, V_p^{\text{max}}, L, \theta_p, \phi)$, we will use the space $(L, \delta)$ with
\[
\delta = \theta_p - \phi, \tag{25}
\]
where $V_e^{\text{max}}$ and $V_p^{\text{max}}$ are fixed. In this case, $M(V_e^{\text{max}}, V_p^{\text{max}}, L, \theta_p, \phi) = 0$ may be written as $M(L, \delta) = 0$. We can rewrite Eqn. (24) as
\[
M(V_e^{\text{max}}, V_p^{\text{max}}, L, \theta_p, \phi) = |u_1^*|g(\phi, \theta_p) - V_p^{\text{max}} (|\cos(\theta_p - \phi)|) = 0. \tag{26}
\]

Using some trigonometric identities, in the proof of Lemma II in the work of Murrieta-Cid et al. (2011) it is shown that, if $\delta \in [0, \pi]$ (including $\delta \in [0, \pi/2]$) and for $u_2^*$ as defined by Eqn. (19), we have
\[
g(\phi, \theta_p, u_2^*) = g(\phi, \theta_p) = \sqrt{1 + \frac{2b}{L} \sin(\theta_p - \phi) + \left(\frac{b}{L}\right)^2}. \tag{27}
\]

Substituting $g(\phi, \theta_p)$ given by Eqn. (27), $u_1^* = V_e^{\text{max}}$ and Eqn. (25) into Eqn. (26), and recalling that $V_e^{\text{max}}$ and $V_p^{\text{max}}$ are fixed, we get
\[
M(L, \delta) = V_e^{\text{max}} \sqrt{1 + \frac{2b}{L} \sin(\delta) + \left(\frac{b}{L}\right)^2} - V_p^{\text{max}} \cos(\delta) = 0. \tag{28}
\]

Figure 3 shows the regions in the $(L, \delta)$ space where $M(L, \delta) > 0$ or $M(L, \delta) < 0$, and the curves (thick lines) where $M(L, \delta) = 0$ for $\delta \in [-\pi, \pi]$.

In Fig. 3 we can observe that the value of $M(L, \delta)$ has some symmetry properties as the value of $\delta$ varies in $[-\pi, \pi]$. Using these properties, we have that the problem can always be reduced to the interval $[0, \pi/2]$, the other quadrants being analogous.

Figure 4 shows the curve representing $M(L, \delta) = 0$ in this interval. In the upper region of the figure are those configurations $(L, \delta)$ where the OA avoids constant distance tracking and $M(L, \delta) > 0$. In the bottom region are those where the DDR maintains tracking and $M(L, \delta) < 0$. The DDR “maintains tracking” in the region $M < 0$, in the sense that, by applying the motion strategy provided below, the DDR can either maintain a constant distance to the evader or achieve any inter-agent distance $L_G$ (plus/minus a small value $\epsilon$), regardless of the actions taken by the evader.

As our analysis will be based on the two regions composing the space $(L, \delta)$, it is important to prove some useful properties of the curve separating those regions.

Lemma 1. Let $\delta^*(L)$ be the curve separating the regions where $M(L, \delta) < 0$ and $M(L, \delta) > 0$.

1. There is a critical value $L = L_o^*$ such that $\delta^*(L_o^*) = 0$.

2. For $L > L_o^*$, $\delta^*(L)$ is a strictly increasing function.

3. If $L \to \infty$, then
\[
\delta^*(L) \to \cos^{-1}\left(\frac{V_e^{\text{max}}}{V_p^{\text{max}}}\right) \leq \frac{\pi}{2}.
\]

4. For $L < \infty$,
\[
\delta^*(L) \leq \cos^{-1}\left(\frac{V_e^{\text{max}}}{V_p^{\text{max}}}\right) \leq \frac{\pi}{2}.
\]

Proof. From Eqn. (28), and recalling that $M(L, \delta^*(L)) = 0$, we have that
\[
M(L, \delta^*(L)) = V_e^{\text{max}} \sqrt{1 + \frac{2b}{L} \sin(\delta^*(L)) + \left(\frac{b}{L}\right)^2} - V_p^{\text{max}} \cos(\delta^*(L)) = 0. \tag{29}
\]
If \( \delta^*(L) = 0 \), then

\[
M(L, 0) = V_e^{\text{max}} \sqrt{1 + \left(\frac{b}{L}\right)^2} - V_p^{\text{max}} = 0. \tag{30}
\]

From the last expression, and by doing some algebra, we obtain the value \( L_o \) such that \( \delta^*(L_o) = 0 \):

\[
L_o^* = \frac{V_e^{\text{max}} b}{\sqrt{(V_p^{\text{max}} - V_e^{\text{max}})(V_p^{\text{max}} + V_e^{\text{max}})}} \tag{31}
\]

which may also be written as

\[
L_o^* = \frac{\rho b}{\sqrt{1 - \rho^2}} \tag{32}
\]

where

\[
\rho = \frac{V_e^{\text{max}}}{V_p^{\text{max}}} \tag{33}
\]

This proves the first part of the lemma.

Note that as \( \rho \to 1, L_o^* \to \infty \), which implies that the OA evader can always break constant distance surveillance, which further means, as shown below, that the OA can always attain an arbitrary distance to the DDR pursuer. On the other hand, for \( \rho \approx 0, L_o^* \to 0 \). In what follows it will always be assumed that \( \rho < 1 \).

From Eqn. (29), we observe that in order to keep a constant value of 0 for \( M(L, \delta) \), if we increase the value of \( L \), then we have also to increase the value of \( \delta \). Therefore, \( \delta^*(L) \) is a strictly increasing function with respect to \( L > L_o^* \), which proves the second part of the lemma. As \( L \to \infty \), we have that Eqn. (29) takes the form

\[
M(L, \delta^*(L)) = V_e^{\text{max}} - V_p^{\text{max}} \cos(\delta^*_\infty) = 0. \tag{34}
\]

By a straightforward manipulation of Eqn. (34), we obtain

\[
\delta^*_\infty = \cos^{-1}(\rho) \tag{35}
\]

Note that \( \delta^*_\infty < \pi/2 \), and for \( \rho \approx 0, \delta^*_\infty \to \pi/2 \). This proves the last part of the lemma.

This lemma implies that \( \delta^*(L) \) is a bounded strictly increasing function with respect to the inter-player distance \( L \). These properties allow us to define the regions in which each player controls the system. The DDR controls the system in the region \( M < 0 \), in the sense that it can achieve any inter-agent distance \( L \in [L_G - \epsilon, L_G + \epsilon] \), regardless of the actions taken by the evader. The OA controls the system in the region \( M > 0 \), in the sense that it can break the constant distance between the players, regardless of the actions taken by the pursuer.

**Remark 1.** From Lemma 1, we have that there is a critical value \( L = L_o^* \) bounding \( \delta^*(L) \) from the left. We have that

\[
L_o^* = \frac{\rho b}{\sqrt{1 - \rho^2}},
\]

where

\[
\rho = \frac{V_e^{\text{max}}}{V_p^{\text{max}}}.
\]

In some cases \( L_o^* < b \). The critical value corresponds to an inter-player distance located inside the robot’s radius. In those cases, we must assume that the curve \( \delta^*(L) \) is bounded by the critical value \( L_o^* = b \), corresponding to configurations where the robot is in collision with the OA evader.

In what follows, we will show that from a given initial configuration \( L_I, \delta_I \), the DDR (depending on the sign of \( M(L_I, \delta_I) \)) will be able to move in such a way that any desired inter-player distance \( L_G \pm \epsilon \) (with certain restrictions) may be obtained in finite time.

### 4.1. DDR pursuer strategy.

The following theorem establishes a strategy with which the DDR can reach a distance \( L_C \in [L_G - \epsilon, L_G + \epsilon] \) in finite time for any \( \epsilon > 0 \), independently of the strategy followed by the OA.

**Theorem 1.** Assume that for the initial configuration \( M(L_I, \delta_I) < 0 \). Given \( \epsilon > 0 \), define \( L_B^* = L_o^* + \epsilon > 0 \). Let \( (L_I, L_G, L_C) > L_B^* + \epsilon \), be the initial, the goal and the current distance between the DDR pursuer and the OA evader, respectively. The DDR can reach a distance \( L_C \in [L_G - \epsilon, L_G + \epsilon] \) in finite time, repeating the following strategy:

1. If \( \delta(L_C) > 0 \), move at constant \( L_C \), changing the DDR’s heading until it is parallel to the orientation of the rod, i.e., make \( \delta(L_C) = 0 \).
2. If \( \delta(L_C) = 0 \), move during a time

\[
\hat{T} = \min(T^*, \frac{|L_C - L_G|}{2V_p^{\text{max}}})
\]

directly towards or away from the position of the OA at time \( t \), depending on the sign of \( L_C - L_G \), with a velocity \( V = \text{sgn}(L_C - L_G) \cdot V_p^{\text{max}} \), where

\[
T^* = \min\left(\frac{\epsilon}{2V_p^{\text{max}}}, \frac{L_B^* \sin(\delta^*(L_B^*))}{V_e^{\text{max}}} \right). \tag{36}
\]

**Proof.** From the results of Murrieta-Cid et al. (2011) it follows directly that if \( M(L_C, \delta) < 0 \), the DDR can yield \( \delta(L_C) = 0 \) in finite time, using the controls \( u_1^*, u_4^* \). For the second part of the strategy, one has to show that, if the DDR moves with a velocity \( V = \text{sgn}(L_C - L_G) \cdot V_p^{\text{max}} \) during a time \( \hat{T} \), \( |L_C - L_G| \) will decrease and the system will remain in the region where \( M(L_C, \delta) < 0 \). To do this, we consider the following two cases:
Case I: ($L_G < L_C$). In this case, we show that in the ($L, \delta$) space, after an incremental motion of duration $\hat{T}$, the new system configuration ($L'_C, \delta'$) falls inside the shaded rectangle of Fig. 5 with $A = L_C - (V_p^{\text{max}} - V_e^{\text{max}})\hat{T}$, $B = L_B^* + \epsilon$ and $C = \delta^*(L_B^*)$, which means that $L_C$ is decreasing as a function of time, and the system never leaves the region where $M(L, \delta) < 0$.

Since the DDR wants to decrease the inter-player distance, it moves toward the OA during a time $\hat{T}$. In order to show that $L'_C < L_C$, it is enough to consider the worst case for the bound $A$, namely, when the OA moves directly away from the DDR at maximum speed, in which case $L'_C = A < L_C$. We use analogous reasoning in order to prove the next cases.

To show that $L'_C > B$, again one considers the worst case for this bound, namely, when the OA moves directly towards the DDR at maximum speed. In this case, from the definitions of $\hat{T}$ and $T^*$, one has that

$$L'_C = L_C - (V_p^{\text{max}} + V_e^{\text{max}})\hat{T} > L_C - 2V_p^{\text{max}}\hat{T} > L_C - \epsilon > B.$$  \hspace{1cm} (37)

Finally, for the bound $C$, the worst case is obtained when the OA moves perpendicularly to the rod at maximum speed (see Fig. 6). In this case, the final angle $\delta'$ satisfies

$$\sin(\delta') \leq \frac{V_e^{\text{max}}T^*}{L_C} < \frac{V_e^{\text{max}}T^*}{L_B^*} \leq \sin(\delta^*(L_B^*)).$$  \hspace{1cm} (38)

where the last inequality follows from Eqn. (36). Since the $\sin(\cdot)$ function is increasing in the interval $[0, \pi/2]$, from Lemma 1 one gets that $\delta' < \delta^*(L_B^*) < \delta^*(L_C^*)$.

Case II: ($L_G > L_C$). In this case there are only two bounds to consider for the new configuration ($L'_C, \delta'$) (see Fig. 7): $D = L_C + (V_p^{\text{max}} - V_e^{\text{max}})\hat{T}$ and $C = \delta^*(L_B^*)$. The first one is obtained when the OA evader moves directly towards the DDR pursuer at maximum speed, in which case $L'_C = D > L_C$, and the other when the DDR moves perpendicularly to the rod at time $t$, as in Case I, in which

$$\sin(\delta') \leq \frac{V_e^{\text{max}}T^*}{L_C} < \frac{V_e^{\text{max}}T^*}{L_B^*} < \sin(\delta^*(L_B^*)),$$  \hspace{1cm} (39)

and therefore $\delta' < \delta^*(L_C^*)$ as above. This completes the proof.

Note that in both cases, since

$$\hat{T} \leq \frac{\epsilon}{2V_p^{\text{max}}} < \frac{\epsilon}{2V_e^{\text{max}}},$$  \hspace{1cm} (40)

the bound on the magnitude of the maximum overshoot (or undershoot) from the target distance is

$$\left(V_p^{\text{max}} + V_e^{\text{max}}\right)\hat{T} < 2V_p^{\text{max}}\hat{T} \leq \epsilon.$$  \hspace{1cm} (41)

Note also that this overshoot or undershoot is due to the assumption that while the pursuer changes the inter-player distance, the motion direction of the evader is unknown. In the worst case, the evader moves in the opposite direction to the one assumed to establish the bounds.

The theorem gives a constructive analysis which yields feasible motions for the DDR pursuer to accomplish its goals. Namely, to go from $L_I$ to $L_G$, the pursuer strategy is to alternate between two sub-strategies: (i) move at constant $L$ to make $\delta = 0$ (which takes a finite amount of time), and (ii) move directly towards (or, away from) the evader for a finite time interval $\hat{T}$.
In Theorem 1, it is established that each time steps (i) and (ii) are applied, \(|L_t - L_G|\) will decrease (or resp. increase) by a finite amount. This implies that the goal \(|L_t - L_G| < \epsilon\) will be attained after a finite number of steps consisting in sub-steps (i) and (ii).

In the second part of the pursuer motion strategy, it moves a discrete time \(\hat{T}\). Thus, the number of sub-motions means the number of times the pursuer moves a finite time \(\hat{T}\) to achieve the desired inter-player distance.

**Remark 2.** In step (ii) of the proposed strategy, \(L_C\) represents the inter-player distance at the instant where \(\hat{T}\) is computed. The strategy then dictates that the DDR moves during time \(\hat{T}\) at maximum speed in a direction that depends on the sign of \(L_C - L_G\). During this time interval the instantaneous value of \(L_C - L_G\) may change sign, but according to the proposed strategy this will not affect the DDR motion until \(\hat{T}\) is re-computed, which will not happen until the current incremental motion of step (ii) is completed, and if needed, \(\delta(L_C)\) is made equal to zero using step (i). Once \(L_C\) is inside the interval \(L_G \pm \epsilon\), i.e., once the control goal has been achieved, depending on the application, it may be more convenient to change the strategy and apply only step (i) to maintain a constant inter-player distance \(L_{G*} \in [L_G - \epsilon, L_G + \epsilon]\).

In the theorem, \(\epsilon\) is a parameter which represents the tolerance of reaching the desired inter-player distance. It allows one to determine a safety margin for not crossing the manifold defining the space partition (recall that the regions of the partition define the winner of the game).

**Remark 3.** On the one hand, a small \(\epsilon\) allows reaching a distance that is closer to the desired inter-player distance \(L_G\) between the players. But on the other hand, as \(\epsilon\) decreases, the number of sub-motions necessary to reach a given configuration increases. Here \(\epsilon\) may be set according to the user requirements, that is, the precision of the desired inter-player distance between the players versus the number of sub-motion (equivalent to the number of pursuer controls’ switches) to reach a desired inter-player distance.

### 4.2. Combining continuous and discrete modeling for reaching \(L_C \in [L_G - \epsilon, L_G + \epsilon]\)

As mentioned above, we model the pursuer-evader system as a hybrid one combining continuous and discrete motion strategies. The first part (continuous) of the pursuer’s strategy is used to maintain a constant distance between the players and to align the pursuer’s heading with the evader’s location. Murrieta-Cid et al. (2011) proved that if \(M < 0\) then

\[|\hat{\phi}(t, u_1^*, u_2^*)| < \max |\hat{\theta}_p(t, u_4^{**})|, \forall t.\]

Therefore, the pursuer aligns its heading with the evader’s location in finite time. Recall that \(\phi\) represents the rate of change of \(\phi\), the direction between the pursuer and evader locations, \(u_1^*\) and \(u_2^*\) denote respectively the evader’s speed and the evader’s motion direction that maximizes the difference \(|\hat{\phi} - \max |\hat{\theta}_p|\), and \(u_4^{**}\) denotes the pursuer’s linear velocity that maintains a constant inter-player distance between the players whenever the evader applies controls \(u_1^*\) and \(u_2^*\).

By Remark 2, \(|L_C - L_G|\) decreases and the system remains in the region \(M(L, \delta) < 0\). Hence, \(L_G \pm \epsilon\) is reached in a finite number of sub-motions. There are direct equivalences between \(\lambda_1\) and the region delimited by \(M < 0\), and also between \(\lambda_2\) and \(\epsilon\). Therefore, the system is stable.

### 5. Simulations

In this section, we present some simulations showing the players’ strategies described before. The first simulation corresponds to the case when the DDR pursuer wants to reduce the inter-player distance. The initial parameters of the system are \(V_p^{\max} = 1 \text{ m/s}, V_e^{\max} = 0.5 \text{ m/s}, b = 1 \text{ m}, v_e = 1 \text{ m}, y_e = 0 \text{ m}, L = 2 \text{ m}, \theta_p = 40^\circ, \phi = 0^\circ, \epsilon = 0.20, L_0 = 1 \text{ m},\) and \(L_G = 1.25 \text{ m}.

Figure 8 shows the system trajectory in the space \((L, \delta)\). The trajectories followed by the players in the Euclidean plane are shown in Fig. 9. In this case, the DDR pursuer first aligns its heading with the rod’s orientation, and then it moves directly towards the OA evader. The OA tries to move directly away from the DDR. We must point out the fact that in this example the DDR moves backwards, and the value of \(u_3\) is negative.

In Fig. 10, we can observe the variation of the inter-player distance \(L\) with respect to time, when the DDR wants to get closer to the OA. Initially, the DDR is aligning its heading with the rod orientation. During this time interval, the distance between both players remains constant. Once the DDR has aligned its heading, it starts moving toward the OA, while this player moves away from the DDR. Both players move in the same direction, but as the DDR moves faster than the OA, the inter-player...
Fig. 8. Representation in \((L, \delta)\) of the case when the DDR decreases the inter-player distance. The thick curve corresponds to \(M(L, \delta) = 0\), and the thick dashed line to the value of \(L_B\). The system is initially at the point \((2, 40^\circ)\).

The thin lines show the trajectory followed by the system. At the end, the system is at the point \((1.25, 0^\circ)\).

The second simulation corresponds to the case when the DDR pursuer increases the inter-player distance. The parameters are the same that in the first simulation, but with the goal distance \(L_G = 3\) m.

Figure 17 shows the system trajectory in the space \((L, \delta)\). The trajectories followed by the players in the Euclidean plane are shown in Fig. 18. The DDR pursuer first aligns its wheels with the rod’s orientation, then it moves away from the evader, and the OA evader tries to get closer to the DDR.

In Fig. 19, we can observe the variation of the inter-player distance \(L\) with respect to time, when the DDR gets farther to the OA.
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Fig. 11. Variation of the control input $u_1$, corresponding to the system trajectory of Fig. 9.

Fig. 12. Variation of the control input $u_2$, corresponding to the system trajectory of Fig. 9.

Fig. 13. Variation of the control input $u_3$, corresponding to the system trajectory of Fig. 9.

Fig. 14. Variation of the control input $u_4$, corresponding to the system trajectory of Fig. 9.

Figures 20 and 21 show the values of the evader’s controls $u_1$ and $u_2$ during the game. We can observe that, as in the previous simulation, the evader moves always at full speed. When the DDR aligns its heading with the evader’s position and starts to move away the evader, the evader also starts to move with $u_2 = \phi$, trying to decrease its distance to the DDR.

In Figs. 22 and 23 we show the pursuer’s inputs $u_3$ and $u_4$. Also, as in the previous simulation, in both figures we can note that during the first part of the game the pursuer is changing the values of both inputs to align its heading with the position of the evader. Once the pursuer’s heading is parallel to the evader’s position, the pursuer moves at full linear speed $u_3$ towards the evader.

Figures 24 and 25 show the variation in $\phi$ and $\delta$, respectively, during the game.

6. Discussion and conclusions

This work proposes an extension of the research presented by Murrieta-Cid et al. (2011). The motion strategies presented in this paper are applicable to several problems related to surveillance or capture.

1. They allow a DDR pursuer to maintain an omnidirectional evader within a limited sensing range defined by a maximal $L_{\text{max}}$ and a minimal $L_{\text{min}}$ sensing distances, provided that the limited sensing range satisfies the restriction imposed by $L_{\text{min}} > L_{o}^* + 2\epsilon$.

2. They allow a DDR pursuer to reduce the distance to the omnidirectional evader, again provided that the desired inter-player distance satisfies the restriction $L_{G} > L_{o}^* + 2\epsilon$. Indeed, the problem of capturing an evader can be established in terms of this inter-player distance. That is, the capture condition is defined as moving the DDR closer than a given distance to the omnidirectional evader.

To our knowledge, this is the first time that a solution is proposed for both problems: tracking and capturing an omnidirectional evader with a differential drive robot.
Fig. 15. Variation of $\dot{\phi}$, corresponding to the system trajectory of Fig. 9.

Fig. 16. Variation of the control input $\delta$, corresponding to the system trajectory of Fig. 9.

It is important to stress that, if $M(L, \delta) < 0$, then the DDR can obtain in finite time an inter-player distance $L \in [L_G - \epsilon, L_G + \epsilon]$, which satisfies $M(L, \delta) < 0$, and such that $L_G > L_0 + 2\epsilon$. In order to obtain the desired inter-player distance $L_G$ (within a tolerance $\epsilon$), the DDR performs the motion strategy described in Theorem 1.

The main drawback of the motion strategies presented in this paper is that they are not necessarily optimal in time. However, note that the presented analysis gives constructive proofs which yield feasible motions for the players to obtain their goals in finite time.

In future work, we will consider acceleration bounds on the pursuer and the evader.

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Fig. 18. DDR pursuer increases the inter-player distance. Again, the trajectories of the players were sub-sampled to show the motion direction of the players.

Fig. 19. Variation of $L$ as time elapses, corresponding to the system trajectory of Fig. 18. The pursuer increases the inter-player distance.

Fig. 20. Variation of the control input $u_1$, corresponding to the system trajectory of Fig. 18.

Fig. 21. Variation of the control input $u_2$, corresponding to the system trajectory of Fig. 18.


Fig. 22. Variation of the control input $u_3$, corresponding to the system trajectory of Fig. 18.


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Received: 25 April 2013
Revised: 9 November 2013
Re-revised: 18 January 2014