The maximal value of the error is the most important criterion in system design. It is also the most difficult one. For that reason there exist many other criteria. The extreme value of the error represents the attainable accuracy which can be obtained and the corresponding extreme time gives information about how fast the transients are. The extreme values of the error and the corresponding time are treated here as functions of the roots of the characteristic equation. The proposed analytical formulae allow designing systems with prescribed dynamic properties.

**Keywords:** extremal dynamic properties, oscillatory systems, extremal time.
The extreme value of the dynamic error is
\[ x(\tau) = \sum_{k=1}^{m} A_k e^{\alpha_k \tau} + \sum_{k=1}^{p} [B_k \cos (\omega_k \tau) + C_k \sin (\omega_k \tau)] e^{\beta_k \tau}. \]

(6)

The extremum of the extreme value of the dynamic error given by Eqn. (6), computed with regard to the parameters \( s_k, \alpha_k \), and \( \omega_k \), is obtained by equating the respective partial derivatives of \( x(\tau) \) to zero.

Denoting by
\[
\left( \frac{\partial x(\tau)}{\partial s_k} \right)^*, \left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^*, \left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^*
\]
the partial derivatives of the expression (6) for constant \( \tau \), we may write
\[
\begin{aligned}
&\left( \frac{\partial x(\tau)}{\partial s_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial s_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^*, \\
&\left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial s_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^*, \\
&\left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial s_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^* + \left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^*.
\end{aligned}
\]

(7)

However, from Eqn. (4) we have
\[ x^{(1)}(t) \bigg|_{t=\tau} = 0, \]
and therefore
\[
\begin{aligned}
&\left( \frac{\partial x(\tau)}{\partial s_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial s_k} \right)^*, \\
&\left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial \alpha_k} \right)^*, \\
&\left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^* = \left( \frac{\partial x(\tau)}{\partial \omega_k} \right)^*.
\end{aligned}
\]

(8)

We obtain the following conditions:
\[
\begin{aligned}
&\sum_{k=1}^{m} \frac{\partial A_k}{\partial s_j} e^{\alpha_k \tau} + A_j \tau e^{\alpha_j \tau} = 0, \\
&\sum_{k=1}^{m} \frac{\partial B_k}{\partial s_j} \cos \omega_k \tau + \frac{\partial C_k}{\partial s_j} \sin \omega_k \tau = 0, \\
&\sum_{k=1}^{m} \frac{\partial A_k}{\partial \alpha_j} e^{\alpha_k \tau} + (B_j \cos \omega_j \tau + C_j \sin \omega_j \tau) e^{\alpha_j \tau} = 0, \\
&\sum_{k=1}^{m} \frac{\partial A_k}{\partial \omega_j} e^{\alpha_k \tau} + (B_j \cos \omega_j \tau - C_j \sin \omega_j \tau) e^{\alpha_j \tau} = 0,
\end{aligned}
\]

(9)

In this way, we have a system of \( n \) linear and homogeneous equations with \( n \) unknowns which are
\[ e^{\alpha_k \tau}, \quad e^{\alpha_k \tau} \sin \omega_k \tau, \quad e^{\alpha_k \tau} \cos \omega_k \tau. \]

The determinant of the system (9) must vanish if there are nontrivial solutions. The same determinant (after being reflected about one of the main diagonals) is
\[
|D + A\tau|, \quad (10)
\]

where \( D \) and \( A \) are matrices determined by the following equations:
\[
\begin{aligned}
&D = \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial A_j}{\partial s_k} E_{jk}, \\
&A = \sum_{j=1}^{m} A_j E_{jj}.
\end{aligned}
\]

(10)

\[
\begin{aligned}
B_j (E_{m+2j-1,m+2j-1} - E_{m+2j,m+2j}) &+ C_j (E_{m+2j-1,m+2j} + E_{m+2j,m+2j-1}), \\
E_{jk} &= (\eta_{\mu,\nu}), \quad \mu, \nu = 1, \ldots, n, \\
\eta_{\mu,\nu} &= \delta_{\mu j} \delta_{\nu k} = \begin{cases} 1 & \text{for } \mu = j, \nu = k, \\
0 & \text{otherwise}.
\end{cases}
\end{aligned}
\]

(11)

Finally, we have
\[
|D + A\tau| = 0, \quad (13)
\]

for the unknown \( \tau \) and the system (9) yields (after some algebraic manipulations) the following equation:
\[
(-1)^n \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2) = 0.
\]

(14)

We obtain the following necessary condition.

**Theorem 1.** (Górecki and Turowicz, 1965) **The necessary condition for the extremal extremum \( x(\tau) \) as the function of \((\tau, s_1, s_2, \ldots, s_n)\) is**
\[
(-1)^n \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2) = 0.
\]
The relation (14) can be fulfilled if at least one of the conditions is met:
\[ \tau = 0, \]  
which means
\[ c_2 = 0, \]  
or
\[ A_k = 0 \]  
or
\[ B_k^2 + C_k^2 = 0. \]  

The conditions (16) or (17) lead to a reduced order of Eqn. (1).

It might be asked whether the time \( \tau \), corresponding to the extreme value of the dynamic error, attains an extreme value with respect to the parameters \( s, \alpha, \omega \). To investigate this, we assume that

\[ \text{We compute the partial derivatives of Eqn. (9), taking into account Eqn. (19):} \]

\[ \frac{\partial \tau}{\partial s_k} = 0, \quad k = 1, \ldots, m, \]
\[ \frac{\partial \tau}{\partial \alpha_k} = 0, \quad k = 1, \ldots, p. \]

Let
\[ \tau = 0, \]
which means
\[ c_2 = 0, \]
or
\[ A_k = 0, \]
or
\[ B_k^2 + C_k^2 = 0. \]

Theorem 2. (Górecki and Turowicz, 1965) The necessary condition for the extreme time \( \tau(s_1, \ldots, s_n) \) is

\[ (-1)^p \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2) \]
\[ \times \left[ \tau + \sum_{k=1}^{m} \frac{1}{r_k} + \sum_{k=1}^{p} \left( \frac{1}{\hat{r}_k} + \frac{1}{\hat{r}_k} \right) \right] = 0. \]

We obtain the following necessary condition for the extreme time \( \tau(s_1, \ldots, s_n) \).

The relation (25) can be fulfilled if at least one of the conditions is satisfied:
\[ \tau = 0, \]
which means
\[ c_2 = 0, \]
or
\[ \left\{ \begin{array}{l} A_k = 0, \\ B_k^2 + C_k^2 = 0 \end{array} \right. \]

or
\[ \left\{ \begin{array}{l} s_k = 0, \\ \alpha_k + \omega_k = 0 \end{array} \right. \]

or, finally and most interestingly,
\[ \tau = -\left[ \sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left( \frac{1}{\hat{r}_k} + \frac{1}{\hat{r}_k} \right) \right]. \]
Using Vieta’s formulae, \( \tau \) from Eqn. (29) is equal to

\[
\tau = \frac{a_{n-1}}{a_n},
\]

(30)

where \( a_{n-1} \) and \( a_n \) are the coefficients of Eqn. (2).

The set of equations (20)–(22) gives also another necessary condition for the extreme time \( \tau(s_1, \ldots, s_n) \), which was presented by Górecki and Turowicz (1966).

**Theorem 3.** The necessary condition for the extreme time \( \tau(s_1, \ldots, s_n) \) is

\[
D_n(\tau) = \begin{vmatrix}
 c_1 & c_2 & c_3 & c_4 & \ldots & c_n \\
 -\frac{a_{n-2}}{a_n} & \tau & -1 & 0 & \ldots & 0 \\
 -\frac{a_{n-3}}{a_n} & 0 & \tau & -2 & \ldots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 -\frac{a_1}{a_n} & 0 & 0 & 0 & \ldots & 2 - n \\
 -\frac{a_0}{a_n} & 0 & 0 & 0 & \ldots & \tau
\end{vmatrix} = 0.
\]

(31)

It is obvious from the condition (31) that there may exist \( n-1 \) values of \( \tau \). Taking into account that \( \tau = a_{n-1}/a_n \), we eventually obtain from Eqn. (31) \( n-2 \) values of \( \tau \). In general if all \( \tau_i > 0 \) (\( i = 1, 2, \ldots, n-1 \)) exist, then all the ratios \( c_i/c_1 \) (\( i = 2, 3, \ldots, n-1 \)) can be determined univocally.

The solution of the algebraic equation (31) for a higher degree may be obtained using additional assumptions (see Górecki, 2009; Górecki and Zaczyk, 2010).

After substitution of \( \tau = a_{n-1}/a_n \) into Eqn. (31), we obtain the relation between the initial conditions \( c_{i+1} \), \( i = 0, 1, \ldots, n-1 \), and coefficients \( a_k \), \( k = 1, 2, \ldots, n \).

\[
D_n = \begin{vmatrix}
 c_1 & c_2 & c_3 & c_4 & \ldots & c_n \\
 a_{n-2} & -a_{n-1} & a_n & 0 & \ldots & 0 \\
 a_{n-3} & 0 & -a_{n-1} & 2a_n & \ldots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 a_1 & 0 & 0 & 0 & \ldots & 0 \\
 1 & 0 & 0 & 0 & \ldots & 0
\end{vmatrix} = 0.
\]

(32)

### 3. Problem solution

**Theorem 4.** (Sędziwy, 1969) The sufficient conditions for the extreme \( \tau(s_1, \ldots, s_n) \) are

\[
\frac{d^2\tau}{ds_k^2} \neq 0, \quad k = 1, \ldots, n,
\]

(33)

\[
\frac{d^2\tau}{ds_k^2} = 0, \quad k \neq j, \quad k = 1, \ldots, n.
\]

(34)

The Hessian \( H_n \neq 0 \), where

\[
H_k = \begin{vmatrix}
 \frac{d^2\tau}{ds_1^2} & 0 & \ldots & 0 \\
 0 & \frac{d^2\tau}{ds_2^2} & \ldots & 0 \\
 \ldots & \ldots & \ldots & \ldots \\
 0 & 0 & \ldots & \frac{d^2\tau}{ds_n^2}
\end{vmatrix} \neq 0,
\]

(35)

If \( H_{2k-1} < 0 \) and \( H_{2k} > 0 \) for \( k = 1, 2, \ldots, n \), then \( \tau \) attains the maximum value with respect to \( s_1, \ldots, s_n \). If

\[
H_{2k-1} > 0 \quad \text{and} \quad H_{2k} > 0
\]

(36)

for \( k = 1, 2, \ldots, n \), then \( \tau \) attains the minimum value with respect to \( s_1, \ldots, s_n \).

**Theorem 5.** The conditions for the existence of \( \tau(s_1, \ldots, s_n, c_1, \ldots, c_{n-1}) \) are

\[
x^{(1)}(\tau) = 0,
\]

(37)

\[
D_n(a_1, \ldots, a_n, c_1, \ldots, c_{n-1}) = 0.
\]

(38)

These two equations, (37) and (38), are linear with respect to the initial conditions \( c_1, \ldots, c_{n-1} \). It is easy to solve them.

**Theorem 6.** The conditions for the existence of \( \tau_2, \tau_3, \ldots, \tau_{n-2} \) are

\[
x^{(1)}(\tau) = 0,
\]

(39)

\[
D_n(\tau) = 0,
\]

(40)

where \( \tau_1 = a_{n-1}/a_n \).

### 4. Particular cases

We illustrate the theorems in the particular cases of the equations.

#### 4.1. Second-order equation \((n = 2)\)

Let us consider the second order differential equations

\[
\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x = 0,
\]

(41)

with the initial conditions

\[
x(0) = c_1, \quad x^{(1)}(0) = c_2.
\]

The characteristic equation of Eqn. (41) is

\[
s^2 + a_1 s + a_2 = 0, \quad a_1, a_2 > 0.
\]

(42)

We denote by \( s_1, s_2 \) the roots of this equation and consider three cases:

1. \( s_1 \neq s_2 \) real and negative,
2. \( s_1 = s_2 \) real and negative,
3. \( s_1 = \alpha + j\omega, \quad s_2 = \alpha - j\omega \) complex with \( \alpha < 0 \).
4.1.1. First case: $s_1 \neq s_2$. The solution of Eqn. (41) is
\[ x(t) = \frac{s_2 c_1 - c_2}{s_2 - s_1} e^{s_1 t} + \frac{s_1 c_1 - c_2}{s_1 - s_2} e^{s_2 t}. \] (43)

The derivative of $x(t)$ is equal to
\[ x'(t) = \frac{s_1 (s_2 c_1 - c_2)}{s_2 - s_1} e^{s_1 t} + \frac{s_2 (s_1 c_1 - c_2)}{s_1 - s_2} e^{s_2 t}. \] (44)

The necessary condition for the extremum $x(t)$ is
\[ x'(\tau) = 0. \] (45)

From the relation (43), using the condition (45), we obtain
\[ e^{(s_1 - s_2)\tau} = \frac{s_2 (s_1 c_1 - c_2)}{s_1 (s_2 c_1 - c_2)}. \] (46)

The necessary conditions for $\tau$ as the function of $(s_1, s_2)$ attains an extremum are
\[ \frac{d\tau}{ds_1} = \frac{1}{s_2 - s_1} \left( \tau - \frac{c_2}{s_1 (s_1 c_1 - c_2)} \right) = 0, \] (47)
\[ \frac{d\tau}{ds_2} = \frac{1}{s_2 - s_1} \left( \tau - \frac{c_2}{s_2 (s_2 c_1 - c_2)} \right) = 0. \] (48)

It is easy to show that there may be at most one value of extreme $\tau$. In consequence, it is required that
\[ \tau = \frac{c_2}{s_1 (s_1 c_1 - c_2)} = \frac{c_2}{s_2 (s_2 c_1 - c_2)}. \] (49)

From (49) we obtain
\[ s_1 + s_2 = \frac{c_2}{c_1}, \quad \frac{c_2}{c_1} < 0. \] (50)

Substitution $c_2$ from (50) into the relation (49) gives
\[ \tau = \frac{c_2}{s_2 (s_2 c_1 - c_2)} = \frac{c_2}{s_2 [s_2 c_1 - (s_1 + s_2) c_1]} = \frac{1}{s_1 + s_2}, \] (51)

Sufficient condition for $\tau(s_1, s_2)$. After differentiating (47) and (48), we obtain
\[ \frac{d^2\tau}{ds_1^2} = \frac{c_2 (2 s_2 c_1 - c_2)}{s_2^3 (s_1 c_1 - c_2)^2} \] (52)
\[ \frac{d^2\tau}{ds_2^2} = \frac{c_2 (2 s_1 c_1 - c_2)}{s_2^3 (s_2 c_1 - c_2)^2}, \] (53)
\[ \frac{d^2\tau}{ds_1 ds_2} = -\frac{1}{(s_2 - s_1)^2}. \] (54)

but $d\tau/ds_2 = 0$ (see (48)) and
\[ \frac{d^2\tau}{ds_1 ds_2} = 0. \] (55)

The Hessian for $\tau = -\left( \frac{1}{s_1} + \frac{1}{s_2} \right)$ is equal to
\[ H = \begin{vmatrix} \frac{d^2\tau}{ds_1^2} & \frac{d^2\tau}{ds_1 ds_2} \\ \frac{d^2\tau}{ds_1 ds_2} & \frac{d^2\tau}{ds_2^2} \end{vmatrix} \] (56)
\[ = \begin{vmatrix} \frac{c_2 (2 s_1 c_1 - c_2)}{s_2^3 (s_1 c_1 - c_2)^2} \\ 0 \end{vmatrix} \begin{vmatrix} \frac{c_2 (2 s_2 c_1 - c_2)}{s_2^3 (s_2 c_1 - c_2)^2} \\ 0 \end{vmatrix} \]
\[ = \frac{c_2^2 (2 s_1 c_1 - c_2)^2 (2 s_2 c_1 - c_2)^2}{s_2^3 (s_1 c_1 - c_2)^2 (s_2 c_1 - c_2)^2}, \]

and, taking into account (50), we finally have
\[ H = \left( \frac{s_2^2 - s_1^2}{s_1 s_2} \right)^2 > 0 \]

This means that if there exists an extremum $\tau(s_1, s_2), s_1 \neq s_2$, then it has to be a minimum.

Existence condition. Substituting $c_2$ from the relation (50) into the relation (46), we obtain
\[ \tau = \frac{1}{s_2 - s_1} \ln \left( \frac{s_1}{s_2} \right)^2. \] (57)

Comparing with $\tau$ from (51), we have the equation
\[ \ln \left( \frac{s_1}{s_2} \right)^2 = \left( \frac{s_2}{s_1} - \frac{s_1}{s_2} \right). \] (58)

The only solution of Eqn. (55) is
\[ s_1 = s_2 = s, \] (59)

which is in contradiction with the assumption that $s_1 \neq s_2$.

We deduce that there does not exist an extremum $\tau$ for real $s_1 \neq s_2$.

4.1.2. Second case: $s_1 = s_2 = s < 0$. The solution of Eqn. (41) is
\[ x(t) = [c_1 + (c_2 - s c_1) t] e^{s t} \] (60)

and its derivative is
\[ \frac{dx(t)}{dt} = [c_2 + (c_2 - s c_1) s_1] e^{s t}. \] (61)

From the necessary condition $x^{(1)}(t) = 0$ and (61) we obtain
\[ \tau = \frac{c_2}{(s c_1 - c_2) s}. \] (62)
The derivative is
\[ \frac{d\tau}{ds} = -\frac{(2sc_1 - c_2)c_2}{s^2(sc_1 - c_2)^2} \] (63)

From the condition \( \frac{d\tau}{ds} = 0 \) we finally have \( c_2 = 0 \). In consequence, \( \tau_1 = 0 \) or
\[ s = \frac{c_2}{2c_1} < 0 \] (64)

and
\[ \tau_2 = -\frac{2}{s} = -\frac{c_1}{c_2} > 0, \]
\[ x(\tau_2) = -c_1e^{-2}, \quad c_1 > 0 \] (65)

\[ \frac{d^2\tau}{ds^2} = -32\left(\frac{c_1}{c_2}\right)^3, \quad c_1c_2 < 0. \] (67)

In conclusion, \( \tau \) has a minimum with respect to \( s \).

### 4.1.3. Third case: \( s_1 = \alpha + j\omega, s_2 = \alpha - j\omega \) are complex and \( \alpha < 0 \). From the relation (46), we have
\[ e^{2j\omega\tau} = \frac{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right] + j\omega c_2}{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right] - j\omega c_2}. \] (68)

From the relation (68), we obtain
\[ \cos(2\omega\tau) = \frac{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right]^2 - \omega^2 c_2^2}{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right]^2 + \omega^2 c_2^2}, \] (69)

\[ \sin(2\omega\tau) = \frac{2\omega c_2[\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right] - \omega^2 c_2]}{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right]^2 + \omega^2 c_2^2}. \] (70)

After division of (69) by (70), we find
\[ \cot(2\omega\tau) = \frac{\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right]^2 - \omega^2 c_2^2}{2\omega c_2[\left[(\alpha^2 + \omega^2)c_1 - \alpha c_2\right] - \omega^2 c_2]]. \] (71)

From the necessary condition
\[ \frac{d\tau}{d\alpha} = 0, \] (72)

we have
\[ 2j\omega c_2(2\alpha c_1 - c_2) \left[(c_2 - c_1\alpha) + j\omega c_1^2(\alpha + j\omega)^2 \right] = 0. \] (73)

From (73) we deduce that \( c_2 = 0 \), then \( \tau = 0 \) or \( \omega = 0 \) or
\[ \alpha = \frac{1}{2} \frac{c_2}{c_1} < 0. \] (74)

After using (74) in (71), we obtain
\[ \cot(2\omega\tau) = \frac{(\omega^2 - \alpha^2)^2 - 4\alpha^2\omega^2}{4\alpha\omega(\omega^2 - \alpha^2)} \] (75)

From the necessary condition
\[ \frac{d\tau}{d\omega} = 0, \] (76)

after differentiating (68), we have
\[ -2\tau \sin(2\omega\tau) = -\frac{\omega c_2^2[c_1(\alpha^2 - \omega^2) - c_2\alpha]}{(c_2 - c_1\alpha)^2 + c_1^2\omega^2]} \times \frac{(c_1(\alpha^2 + \omega^2) - c_2\alpha)}{(\alpha^2 + \omega^2)} \] (77)

After elimination of \( c_2 \), using (74), we get
\[ -2\tau \sin(2\omega\tau) = \frac{-16(\alpha - \omega)(\alpha + \omega)\omega^2}{(\alpha^2 + \omega^2)^3} \] (78)

and
\[ 2\tau \cos(2\omega\tau) = \frac{2c_2[c_1(\alpha^2 - \omega^2) - c_2\alpha]}{(\alpha^2 + \omega^2)^2} \times \frac{c_1(\alpha^2 + \omega^2) - c_2(\alpha + \omega)}{[(c_2 - c_1\alpha)^2 + c_1^2\omega^2]} \times \frac{c_1(\alpha^2 + \omega^2) + c_2(\alpha - \omega)}{[c_1(\alpha^2 + \omega^2) + c_2(\alpha - \omega)]}. \] (79)

After elimination of \( c_2 \) from (74),
\[ 2\tau \cos(2\omega\tau) = \frac{-4\alpha(\alpha^2 - 2\alpha\omega - \omega^2)(\alpha^2 + 2\alpha\omega - \omega^2)}{(\alpha^2 + \omega^2)^3}. \] (80)

From (77) and (80),
\[ 4\tau^2 = \frac{2c_2[c_1(\alpha^2 - \omega^2) - c_2\alpha]^2}{(\alpha^2 + \omega^2)[(c_2 - c_1\alpha)^2 + c_1^2\omega^2]}. \] (81)

After elimination of \( c_2 \),
\[ \tau^2 = \frac{(2\alpha)^2}{(\alpha^2 + \omega^2)^2}, \] (82)

and, finally, for \( \tau > 0 \),
\[ \tau = -\frac{2\alpha}{\alpha^2 + \omega^2}, \quad \alpha < 0. \] (83)

The determinant (56) in this case for \( s_1 = \alpha + j\omega \) and \( s_2 = \alpha - j\omega \) is
\[ H = \left[\frac{(s_2^2 - s_1^2)^2}{s_1^2 s_2^2}\right] = -\frac{4\alpha\omega}{\alpha^2 + \omega^2} < 0. \] (84)

It is obvious that \( \tau \) has a maximum with respect to \( \omega \).

**Sufficient condition.** After dividing both the sides of (79) by (77), we have
\[ \cot(2\omega\tau) = \frac{1}{4} \frac{(\alpha^2 - 2\alpha\omega - \omega^2)(\alpha^2 + 2\alpha\omega - \omega^2)}{\alpha\omega(\alpha - \omega)(\alpha + \omega)}. \] (85)

Comparing (71) with (85), for \( \alpha = \frac{1}{2} c_2/c_1 \) we obtain
\[ \omega = \pm \alpha. \] (86)

Substitution of (85) into (83) gives
\[ \tau = \frac{1}{\alpha}, \] (87)

which, together with (74), yields
\[ \tau = -2\frac{c_1}{c_2}, \quad \frac{c_1}{c_2} < 0. \] (88)
4.2. Third-order equation \((n = 3)\). Consider the following equation (Gorecki and Zaczyk, 2013):

\[
d^3x(t)/dt^3 + a_1 d^2x(t)/dt^2 + a_2 dx(t)/dt + a_3 x(t) = 0. \tag{89}
\]

The initial conditions are

\[
\begin{align*}
x(0) &= c_1, \\
x'(0) &= c_2, \\
x''(0) &= c_3.
\end{align*}
\]

The characteristic equation is

\[
s^3 + a_1 s^2 + a_2 s + a_3 = 0. \tag{90}
\]

We assume that the roots of \((90)\) are

\[
s_1, \quad s_2 = \alpha + j\omega, \quad s_3 = \alpha - j\omega,
\]

where \(\alpha < 0\).

The solution of Eqn. \((89)\) is

\[
x(t) = \frac{c_1 - c_2(s_2 + s_3) + c_1 s_2 s_3}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{c_2 - c_2(s_2 + s_1) + c_1 s_2 s_1}{(s_2 - s_3)(s_2 - s_1)} e^{s_2 t} + \frac{c_3 - c_2(s_1 + s_2) + c_1 s_3 s_2}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t}. \tag{91}
\]

The derivative of \(x(t)\) is equal to

\[
x'(t) = \frac{s_1[c_1 - c_2(s_2 + s_3) + c_1 s_2 s_3]}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{s_2[c_2 - c_2(s_2 + s_1) + c_1 s_2 s_1]}{(s_2 - s_3)(s_2 - s_1)} e^{s_2 t} + \frac{s_3[c_3 - c_2(s_1 + s_2) + c_1 s_3 s_2]}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t}. \tag{92}
\]

The necessary condition for the extremum \(x(t)\) is

\[
x'(t) \bigg|_{t = \tau} = 0. \tag{93}
\]

After substitution of \(s_1, \ s_2 = \alpha + j\omega, \text{ and } s_3 = \alpha - j\omega\) into \((92)\) and using \((93)\), Eqn. \((92)\) takes the form

\[
x'(\tau) = -\frac{1}{2} \left[ -4js_1 c_2 \alpha \omega + 2js_1 \omega^3 c_1 + 2js_1 c_1 \alpha^2 \omega \right] \left( \alpha - j\omega - s_1 \right) (\alpha + j\omega - s_1) \omega \nonumber
\]

\[
+\frac{2jc_2 \omega s_1}{\left( \alpha - j\omega - s_1 \right) (\alpha + j\omega - s_1) \omega} e^{s_1 \tau} - \frac{1}{2} \left[ -c_3 \alpha s_1 + s_1 c_1 \alpha^2 \omega - j c_2 \omega s_1 \right] \left( \alpha - j\omega - s_1 \right) (\alpha + j\omega - s_1) \omega
\]

\[
+ s_1^2 c_2 \alpha + c_3 \alpha^2 - j s_1 c_1 \alpha^2 \omega + j c_2 \omega^3 + j s_1^2 c_2 \omega - c_2 \alpha^3
\]

\[
\left( \alpha - j\omega - s_1 \right) (\alpha + j\omega - s_1) \omega
\]

\[
+ \frac{j c_2 \alpha^2 \omega - c_2 \omega^2 \alpha - s_1^2 c_1 \alpha^2 - s_1^2 c_1 \omega^2 + s_1 c_1 \alpha^3}{(\alpha - j\omega - s_1) (\alpha + j\omega - s_1) \omega} e^{\alpha + j\omega} \tau = 0. \tag{94}
\]

The derivatives of \(\tau\), determined by Eqn. \((94)\), with respect to \(s_1, \alpha \) and \(\omega\), yield the necessary conditions for the extreme \(\tau\):

\[
\frac{d\tau}{ds_1} = 0, \tag{95}
\]

\[
\frac{d\tau}{d\alpha} = 0, \tag{96}
\]

\[
\frac{d\tau}{d\omega} = 0. \tag{97}
\]

We get

\[
e^{(-\alpha + \tau) \tau} \cos(\omega \tau) \nonumber
\]

\[
= \left( \frac{\tau \alpha s_1^2 - \tau s_1 \alpha^2 + \tau s_1 \omega^2}{s_1 (\tau^2 \alpha^2 \omega^2 + (\alpha + \tau \omega)^2)} + \frac{0.5 \tau^2 s_1^2 \omega^2}{s_1 (\tau^2 \omega^2 \alpha^2 + (\alpha + \tau \omega)^2)} \right) \times (\alpha + \tau \omega)^2, \tag{98}
\]
Substitution of (106) into (101) gives

\[ e^{(-\alpha+\tau)s}\sin(\omega \tau) = \left( \frac{\tau \alpha s^2 - \tau s_1 \alpha^2 + \tau s_1 \omega^2}{s_1(\tau^2 \omega^2 \alpha^2 + (\alpha + \tau \omega^2)^2)} \right) + \frac{0.5 \tau^2 \alpha^2 + 0.5 \tau^2 \omega^2}{s_1(\tau^2 \omega^2 \alpha^2 + (\alpha + \tau \omega^2)^2)} + \frac{0.5 \tau^2 \alpha^4 + \tau^2 \omega^2 \alpha^2 + 0.5 \tau^2 \omega^4 + s_1 \alpha}{s_1(\tau^2 \omega^2 \alpha^2 + (\alpha + \tau \omega^2)^2)} \right) (\alpha \tau \omega), \]  

(99)

From Eqn. (100), we have

\[ \cos(2\omega \tau) = \frac{(\alpha + \tau \omega^2)^2 - \alpha^2 \omega^2 \tau^2}{(\alpha + \tau \omega^2)^2 + \alpha^2 \omega^2 \tau^2} \]  

(101)

and

\[ \sin(2\omega \tau) = \frac{2(\alpha + \tau \omega^2)\alpha \tau \omega}{(\alpha + \tau \omega^2)^2 + \alpha^2 \omega^2 \tau^2} \]  

(102)

The relations (98) and (99) lead to the assumption that

\[ s_1 = \alpha. \]  

(103)

In this case, the necessary condition for the extreme \( \tau \) is

\[ \tau = -\left( \frac{1}{s_1} + \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega} \right) \]  

\[ = \frac{-3 \alpha^2 + \omega^2}{\alpha(\alpha^2 + \omega^2)}. \]  

(104)

Substitution of \( \tau \) from the relation (104) into the relation (102) gives

\[ \sin(2\omega \tau) = -\frac{2 \alpha \omega(3 \alpha^2 + \omega^2)(\alpha^4 - 2 \alpha^2 \omega^2 - \omega^4)}{(\alpha^2 + \omega^2)(\alpha^4 + 3 \alpha^2 \omega^2 + \omega^4)}. \]  

(105)

One of the solutions of Eqn. (105) is

\[ \omega = \pm \alpha \sqrt{2} - 1. \]  

(106)

Then

\[ \sin(2\omega \tau) = 0. \]  

(107)

Substitution of (106) into (101) gives

\[ \cos(2\omega \tau) = 1. \]  

(108)

Taking into account (103) and (106) in the relation (104), we finally obtain

\[ \tau = \frac{1 + \sqrt{2}}{\alpha}, \quad \alpha < 0. \]  

(109)

Substituting \( s_1 = \alpha, \) \( s_{2,3} = \alpha \pm j \sqrt{2} - 1 \alpha \) into (99), we finally obtain that

\[ \alpha = \frac{a_3(9 - 4 \sqrt{2}) - a_1 a_2}{2 a_1^2 + 2 a_2(1 - 2 \sqrt{2})} \]  

(110)

**Sufficient conditions.** Calculation of the second derivatives of \( \tau \) with respect to \( s_1, s_2, s_3 \) gives the following results for \( \tau = -\left( \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} \right) \):

\[ \frac{d^2 \tau}{ds_1^2} = \exp \left( \frac{-s_1 s_3 + s_1 s_2 + s_2 s_3}{s_1 s_2} \right) \times \frac{-s_2 s_3 c_1 + s_3 c_2 - c_3 + s_2 c_2}{s_1^2 s_2^2 (s_1 s_3 - s_3^2 + s_1 s_2 + s_2 s_3)} \times \frac{1}{(s_1 s_3 + s_1 s_2 + s_2 s_3)^2} \]  

(111)

\[ \frac{d^2 \tau}{ds_2^2} = \exp \left( \frac{-s_1 s_3 + s_1 s_2 + s_2 s_3}{s_1 s_2} \right) \times \frac{-s_1 s_3 c_1 + s_3 c_2 - c_3 + s_2 c_2}{s_1 s_2^2 (s_1 s_3 - s_3^2 + s_1 s_2 + s_2 s_3)} \times \frac{1}{(s_1 s_3 + s_1 s_2 + s_2 s_3)^2} \]  

(112)

\[ \frac{d^2 \tau}{ds_3^2} = \exp \left( \frac{-s_1 s_3 + s_1 s_2 + s_2 s_3}{s_1 s_2} \right) \times \frac{-s_1 s_2 c_1 + s_2 c_2 - c_3 + s_1 c_2}{s_1^2 s_2^2 (s_1 s_3 - s_3^2 + s_1 s_2 + s_2 s_3)} \times \frac{1}{(s_1 s_3 + s_1 s_2 + s_2 s_3)^2} \]  

(113)

The Hessian is equal to

\[ H = \begin{vmatrix} \frac{d^2 \tau}{ds_1^2} & 0 & 0 \\ 0 & \frac{d^2 \tau}{ds_2^2} & 0 \\ 0 & 0 & \frac{d^2 \tau}{ds_3^2} \end{vmatrix}. \]  

(117)

From Eqns. (111), (112) and (113), we obtain that

\[ H_1 = \exp \left( \frac{-s_1 s_3 + s_1 s_2 + s_2 s_3}{s_1 s_2} (s_1 + s_2 + s_3) \right) \times \frac{(s_1 s_3 + s_1 s_2 + s_2 s_3)^6}{s_1^2 s_2^2 (s_1 s_3 - s_3^2 + s_1 s_2 + s_2 s_3)} \times \frac{(c_3 + s_2 s_3 c_1 - s_2 c_2 - s_3 c_2)}{s_1^2 - s_2 s_3 - s_1 s_3 - s_2 s_3} \times \frac{(c_3 + s_1 s_3 c_2 - s_1 c_2 - s_3 c_2)}{(s_2^2 + s_3^2 + s_1 s_3 + s_1 s_2)} \times \frac{(c_3 + s_1 s_2 c_1 - s_1 c_2 - s_2 c_2)}{(s_1^2 + s_3^2 + s_1 s_2 + s_2 s_3)}, \]  

(118)
or, after symmetrization,

\[ H_1^2 = \left( -\alpha_3^2 \left( a_3^2 c_1^2 + c_3^2 a_3 a_1 + 2 c_1^2 c_2 a_3 a_3 \right) \left( a_3^2 (-4 a_1^2 + a_3^2 + 2 a_1 a_2 a_3 + a_2^2 a_1^2) \right) + 3 c_1 c_2 c_3 a_1 + c_1 c_2 a_3 a_2 + c_1 c_3^2 a_1 a_1 + c_1 c_3^2 a_2^2 \right) \left( a_3^2 (-4 a_1^2 + a_3^2 + 2 a_1 a_2 a_3 + a_2^2 a_1^2) \right) + c_3 c_2^2 a_2 + c_3 c_2^2 a_1^2 + 2 a_1 c_3^2 c_2 + c_1^2 \right) \right) \left( a_3^2 (-4 a_1^2 + a_3^2 + 2 a_1 a_2 a_3 + a_2^2 a_1^2) \right) \times \exp \left( -\frac{\alpha_1 a_2}{a_3} \right), \]

\[ H_2 = H_1^2, \quad (119) \]

\[ H_3 = H_1^3, \quad (120) \]

**Sufficient conditions.** From \(118\), \(119\) and \(120\), we finally find that

\[ H = \begin{vmatrix} H_1 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & H_1 \end{vmatrix}. \quad (121) \]

From \(121\) we deduce that, if

\[ H_1 > 0, \quad (122) \]

it is a minimum \(\tau\) with respect to \(s_1, s_2, s_3\), and if

\[ H_1 < 0, \quad (123) \]

\(\tau\) has a maximum, according to \(56\), with respect to \(s_1, s_2, s_3\),

\[ H = \begin{vmatrix} \frac{d^2 \tau}{d s_1^2} & 0 & 0 \\ 0 & \frac{d^2 \tau}{d s_2^2} & 0 \\ 0 & 0 & \frac{d^2 \tau}{d s_3^2} \end{vmatrix} \]

\[ = \begin{vmatrix} \frac{d^2 \tau}{d s_1^2} & 0 & 0 \\ 0 & \frac{d^2 \tau}{d s_2^2} & 0 \\ 0 & 0 & \frac{d^2 \tau}{d s_3^2} \end{vmatrix} \]

This indicates that for \(\alpha\) there is a minimum of \(\tau\) and for \(\omega\) there is a maximum of \(\tau\).

**Existence conditions.** Substituting \(\tau\) from \(109\) and \(\omega\) from \(106\) into \(24\), we obtain the relation

\[ x^{(1)}(t) \bigg|_{t = \tau} = 0.0800187074 a_1^2 
- 0.07571678 a_2^2 c_1 + 0.1954085 a_3 c_1 = 0. \quad (125) \]

The second equation for the determined \(c_2/c_1\) and \(c_3/c_1, c_1 \neq 0\) is obtained from

\[ \frac{a_2^2}{a_1} + (a_3 + a_1 a_2 c_2^2 c_1 + a_2 c_3^2 c_1 = 0. \quad (126) \]

4.3. **Fourth-order equation** \((n = 4)\). Consider

\[ \frac{d^4 x(t)}{dt^4} + a_1 \frac{d^3 x(t)}{dt^3} + a_2 \frac{d^2 x(t)}{dt^2} + a_3 \frac{dx(t)}{dt} + a_4 x(t) = 0. \quad (127) \]

The initial conditions are

\[ x(0) = c_1, \quad x^{(1)}(0) = c_2, \]

\[ x^{(2)}(0) = c_3, \quad x^{(3)}(0) = c_4. \]

The characteristic equation is

\[ s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0. \quad (128) \]

The first derivative of the solution of Eqn. \(127\) is

\[ \frac{dx}{dt} \bigg|_{t = \tau} = \frac{\left( a_2 + a_3 + a_2 a_3 a_1 + a_2 a_3 a_2 + a_2 a_3 a_3 \right)}{(s_2 c_3 + s_3 c_3 - s_4 - s_4 s_4 c_3 - s_2 s_4 c_2 - s_2 s_4 c_2)} \left( s_1 - s_4 \right) \left( s_1 - s_2 \right) \left( s_1 - s_3 \right) + s_4 c_3 + s_4 s_4 c_1 + s_4 c_2 s_1 e^{s_1 \tau} + \frac{(s_1 c_3 + s_1 c_3 - s_1 s_2 c_2 - s_4 - s_3 s_4 c_2)}{(s_2 - s_3)(s_1 - s_2)(s_1 - s_4)} + s_4 c_3 + s_4 s_4 c_1 + s_4 c_2 s_1 e^{s_1 \tau} + \frac{(s_2 - s_3)(s_1 - s_2)(s_1 - s_4)}{(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} + s_1 s_2 c_4 + s_2 s_4 c_2 + s_2 c_4 s_1 e^{s_1 \tau} + \frac{(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)}{(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + s_2 s_3 c_2 - s_2 s_4 c_2 - c_4 s_4 e^{s_1 \tau} + \frac{(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)}{(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} = 0. \quad (129) \]

Derivatives of \(\tau\) determined by \(129\) with respect to \(s_1, s_2, s_3\) and \(s_4\) give the following necessary conditions:

\[ e^{(s_1 - s_4) \tau} \]

\[ = -s_4(-s_3 s_2 s_1 \tau^2 + s_2^2 s_4^2) - s_1 s_2^2 \tau - s_3 s_2^2 \tau + s_1 s_2^2 \tau + 2s_2 s_3 - s_1 s_2^2 \tau - 2s_2 s_3 + s_2 s_3 \tau^2 + 2s_1 s_2 s_3 \tau + 2s_1 s_3 + s_2^2 s_3 - s_1 s_2 s_3 \tau = 0, \quad (130) \]
In the special case, when \( g \) gives \( n = 1 \), we have that
\[
e^{-s_4} = e^{-s_3} = e^{-s_2} = e^{-s_1} = 0.
\]

We assume that
\[
s_1 = s_2 = \alpha, \quad s_3 = \alpha + j\omega, \quad s_4 = \alpha - j\omega.
\]

The optimal value of \( \tau \) is
\[
\tau = -\left( \frac{1}{s_1} + \frac{1}{s_2} + \frac{2\alpha}{s_3 + \alpha^2 + \omega^2} \right) = -2\frac{s\alpha + \alpha^2 + \omega^2}{s_3(\alpha^2 + \omega^2)}.
\]

From Eqn. (132), we obtain (134) and its solution gives
\[
\omega = \pm \alpha \sqrt{-3}.
\]

In the special case, when
\[
s_1 = s_3 = \alpha + j\omega, \quad s_2 = s_4 = \alpha - j\omega,
\]
we obtain
\[
\tau = -4 \frac{\alpha}{\alpha^2 + \omega^2},
\]
and from the equation
\[
\sin(2\omega\tau) = -8 \frac{(-\omega^2 + 3\alpha^2)}{(9\alpha^4 + 42\alpha^2\omega^2 + \omega^4)} \times \frac{(-\omega^4 + 14\alpha^2\omega^2 + 3\alpha^4)\omega}{(\alpha^2 + \omega^2)^2}
\]
we have that
\[
\omega = \pm \alpha \sqrt{-3}.
\]

4.4. Fifth-order equation (\( n = 5 \)). Consider
\[
\frac{d^5x(t)}{dt^5} + a_1\frac{d^4x(t)}{dt^4} + a_2\frac{d^3x(t)}{dt^3} + a_3\frac{d^2x(t)}{dt^2} + a_4\frac{dx(t)}{dt} + a_5x(t) = 0.
\]

We assume one real root and a double pair of the complex roots:
\[
\begin{cases}
s_1 = \alpha, & s_2 = s_4 = \alpha + j\omega, \\
s_3 = s_5 = \alpha - j\omega.
\end{cases}
\]

In the same way, we obtain (141), from which we have
\[
\omega = \pm \alpha.
\]

Last example of the fifth-order equation. We assume that
\[
\begin{cases}
s_1 = s_2 = s_3 = \alpha, s_4 = \alpha + j\omega, \\
s_5 = \alpha - j\omega.
\end{cases}
\]

In this case, we obtain (144) and its solution is
\[
\omega = 0.7606336797 \alpha.
\]

5. Basic results

Theorem 7. If the characteristic equation (21) has complex-conjugate roots, then the optimal time \( \tau \) can be computed numerically from the system of equations (109), (135), (138), (142), (145).

Theorem 8. The optimal times \( \tau_i > 0, i = 1, 2, \ldots, (n - 2) \), \( n \geq 3 \) are determined by \( D_n(\tau) = 0 \). If they exist, and the equation \( \frac{dx(t)}{dt} \big|_{t=\tau} = 0 \), Here (42) gives \( n - 1 \) linear algebraic equations for the initial conditions \( c_2/c_1, c_3/c_1, \ldots, c_{n-1}/c_1, c_1 \neq 0 \). This set of equations represents the solution of the problem.

6. Numerical examples

6.1. Third-order equation. Consider
\[
\frac{d^3x(t)}{dt^3} + a_1\frac{d^2x(t)}{dt^2} + a_2\frac{dx(t)}{dt} + a_3x(t) = 0.
\]

We assume that
\[
\begin{cases}
s_1 = \alpha = -1, \\
s_2 = \alpha + j\omega, \\
s_3 = \alpha - j\omega,
\end{cases}
\]
and, according to the relation (106), we have
\[
\omega = \pm \alpha \sqrt{-1} = \pm 0.6435942526.
\]

From (109), we get
\[
\tau = \frac{1 + \sqrt{2}}{\alpha} = 2.414213563,
\]
and
\[
\frac{dx(t)}{dt} \big|_{t=\tau} = -0.07330051053c_3 - 0.2840241129c_2 - 0.3001615506c_1 = 0.
\]

From the relation (126) we get
\[
D_3 = a_2^2c_1 + (a_3 + a_1a_2)c_2 + a_2c_3 = 0.
\]
6.2. Fourth-order equation. Consider

\[
\sin(2\omega t) = -4\left[\frac{-\omega^4 + 3\alpha^2\omega(2\alpha^2 + \omega^2)}{(\alpha^2 + \omega^2)^2}\right]
\]

\[
\sin(2\omega t) = \frac{-5\alpha^6 - 17\alpha^2\omega^4 - 13\alpha^4\omega^2 + 3\alpha^6}{(9\alpha^{10} + 48\omega^2\alpha^8 + 106\omega^4\alpha^6 + 92\omega^6\alpha^4 + 33\alpha^2\omega^8 + 4\omega^{10})}
\]

\[
\sin(2\omega t) = \frac{-2\alpha^2 - 5\alpha^2\omega^2 + 3\alpha^2}{(9\alpha^{10} + 63\alpha^8\omega^2 + 153\alpha^6\omega^4 + 82\alpha^4\omega^6 + 16\alpha^2\omega^8 + \omega^{10})}
\]

\[
\sin(2\omega t) = \frac{-2(9\omega^{10} + 12\alpha^2\omega^6 - 72\alpha^4\omega^4 - 172\alpha^6\omega^2 + 93\alpha^8\omega^2 + 12\alpha^{10})((9\omega^{10} - 12\alpha^2\omega^4 + 5\alpha^4\omega^2 + 12\alpha^6)(5\alpha^2 + 3\omega^2)\alpha\omega)}{(9\omega^{10} + 18\alpha^2\omega^4 + 9\alpha^4\omega^2 + 4\alpha^6)(9\omega^{10} + 69\alpha^2\omega^8 + 208\alpha^4\omega^6 + 297\alpha^6\omega^4 + 189\alpha^8\omega^2 + 36\alpha^{10})(\alpha^2 + \omega^2)^2}
\]

and

\[
3.414213562c_3 + 11.65685425c_2 + 11.65685425c_1 = 0, \quad (151)
\]

where

\[
\begin{align*}
a_1 &= 3, \\
a_2 &= 3.414213562, \\
a_3 &= 1.414213562.
\end{align*}
\]

From (150) and (151), assuming \(c_1 = 1\), we have

\[
\begin{align*}
c_2 &= -1.477984236, \\
c_3 &= 1.631940262.
\end{align*}
\]

We finally obtain

\[
x(t) = 0.2177267e^{-t} + 0.7822733e^{-t}\cos(0.6435942526t) - 0.74267947e^{-t}\sin(0.6435942526t).
\]

In Fig. 1 we present the optimal transient of \(x(t)\).

6.2. Fourth-order equation. Consider

\[
\frac{d^4x(t)}{dt^4} + a_1 \frac{d^3x(t)}{dt^3} + a_2 \frac{d^2x(t)}{dt^2} + a_3 \frac{dx(t)}{dt} + a_4 x(t) = 0. \quad (155)
\]

We shall analyse two cases:

- one double real root and one pair of complex-conjugate roots,
- one double pair of complex roots.

Fig. 1. Optimal transient of \(x(t)\) (one real root and one pair of complex roots).

6.2.1. Case 1. Assume that

\[
\begin{align*}
s_1 &= s_2 = \alpha = -1, \\
s_3 &= \alpha + j\omega, \\
s_4 &= \alpha - j\omega,
\end{align*}
\]

and, according to the relation (133), we have

\[
\omega = \pm\alpha \sqrt{3} = \pm1.316074013. \quad (157)
\]

From (133), we get

\[
\tau = 2.732050808, \quad (158)
\]

\[
\left. \frac{dx}{dt} \right|_{t=\tau} = -0.04383663c_4 - 0.224546377c_3 - 0.430314558c_2 - 0.314690486c_1 = 0. \quad (159)
\]
Let \( c_2 = 0 \). Then
\[
\frac{dx}{dt} = -0.0438366299 c_1 - 0.224546377 c_3 - 0.31469047 c_1 = 0.
\]
(160)

From the relation (162), we obtain
\[
D_4 = c_1 a_3^4 + (a_2^4 a_2 + a_1 a_3 a_4 + 2a_4^2) c_2 + (2a_3 a_4 + a_1 a_3^3) c_3 + a_2^2 c_4 = 0.
\]
(161)

For \( c_2 = 0 \) and \( \alpha = -1 \), \( \omega = -1.316074013 \), from (161) we have
\[
D_4 = 415.8460971 c_1 + 263.63586 c_3 + 55.7128129 c_4 = 0.
\]
(162)

From (160) and (162), we finally have \( c_3 = 0.7312184409 c_1 \), \( c_4 = -10.92426443 c_1 \), and for \( c_1 = 1 \),
\[
x(t) = -3.463269 e^{-t} + 1.999519 e^{-t} - 0.999519433 \cos(1.316074t) e^{-t} + 3.39135125 \sin(1.316074013t) e^{-t}.
\]
(163)

In Fig. 2 we present the optimal transient of \( x(t) \).

6.2.2. Case 2. Assume that
\[
\begin{align*}
s_1 &= s_3 = \alpha + j \omega, \\
s_2 &= s_4 = \alpha - j \omega.
\end{align*}
\]
(164)

Then the optimal time is
\[
\tau = -\frac{1}{a^2 + \omega^2}.
\]
(165)

From (167) we obtain that
\[
\omega = \pm \alpha \sqrt{3}.
\]
(166)

For
\[
\begin{align*}
\alpha &= -1, \\
\omega &= \pm 1.732050808
\end{align*}
\]
(167)

we get the coefficients
\[
\begin{align*}
a_1 &= 4, \\
a_2 &= 2, \\
a_3 &= 12, \\
a_4 &= 16.
\end{align*}
\]
(168)

and from (165),
\[
\tau = 1.
\]
(169)

In much the same way as in to the previous case, we assume \( c_2 = 0 \) and obtain that
\[
\frac{dx}{dt} \bigg|_{t=\tau} = -0.7165473715 c_1 + 0.1505743654 c_3 + 0.06003569669 c_4 = 0.
\]
(170)

From (161), we get
\[
4096.000008 c_1 + 1536.000002 c_3 + 256.000003 c_4 = 0.
\]
(171)

The solution of (170) and (171) is
\[
\begin{align*}
c_3 &= -8.00000005 c_1, \\
c_4 &= 32.00000002 c_1.
\end{align*}
\]
(172)

For \( c_1 = 1 \), we get
\[
x(t) = -2 \cos(1.732050808 t) e^{-t} + \cos(1.732050808 t) e^{-t} - 1.154700539 \sin(1.732050808 t) e^{-t} + 1.732050808 \sin(1.732050808 t) e^{-t}.
\]
(173)

In Fig. 3 we present the transient of \( x(t) \).

6.3. Fifth-order equation. Consider
\[
\begin{align*}
\frac{d^5 x(t)}{dt^5} + a_1 \frac{d^4 x(t)}{dt^4} + a_2 \frac{d^3 x(t)}{dt^3} \\
+ a_3 \frac{d^2 x(t)}{dt^2} + a_4 \frac{dx(t)}{dt} + a_5 x(t) &= 0.
\end{align*}
\]
(174)

We consider the case of one real root and double pair of complex roots,
\[
\begin{align*}
s_1 &= \alpha, \\
s_2 &= s_4 = \alpha + j \omega, \\
s_3 &= s_5 = \alpha - j \omega.
\end{align*}
\]
(175)

From (124), we have
\[
\omega = \pm \alpha.
\]
(176)
For \( \alpha = -1, \) and \( c_2 = 0, \ c_3 = 0, \) we obtain the following results:

\[
\begin{align*}
 a_1 &= 5, \quad a_2 = 12, \\
 a_3 &= 16, \quad a_4 = 12, \\
 a_5 &= 4,
\end{align*}
\]

and the optimal time

\[
\tau = \frac{a_1}{a_5} = 3.
\]

From the equations

\[
\left. \frac{dx}{dt} \right|_{t=\tau} = -0.3541478601c_1 - 0.1112703c_4 - 0.1109075c_5 = 0
\]

and from

\[
D_5 = 20736c_1 + 10368c_4 + 1728c_5 = 0,
\]

the solution is

\[
\begin{align*}
 c_4 &= -4.942537184c_1, \\
 c_5 &= 17.6552231c_1.
\end{align*}
\]

The optimal transient \( x(t), \) for \( c_1 = 1, \) is

\[
x(t) = 1.885074362e^{-t} - 0.8850743624e^{-t}\cos(t) + 1.471268592e^{-t}\cos(t)t - 0.471268592e^{-t}\sin(t) + 0.0574628215e^{-t}\sin(t)t.
\]

In Fig. 4, \( x(t) \) is presented.

In the same way, for \( c_2 = 0, \ c_4 = 0 \) we obtain

\[
\begin{align*}
 c_3 &= -1.145649523c_1, \\
 c_5 &= 6.33039236c_1,
\end{align*}
\]

and for \( c_1 = 1 \) the optimal transient is

\[
x(t) = 1.1651196176e^{-t} + 0.7184742831e^{-t}\cos(t)t - 0.1651961744e^{-t}\cos(t) - 0.1554282492e^{-t}\sin(t)t + 0.2815257151e^{-t}\sin(t),
\]

which is presented in Fig. 5.

For \( c_4 = 0, \ c_5 = 0 \) we obtain

\[
\begin{align*}
 c_2 &= -0.9458611703c_1, \\
 c_3 &= 0.5111482271c_1,
\end{align*}
\]

and for \( c_1 = 1 \) the transient is

\[
x(t) = 0.5223e^{-t} + 0.125e^{-t}\cos(t)t + 0.4777035489e^{-t}\cos(t) + 0.048564717e^{-t}\sin(t)t - 0.07086117224e^{-t}\sin(t),
\]

which is presented in Fig. 6.

### 7. Conclusion

Our basic theorems derive the solution of the problem of determining an optimal time \( \tau. \) The presented examples of the differential equations of the order \( n = 2, 3, 4, 5 \) illustrate the solution method. We stress that for the differential equation of the \( n \)-th order it is in general necessary to determine \( n - 2 \) values of \( \tau_i > 0, \ i = 1, 2, \ldots, n - 1. \)

**Remark 1.** The functions \( e^s, \sin(s), \cos(s) \) are analytic in the whole domain and have all derivatives. For that reason it is sufficient to consider the real, negative roots \( s. \)
Fig. 5. Optimal transient of $x(t)$ for $c_2 = 0, c_4 = 0$.

Fig. 6. Optimal transient of $x(t)$ for $c_4 = 0, c_5 = 0$.

References


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