Isomorphisms of Direct Products of Cyclic Groups of Prime Power Order

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Summary. In this paper we formalized some theorems concerning the cyclic groups of prime power order. We formalize that every commutative cyclic group of prime power order is isomorphic to a direct product of family of cyclic groups \[1\], \[18\].

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The notation and terminology used in this paper have been introduced in the following articles: \[2\], \[20\], \[6\], \[11\], \[7\], \[18\], \[24\], \[18\], \[25\], \[26\], \[27\], \[28\], \[13\], \[23\], \[19\], \[5\], \[12\], \[30\], \[31\], \[14\], \[29\], and \[10\].

1. **Basic Properties of Cyclic Groups of Prime Power Order**

Let \( G \) be a finite group. The functor \( \text{Ordset}(G) \) yielding a subset of \( \mathbb{N} \) is defined by the term

(Def. 1) the set of all \( \text{ord}(a) \) where \( a \) is an element of \( G \).

One can check that \( \text{Ordset}(G) \) is finite and non empty.

Now we state the propositions:

(1) Let us consider a finite group \( G \). Then there exists an element \( g \) of \( G \) such that \( \text{ord}(g) = \sup \text{Ordset}(G) \).

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(2) Let us consider a strict group $G$ and a strict normal subgroup $N$ of $G$. If $G$ is commutative, then $G/N$ is commutative.

(3) Let us consider a finite group $G$ and elements $a, b$ of $G$. Then $b \in \text{gr}(\{a\})$ if and only if there exists an element $p$ of $\mathbb{N}$ such that $b = a^p$.

(4) Let us consider a finite group $G$, an element $a$ of $G$, and elements $n, p, s$ of $\mathbb{N}$. Suppose

(i) $\text{gr}(\{a\}) = n$, and

(ii) $n = p \cdot s$.

Then $\text{ord}(a^p) = s$.

Let us consider an element $k$ of $\mathbb{N}$, a finite group $G$, and an element $a$ of $G$. Now we state the propositions:

(5) $\text{gr}(\{a\}) = \text{gr}(\{a^k\})$ if and only if $\text{gcd}(k, \text{ord}(a)) = 1$.

(6) If $\text{gcd}(k, \text{ord}(a)) = 1$, then $\text{ord}(a) = \text{ord}(a^k)$.

(7) $\text{ord}(a) | k \cdot \text{ord}(a^k)$.

Now we state the proposition:

(8) Let us consider a group $G$ and elements $a, b$ of $G$. Suppose $b \in \text{gr}(\{a\})$. Then $\text{gr}(\{b\})$ is a strict subgroup of $\text{gr}(\{a\})$.

Let $G$ be a strict commutative group and $x$ be an element of $\text{SubGr}_G$. The functor $\text{NormSp}_\mathbb{R}(x)$ yielding a normal strict subgroup of $G$ is defined by the term

(Def. 2) $x$.

Now we state the propositions:

(9) Let us consider groups $G, H$, a subgroup $K$ of $H$, and a homomorphism $f$ from $G$ to $H$. Then there exists a strict subgroup $J$ of $G$ such that the carrier of $J = f^{-1}(\text{the carrier of } K)$. PROOF: Reconsider $I_3 = f^{-1}(\text{the carrier of } K)$ as a non empty subset of the carrier of $G$. For every elements $g_1, g_2$ of $G$ such that $g_1, g_2 \in I_3$ holds $g_1 \cdot g_2 \in I_3$ by [8 (38)], [25 (50)]. For every element $g$ of $G$ such that $g \in I_3$ holds $g^{-1} \in I_3$ by [8 (38)], [25 (51)], [28 (32)]. Consider $J$ being a strict subgroup of $G$ such that the carrier of $J = f^{-1}(\text{the carrier of } K)$. □

(10) Let us consider a natural number $p$, a finite group $G$, and elements $x, d$ of $G$. Suppose

(i) $\text{ord}(d) = p$, and

(ii) $p$ is prime, and

(iii) $x \in \text{gr}(\{d\})$.

Then

(iv) $x = 1_G$, or

(v) $\text{gr}(\{x\}) = \text{gr}(\{d\})$. 
The theorem is a consequence of (8). **Proof:** If $\text{gr}(\{x\}) = \{1\}_{\text{gr}(\{d\})}$, then $x = 1_G$ by [109 (2)], [20 (44)]. □

(11) Let us consider a group $G$ and normal subgroups $H$, $K$ of $G$. Suppose $(\text{the carrier of } H) \cap (\text{the carrier of } K) = \{1_G\}$. Then (the canonical homomorphism onto cosets of $H$)\(\lceil(\text{the carrier of } K)\rceil\) is one-to-one. **Proof:** Set $f = \text{the canonical homomorphism onto cosets of } H$. For every elements $x_1$, $x_2$ such that $x_1$, $x_2 \in \text{dom } g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$ by [30 (57)], [7 (49)], [20 (46), (103), (51)]. □

Let us consider finite commutative groups $G$, $F$, an element $a$ of $G$, and a homomorphism $f$ from $G$ to $F$. Now we state the propositions:

(12) The carrier of $\text{gr}(\{f(a)\}) = f^\circ \text{the carrier of } \text{gr}(\{a\})$.

(13) $\text{ord}(f(a)) \leq \text{ord}(a)$.

(14) If $f$ is one-to-one, then $\text{ord}(f(a)) = \text{ord}(a)$.

Now we state the propositions:

(15) Let us consider groups $G$, $F$, a subgroup $H$ of $G$, and a homomorphism $f$ from $G$ to $F$. Suppose $f|\text{the carrier of } H$ as a function from the carrier of $H$ into the carrier of $F$. For every elements $a$, $b$ of $H$, $g(a \cdot b) = g(a) \cdot g(b)$ by [25 (40)], [7 (49)], [20 (43)]. □

(16) Let us consider finite commutative groups $G$, $F$, an element $a$ of $G$, and a homomorphism $f$ from $G$ to $F$. Suppose $f|\text{the carrier of } \text{gr}(\{a\})$ is one-to-one. Then $\text{ord}(f(a)) = \text{ord}(a)$. The theorem is a consequence of (15) and (14).

(17) Let us consider a finite commutative group $G$, a prime number $p$, a natural number $n$, and an element $a$ of $G$. Suppose

(i) $\overline{G} = p^n$, and

(ii) $a \neq 1_G$.

Then there exists a natural number $n$ such that $\text{ord}(a) = p^{n+1}$.

Let us consider a prime number $p$ and natural numbers $j$, $m$, $k$. If $m = p^k$ and $p \nmid j$, then $\gcd(j, m) = 1$.

2. **Isomorphism of Cyclic Groups of Prime Power Order**

Let us consider a strict finite commutative group $G$, a prime number $p$, and a natural number $m$. Now we state the propositions:

(19) Suppose $\overline{G} = p^m$. Then there exists a normal strict subgroup $K$ of $G$ and there exist natural numbers $n$, $k$ and there exists an element $g$ of $G$ such that $\text{ord}(g) = \sup \text{Ordset}(G)$ and $K$ is finite and commutative and
(the carrier of $K$) $\cap$ (the carrier of $\text{gr} \{g\}) = \{1_G\}$ and for every element $x$ of $G$, there exist elements $b_1$, $a_1$ of $G$ such that $b_1 \in K$ and $a_1 \in \text{gr} \{g\}$ and $x = b_1 \cdot a_1$ and $\text{ord} (g) = p^n$ and $k = m - n$ and $n \leq m$ and $\overline{K} = p^k$ and there exists a homomorphism $F$ from $\prod (K, \text{gr} \{g\})$ to $G$ such that $F$ is bijective and for every elements $a$, $b$ of $G$ such that $a \in K$ and $b \in \text{gr} \{g\}$ holds $F((a, b)) = a \cdot b$.

(20) Suppose $\overline{G} = p^m$. Then there exists a non zero natural number $k$ and there exists a $k$-element finite sequence $a$ of elements of $G$ and there exists a $k$-element finite sequence $I_2$ of elements of $N$ and there exists an associative group-like commutative multiplicative magma family $F$ of Seg $k$ and there exists a homomorphism $H_1$ from $\prod F$ to $G$ such that for every natural number $i$ such that $i \in \text{Seg} k$ there exists an element $a_2$ of $G$ such that $a_2 = a(i)$ and $F(i) = \text{gr} \{a_2\}$ and $\text{ord} (a_2) = p^{I_2(i)}$ and for every natural number $i$ such that $1 \leq i \leq k - 1$ holds $I_2(i) \leq I_2(i + 1)$ and for every elements $p$, $q$ of Seg $k$ such that $p \neq q$ holds $(\text{the carrier of } F(p)) \cap (\text{the carrier of } F(q)) = \{1_G\}$ and $H_1$ is bijective and for every $(\text{the carrier of } G)$-valued total Seg $k$-defined function $x$ such that for every element $p$ of Seg $k$, $x(p) \in F(p)$ holds $x \in \prod F$ and $H_1(x) = \prod x$.

(21) Suppose $\overline{G} = p^m$. Then there exists a non zero natural number $k$ and there exists a $k$-element finite sequence $a$ of elements of $G$ and there exists a $k$-element finite sequence $I_2$ of elements of $N$ and there exists an associative group-like commutative multiplicative magma family $F$ of Seg $k$ such that for every natural number $i$ such that $i \in \text{Seg} k$ there exists an element $a_2$ of $G$ such that $a_2 = a(i)$ and $F(i) = \text{gr} \{a_2\}$ and $\text{ord} (a_2) = p^{I_2(i)}$ and for every natural number $i$ such that $1 \leq i \leq k - 1$ holds $I_2(i) \leq I_2(i + 1)$ and for every elements $p$, $q$ of Seg $k$ such that $p \neq q$ holds $(\text{the carrier of } F(p)) \cap (\text{the carrier of } F(q)) = \{1_G\}$ and for every element $y$ of $G$, there exists a $(\text{the carrier of } G)$-valued total Seg $k$-defined function $x$ such that for every element $p$ of Seg $k$, $x(p) \in F(p)$ and $y = \prod x$ and for every $(\text{the carrier of } G)$-valued total Seg $k$-defined functions $x_1$, $x_2$ such that for every element $p$ of Seg $k$, $x_1(p) \in F(p)$ and for every element $p$ of Seg $k$, $x_2(p) \in F(p)$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

References


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