Random Variables and Product of Probability Spaces

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. We have been working on the formalization of the probability and the randomness. In [15] and [16], we formalized some theorems concerning the real-valued random variables and the product of two probability spaces. In this article, we present the generalized formalization of [15] and [16]. First, we formalize the random variables of arbitrary set and prove the equivalence between random variable on Σ, Borel sets and a real-valued random variable on Σ. Next, we formalize the product of countably infinite probability spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [14], [12], [4], [11], [18], [7], [8], [5], [2], [3], [9], [13], [22], [15], [16], [20], [21], [17], [19], [6], and [10].

1. Random Variables

In this paper Ω, Ω₁, Ω₂ denote non empty sets, Σ denotes a σ-field of subsets of Ω, S₁ denotes a σ-field of subsets of Ω₁, and S₂ denotes a σ-field of subsets of Ω₂.

Now we state the proposition:

(1) Let us consider a non empty set B and a function f. Then \( f^{-1}(\bigcup B) = \bigcup \{f^{-1}(Y)\} \text{ where } Y \text{ is an element of } B : \text{ not contradiction}\).

Let us consider a function f from Ω₁ into Ω₂, a sequence B of subsets of Ω₂, and a sequence D of subsets of Ω₁. Now we state the propositions:

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If for every element \( n \) of \( \mathbb{N} \), \( D(n) = f^{-1}(B(n)) \), then \( f^{-1}(\bigcup B) = \bigcup D \).

If for every element \( n \) of \( \mathbb{N} \), \( D(n) = f^{-1}(B(n)) \), then \( f^{-1}(\bigcap B) = \bigcap D \).

Now we state the propositions:

Let us consider a function \( F \) from \( \Omega \) into \( \mathbb{R} \) and a real number \( r \). Suppose \( F \) is a real-valued random variable on \( \Sigma \). Then \( F^{-1}(-\infty, r] \in \Sigma \).

**Proof:** Consider \( X \) being an element of \( \Sigma \) such that \( X = \Omega \) and \( F \) is measurable on \( X \). For every element \( z \), \( z \in F^{-1}(-\infty, r] \) iff \( z \in \Omega \cap \text{LE-dom}(F, r) \).

Let us consider a function \( F \) from \( \Omega \) into \( \mathbb{R} \). Suppose \( F \) is a real-valued random variable on \( \Sigma \). Then \( \{ x \text{ where } x \text{ is an element of the Borel sets: } f^{-1}(x) \text{ is an element of } \Sigma \} = \text{the Borel sets} \).

Let us consider \( \Omega_1, \Omega_2, S_1, \) and \( S_2 \). Let \( F \) be a function from \( \Omega_1 \) into \( \Omega_2 \). We say that \( F \) is \((S_1, S_2)\)-random variable-like if and only if

**Def. 1** \( F \) is random variable on \( S_1 \) and \( S_2 \).

Observe that there exists a function from \( \Omega_1 \) into \( \Omega_2 \) which is \((S_1, S_2)\)-random variable-like.

A random variable of \( S_1 \) and \( S_2 \) is an \((S_1, S_2)\)-random variable-like function from \( \Omega_1 \) into \( \Omega_2 \). Now we state the proposition:

Let us consider a function \( f \) from \( \Omega \) into \( \mathbb{R} \). Then \( f \) is a random variable of \( \Sigma \) and the Borel sets if and only if \( f \) is a real-valued random variable on \( \Sigma \).

Let \( F \) be a function. We say that \( F \) is random variable family-like if and only if

**Def. 2** Let us consider a set \( x \). Suppose \( x \in \text{dom } F \). Then there exist non empty sets \( \Omega_1, \Omega_2 \) and there exists a \( \sigma \)-field \( S_1 \) of subsets of \( \Omega_1 \) and there exists
a σ-field $S_2$ of subsets of $\Omega_2$ and there exists a random variable $f$ of $S_1$ and $S_2$ such that $F(x) = f$.

One can verify that there exists a function which is random variable family-like.

A random variable family is a random variable family-like function. In this paper $F$ denotes a random variable of $S_1$ and $S_2$.

Let $Y$ be a non empty set, $S$ be a σ-field of subsets of $Y$, and $F$ be a function. We say that $F$ is $S$-measure valued if and only if

(Def. 3) Let us consider a set $x$. If $x \in \text{dom } F$, then there exists a σ-measure $M$ on $S$ such that $F(x) = M$.

Note that there exists a function which is $S$-measure valued.

Let $F$ be a function. We say that $F$ is $S$-probability valued if and only if

(Def. 4) Let us consider a set $x$. If $x \in \text{dom } F$, then there exists a probability $P$ on $S$ such that $F(x) = P$.

Let us note that there exists a function which is $S$-probability valued.

Let $X, Y$ be non empty sets. One can verify that there exists an $S$-probability valued function which is $X$-defined.

One can verify that there exists an $X$-defined $S$-probability valued function which is total.

Let $Y$ be a non empty set. Let us note that every function which is $S$-probability valued is also $S$-measure valued.

Let $F$ be a function. We say that $F$ is $S$-random variable family if and only if

(Def. 5) Let us consider a set $x$. Suppose $x \in \text{dom } F$. Then there exists a real-valued random variable $Z$ on $S$ such that $F(x) = Z$.

Observe that there exists a function which is $S$-random variable family.

Now we state the propositions:

(10) Let us consider an element $y$ of $S_2$. Suppose $y \neq \emptyset$. Then \{z where $z$ is an element of $\Omega_1 : F(z)$ is an element of $y}\} = F\{y\}$. Proof: Set $D = \{z$ where $z$ is an element of $\Omega_1 : F(z)$ is an element of $y\}$. For every element $x, x \in D$ iff $x \in F\{y\}$. □

(11) Let us consider a random variable $F$ of $S_1$ and $S_2$. Then

(i) \{x where $x$ is a subset of $\Omega_1 :$ there exists an element $y$ of $S_2$ such that $x = F^{-1}(y)\} \subseteq S_1$, and

(ii) \{x where $x$ is a subset of $\Omega_1 :$ there exists an element $y$ of $S_2$ such that $x = F^{-1}(y)\}$ is a σ-field of subsets of $\Omega_1$.

The theorem is a consequence of (3). Proof: Set $S = \{x$ where $x$ is a subset of $\Omega_1 :$ there exists an element $y$ of $S_2$ such that $x = F^{-1}(y)\}$. For every element $x$ such that $x \in S$ holds $x \in S_1$. For every subset $A$ of
\[\Omega_1 \text{ such that } A \in S \text{ holds } A^c \in S. \text{ For every sequence } A_1 \text{ of subsets of } \Omega_1 \text{ such that } \operatorname{rng} A_1 \subseteq S \text{ holds Intersection } A_1 \in S. \square\]

Let us consider \(\Omega_1, \Omega_2, S_1, \) and \(S_2. \) Let \(M\) be a measure on \(S_1\) and \(F\) be a random variable of \(S_1\) and \(S_2. \) The functor the image measure of \(F\) and \(M\) yielding a measure on \(S_2\) is defined by

**Def. 6** Let us consider an element \(y\) of \(S_2. \) Then \(it(y) = M(F^{-1}(y)).\)

Let \(M\) be a \(\sigma\)-measure on \(S_1. \) Note that the image measure of \(F\) and \(M\) is \(\sigma\)-additive.

Now we state the proposition:

(12) Let us consider a probability \(P\) on \(S_1\) and a random variable \(F\) of \(S_1\) and \(S_2. \) Then (the image measure of \(F\) and \(P\))\((\Omega_2) = 1.\)

Let us consider \(\Omega_1, \Omega_2, S_1, \) and \(S_2. \) Let \(P\) be a probability on \(S_1\) and \(F\) be a random variable of \(S_1\) and \(S_2. \) The functor probability\((F, P)\) yielding a probability on \(S_2\) is defined by the term

**Def. 7** \(M2P\) the image measure of \(F\) and \(P2M.\)

Now we state the propositions:

(13) Let us consider a probability \(P\) on \(S_1\) and a random variable \(F\) of \(S_1\) and \(S_2. \) Then \(\text{probability}(F, P) = \text{the image measure of } F \text{ and } P2M.\)

The theorem is a consequence of (12).

(14) Let us consider a probability \(P\) on \(S_1, \) a random variable \(F\) of \(S_1\) and \(S_2, \) and a set \(y. \) If \(y \in S_2, \) then \((\text{probability}(F, P))(y) = P(F^{-1}(y)).\) The theorem is a consequence of (13).

(15) Every function from \(\Omega_1\) into \(\Omega_2\) is a random variable of the trivial \(\sigma\)-field of \(\Omega_1\) and the trivial \(\sigma\)-field of \(\Omega_2.\)

(16) Let us consider a non empty set \(S. \) Then every non empty finite sequence of elements of \(S\) is a random variable of the trivial \(\sigma\)-field of \(\operatorname{Seg} len F\) and the trivial \(\sigma\)-field of \(S.\) The theorem is a consequence of (15).

(17) Let us consider finite non empty sets \(V, S, \) a random variable \(G\) of the trivial \(\sigma\)-field of \(V\) and the trivial \(\sigma\)-field of \(S, \) and a set \(y. \) Suppose \(y \in \text{the trivial } \sigma\text{-field of } S.\) Then \((\text{probability}(G, \text{the trivial probability of } V))(y) = \frac{G^{-1}(y)}{V}.\) The theorem is a consequence of (14).

(18) Let us consider a finite non empty set \(S, \) a non empty finite sequence \(s\) of elements of \(S, \) and a set \(x. \) Suppose \(x \in S.\) Then there exists a random variable \(G\) of the trivial \(\sigma\)-field of \(\operatorname{Seg} len s\) and the trivial \(\sigma\)-field of \(S\) such that

(i) \(G = s, \) and

(ii) \((\text{probability}(G, \text{the trivial probability of } \operatorname{Seg} len s))(\{x\}) = \text{Prob}_D(x, s).\)

The theorem is a consequence of (16) and (17).
2. Product of Probability Spaces

Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$ and $n$ be a natural number. One can check that $D(n)$ is non empty.

Let $S$, $F$ be many sorted sets indexed by $\mathbb{N}$. We say that $F$ is $\sigma$-field $S$-sequence-like if and only if

(Def. 8) Let us consider a natural number $n$. Then $F(n)$ is a $\sigma$-field of subsets of $S(n)$.

Let $S$ be a many sorted set indexed by $\mathbb{N}$. Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is $\sigma$-field $S$-sequence-like.

Let $D$ be a many sorted set indexed by $\mathbb{N}$. A $\sigma$-field sequence of $D$ is a $\sigma$-field $D$-sequence-like many sorted set indexed by $\mathbb{N}$. Let $S$ be a $\sigma$-field sequence of $D$ and $n$ be a natural number. Note that the functor $S(n)$ yields a $\sigma$-field of subsets of $D(n)$. Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$. Let $M$ be a many sorted set indexed by $\mathbb{N}$. We say that $M$ is $S$-probability sequence-like if and only if

(Def. 9) Let us consider a natural number $n$. Then $M(n)$ is a probability on $S(n)$.

Observe that there exists a many sorted set indexed by $\mathbb{N}$ which is $S$-probability sequence-like.

A probability sequence of $S$ is an $S$-probability sequence-like many sorted set indexed by $\mathbb{N}$. Let $P$ be a probability sequence of $S$ and $n$ be a natural number. One can verify that the functor $P(n)$ yields a probability on $S(n)$. Let $D$ be a many sorted set indexed by $\mathbb{N}$. The functor the product domain $D$ yielding a many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 10) (i) $it(0) = D(0)$, and

(ii) for every natural number $i$, $it(i + 1) = it(i) \times D(i + 1)$.

Now we state the proposition:

(19) Let us consider a many sorted set $D$ indexed by $\mathbb{N}$. Then

(i) (the product domain $D)(0) = D(0)$, and

(ii) (the product domain $D)(1) = D(0) \times D(1)$, and

(iii) (the product domain $D)(2) = D(0) \times D(1) \times D(2)$, and

(iv) (the product domain $D)(3) = D(0) \times D(1) \times D(2) \times D(3)$.

Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$. Let us note that the product domain $D$ is non-empty.

Let $D$ be a finite-yielding many sorted set indexed by $\mathbb{N}$. One can check that the product domain $D$ is finite-yielding.

Let us consider $\Omega$ and $\Sigma$. Let $P$ be a set. Assume $P$ is a probability on $\Sigma$.

The functor $\text{modetrans}(P, \Sigma)$ yielding a probability on $\Sigma$ is defined by the term

(Def. 11) $P$. 
Let $D$ be a finite-yielding non-empty many sorted set indexed by $\mathbb{N}$. The functor the trivial $\sigma$-field sequence $D$ yielding a $\sigma$-field sequence of $D$ is defined by

\begin{equation}
(\text{Def. 12}) \quad \text{Let us consider a natural number } n. \text{ Then } \it(n) = \text{the trivial } \sigma\text{-field of } D(n).
\end{equation}

Let $P$ be a probability sequence of the trivial $\sigma$-field sequence $D$ and $n$ be a natural number. One can check that the functor $P(n)$ yields a probability on the trivial $\sigma$-field of $D(n)$. The functor $\text{ProductProbability}(P, D)$ yielding a many sorted set indexed by $\mathbb{N}$ is defined by

\begin{equation}
(\text{Def. 13}) \quad \text{(i)} \quad \it(0) = P(0), \text{ and}
\end{equation}

\begin{equation}
\text{(ii)} \quad \text{for every natural number } i, \it(i + 1) = \text{ProductProbability((the product domain } D)(i), D(i + 1), \modetrans (\it(i), \text{the trivial } \sigma\text{-field of (the product domain } D)(i)), P(i + 1)).
\end{equation}

Let us consider a finite-yielding non-empty many sorted set $D$ indexed by $\mathbb{N}$, a probability sequence $P$ of the trivial $\sigma$-field sequence $D$, and a natural number $n$. Now we state the propositions:

\begin{equation}
(20) \quad (\text{ProductProbability}(P, D))(n) \text{ is a probability on the trivial } \sigma\text{-field of (the product domain } D)(n).
\end{equation}

\begin{equation}
(21) \quad \text{There exists a probability } P_4 \text{ on the trivial } \sigma\text{-field of (the product domain } D)(n) \text{ such that}
\end{equation}

\begin{enumerate}
\item[(i)] $P_4 = (\text{ProductProbability}(P, D))(n)$, and
\item[(ii)] $(\text{ProductProbability}(P, D))(n + 1) = \text{ProductProbability((the product domain } D)(n), D(n + 1), P_4, P(n + 1))$.
\end{enumerate}

Now we state the proposition:

\begin{equation}
(22) \quad \text{Let us consider a finite-yielding non-empty many sorted set } D \text{ indexed by } \mathbb{N} \text{ and a probability sequence } P \text{ of the trivial } \sigma\text{-field sequence } D. \text{ Then}
\end{equation}

\begin{enumerate}
\item[(i)] $(\text{ProductProbability}(P, D))(0) = P(0)$, and
\item[(ii)] $(\text{ProductProbability}(P, D))(1) = \text{ProductProbability}(D(0), D(1), P(0), P(1))$, and
\item[(iii)] there exists a probability $P_1$ on the trivial $\sigma$-field of $D(0) \times D(1)$ such that $P_1 = (\text{ProductProbability}(P, D))(1)$ and $(\text{ProductProbability}(P, D))(2) = \text{ProductProbability}(D(0) \times D(1), D(2), P_1, P(2))$, and
\item[(iv)] there exists a probability $P_2$ on the trivial $\sigma$-field of $D(0) \times D(1) \times D(2)$ such that $P_2 = (\text{ProductProbability}(P, D))(2)$ and $(\text{ProductProbability}(P, D))(3) = \text{ProductProbability}(D(0) \times D(1) \times D(2), D(3), P_2, P(3))$, and
\item[(v)] there exists a probability $P_3$ on the trivial $\sigma$-field of $D(0) \times D(1) \times D(2) \times D(3)$ such that $P_3 = (\text{ProductProbability}(P, D))(3)$ and
\end{enumerate}
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(ProductProbability($P, D$))(4) = Product-Probability($D(0) \times D(1) \times D(2) \times D(3), D(4), P_3, P(4)$).

The theorem is a consequence of (19) and (21).

References


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