Numerical analysis of controllability of a parabolic system with the delayed controls and non-zero boundary conditions

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Abstract
The aim of the article is the numerical analysis of the controllability of a parabolic system with the delayed controls and non-zero boundary conditions. The main novelty of the article are just non-zero Dirichlet conditions for the PDE. As the result the conditions of the controllability for the parabolic PDE with delayed controls are determined.

1. Introduction
The aim of the article is the numerical analysis of the controllability of a parabolic system with the delayed controls and non-zero boundary conditions. The main novelty of the article are just non–zero Dirichlet conditions for the PDE. Such a form of boundary conditions, after transformation of the parabolic PDE to the abstract differential equation, leads to the non-self-adjointed state operators, which is mathematically troublesome. Thus in this article we propose a new tool for the controllability analysis i.e. the numerical line method

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for the PDEs decomposition. As the result the conditions of the controllability for the parabolic PDE with the delayed controls are determined which is a novelty in the controllability literature. In paper [1] we analyzed the 2nd order system with the self-adjoined state operators-which is equivalent to the zero boundary condition. In paper [2] we analyzed the parabolic equation defined in the n-dimensional rectangular prism. Thus in this article we make a step forward in the parabolic PDEs numerical controllability analysis considering non-zero values at the domain boundary.

2. Basic notions

Following the aim of analyzing the approximate controllability of infinite dimensional system with delays, at first let us present this notion in the case of finite dimensional systems. We consider linear the stationary dynamical system described by the differential equation without delays in control [3] pp. 5:

$$\dot{x}(t) = A_0 x(t) + B_0 u(t), \quad t \geq 0.$$ 

(1)

**Definition 1.** (cf. [3]) The dynamical system (1) is said to be controllable, if and only if there exists such a control $u(t)$, which will transfer the system from any given initial state to any final state in the control space in the finite time.

**Theorem 1.** (cf. [3] pp.16, [4] pp. 70) Dynamical system (1) is controllable if and only if condition (2) holds true:

$$\text{rank}[B_0 | A_0 B_0 | A_0^2 B_0 | \ldots | A_0^{n-1} B_0] = n$$  

(2)

Now let us consider the linear stationary dynamical system described by the differential equation with delays in control (3) [3] pp. 196:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=0}^{M} B_{0k} u(t - h_k), \quad t \geq 0,$$

(3)

where $A_0, B_0$ are the constant matrices with the dimensions $n \times n, n \times p$, respectively.

For the dynamical system of form (3) besides the instantaneous state $x(t) \in R^n$, we introduce also the notion of the so-called complete state at time $t$, $z(t) = \{x(t), u(t)\}$, where $u_t(s) = u(s)$ for $s \in [t-h_M, t]$ [3]. Therefore we distinguish two basic notions of controllability for dynamical systems (3), namely: the relative controllability and the absolute controllability [3] pp. 195. Definitions 2 and 3 taken from position [3] pp. 195 are adapted to dynamical system (3) i.e. with the multiple, lumped time-invariant delays in control.
Definition 2. (cf. [3] pp. 195) Dynamical system (3) is said to be relatively controllable in $[t_0, t_1]$, if for any initial complete state $z(t_0)$ and any vector $x_1 \in \mathbb{R}^n$, there exists a control $u \in L^2([t_0, t_1], \mathbb{R}^p)$ such that the corresponding trajectory $x(t, z(t_0), u)$ of dynamical system (3) satisfies the following condition (4):

$$x(t_1, z(t_0), u) = x_1.$$  

Definition 3. (cf. [3] pp. 195) Dynamical system (3) is said to be absolutely controllable in $[t_0, t_1]$, if for any initial complete state $z(t_0)$, any vector $x_1 \in \mathbb{R}^n$ and an arbitrary function $w \in L^2([0, h_M], \mathbb{R}^p)$ there exists a control $u \in L^2([t_0, h_M], \mathbb{R}^p)$ such that the complete state at time $t_1$ of dynamical system (3) satisfies the following condition (5):

$$z(t_1) = \{x_1, w\}.$$ 

There are some known theorems for verifying the relative and absolute controllabilities of the linear time varying systems with delays and control. Let us present two main theorems adapted to the stationary dynamical system of form (3).

Theorem 2. (cf. [3] pp. 202) Dynamical system (3) is relatively controllable in $[t_0, t_1]$, if and only if the dynamical system without delays in control, of the form

$$\dot{x}(t) = A_0 x(t) + \tilde{B}_0 w(t), \quad t \in [t_0, t_1]\]$$

where

$$\tilde{B} = [B_{00} | B_{01} | \ldots | B_{0(k-1)}], \quad t \in [t_0, t_1], \quad w \in \mathbb{R}^{kp}$$

is controllable in $[t_0 + h_{k-1}, t_1]$.

Theorem 3. (cf. [3] pp. 207) Dynamical system (3) is absolutely controllable in $[t_0, t_1]$ if and only if the dynamical system without delays in control of the form

$$\dot{x}(t) = A_0 x(t) + \hat{B}_0 u(t), \quad t \in [t_0, t_1]\]$$

where

$$\hat{B}_0 = \sum_{k=0}^{M} e^{-A_0 h_k} B_{k0}$$

is controllable in $[t_0, t_1 - h_M]$. 

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2.1. Chen’s controllability theorem.

The Chen’s theorem plays a fundamental role in verifying of the unconstrained controllability of the dynamical system given in the Jordan canonical form. Let us assume that the Jordan canonical form of dynamical system (1) is represented by the matrices $J$ and $G = T^{-1}B_0$, which are given by the following equalities:

$$J = \begin{bmatrix} J_1 & 0 \\ J_2 & \ddots \\ 0 & \ddots & J_k \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_k \end{bmatrix}$$

$$J_i = \begin{bmatrix} J_{i1} & 0 \\ J_{i2} & \ddots \\ 0 & \ddots & J_{i(r(i))} \end{bmatrix}, \quad G_i = \begin{bmatrix} G_{i1} \\ G_{i2} \\ \vdots \\ G_{i(r(i))} \end{bmatrix}, \quad i = 1, 2, \ldots, k$$

$$J_{ij} = \begin{bmatrix} s_i & 1 & 0 \\ s_i & \ddots & \ddots \\ 0 & \ddots & s_i \\ & \ddots & \ddots \\ & & s_i \end{bmatrix}, \quad G_{ij} = \begin{bmatrix} g_{ij1} \\ g_{ij2} \\ \vdots \\ g_{ijn_{(i)}} \end{bmatrix}$$

$i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, r(i)$

where $s_1, s_2, \ldots, s_k$ are the distinct eigenvalues of the matrix $A_0$ with the multiplicities $n_i, i = 1, 2, \ldots, k$; $J_i, i = 1, 2, \ldots, k$ are $n_i \times n_i$ - the dimensional matrices containing all the Jordan blocks associated with the eigenvalues $s_i$; $J_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, r(i)$ are $(n_{ij} \times n_{ij})$ - dimensional Jordan blocks in $J_i$; $r(i)$ is the number of Jordan blocks in the submatrix $J_i, i = 1, 2, \ldots, k$; $G_i, i = 1, 2, \ldots, k$ are $(n_i \times m)$ - dimensional submatrices of the matrix $G$ corresponding to the submatrices $J_i$; $G_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, r(i)$ are $(n_{ij} \times m)$ - dimensional submatrices of the matrix $G_i$ corresponding to the Jordan blocks $J_{ij}$; $g_{ijn_{(i)}}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, r(i)$ are the rows of the submatrix $G_{ij}$ corresponding to the rows of the Jordan blocks $J_{ij}$.

Theorem 4. (cf. [3] pp. 25, [5] pp. 511) Dynamical system (3) is controllable if and only if for each $i = 1, 2, \ldots, k$ the rows $g_{i1n_{i1}}, g_{i2n_{i2}}, \ldots, g_{ir(i)n_{ir(i)}}$ of the matrix $G$ are linearly independent over the field of the complex numbers.
3. Main analysis

3.1. System model.

Let us consider the following distributed parameter system with the delayed controls, described by the parabolic type partial differential equation:

\[
\frac{\partial x(z, t)}{\partial t} - \frac{\partial^2 x(z, t)}{\partial z^2} = \sum_{k=0}^{M} B_k(z) \overline{u}(t - h_k), \quad t \geq t_0
\]  

(9)

defined in the domain:

\[
D = \{(z, t) : 0 \leq z \leq L, t_0 \leq t \leq t_0 + T\}
\]

the input functions \(\overline{b}_k(z)\) and the control vectors \(\overline{u}(t - h_k)\) are of the form (10), (11), respectively:

\[
\overline{b}_k(z) = \begin{bmatrix} b_{k1}(z) & b_{k2}(z) & \cdots & b_{kp}(z) \end{bmatrix}, \quad k = 0, 1, \ldots, M
\]  

(10)

\[
\overline{u}(t - h_k) = \begin{bmatrix} u_1(t - h_k) & u_2(t - h_k) & \cdots & u_p(t - h_k) \end{bmatrix}^T, \quad k = 0, 1, \ldots, M.
\]  

(11)

The delay fulfills the inequality:

\[
0 = h_0 < h_1 < \ldots < h_k < \ldots h_M.
\]

The controls are assumed to belong to the class \(u_1 \in L^2_{loc}([t_0, \infty), R)\). The initial conditions have the form:

\[
x(z, t_0) = g(z), \quad 0 \leq z \leq L.
\]

We assume the following, non–zero Dirichlet–type boundary conditions:

\[
x(0, t) = \varphi, \quad x(L, t) = \psi, \quad 0 \leq t \leq T, \quad \varphi, \psi \in R.
\]

3.2. Line decomposition of the system.

Let us decompose the partial differential equation (9) making the use of the line method [6]. We can obtain the following, finite dimensional linear system (12) with the delayed controls:

\[
\frac{dX(t)}{dt} = \frac{1}{h^2} A_0 X(t) + \sum_{k=0}^{M} B_k \overline{u}(t - h_k) + C, \quad t \geq t_0, \quad X(t_0) = G_1.
\]  

(12)

All the lines, decomposing the system (9), are equidistant with the grid size \(h = L/N\):

\[
z_i = ih, \quad i = 0, 1, \ldots, N.
\]
The state matrix $A_0$ is the tridiagonal symmetrical Metzler matrix:

$$
A_0 = \begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & \ddots \\
& \ddots & \ddots \\
& & -2 & 1 \\
0 & 1 & -2
\end{bmatrix} \in \mathbb{R}^{(N-1) \times (N-1)}. 
$$  \hspace{1cm} (13)

The input matrix has the form:

$$
B_k = \begin{bmatrix}
\bar{b}_k(z_1) \\
\vdots \\
\bar{b}_k(z_{N-1})
\end{bmatrix}. 
$$  \hspace{1cm} (14)

The state vector $X(t)$, vector $C$ and the $G_1$ vector, corresponding to the boundary conditions are of the form:

$$
X(t) = \begin{bmatrix}
x(z_1, t) \\
x(z_2, t) \\
\vdots \\
x(z_{N-2}, t) \\
x(z_{N-1}, t)
\end{bmatrix}, 
C = \begin{bmatrix}
\frac{1}{h^2} \varphi \\
0 \\
\vdots \\
0 \\
\frac{1}{h^2} \psi
\end{bmatrix}, 
G_1 = \begin{bmatrix}
g_1(z_1) \\
g_1(z_2) \\
\vdots \\
g_1(z_{N-2}) \\
g_1(z_{N-1})
\end{bmatrix}.
$$

Making in equation (12) the substitution $X(t) = V(t) - A^{-1}C$, $A = \frac{1}{h^2} A_0$ we can obtain:

$$
\frac{dV}{dt} = AV(t) + \sum^{M}_{k=0} B_k \pi(t - h_k), \quad t \geq t_0, \quad V(t_0) = G_1 + A^{-1}C. 
$$  \hspace{1cm} (15)

Equation (15) is the basis for further controllability analysis. In the next points the absolute and relative controllabilities of the parabolic system (9), decomposed by the line method to the form (12) are analyzed.

Such a decomposition enabled us to employ Theorems 2 and 3 determining the conditions for the absolute and relative controllabilities for the linear, finite dimensional systems with the delayed controls and the Chen’s Theorem 4.

### 3.3. The absolute controllability analysis.

As already mentioned, the absolute controllability makes a sense only in a sufficiently broad time interval i.e. $t_1 > t_0 + h_M$. The first step in the absolute controllability analysis of the system in the finite dimensional form (12) is the transformation of the system (12) to the corresponding form (7).
without delays in controls. Following this aim, let us calculate the input matrix \( \hat{B} \). Based on formula (8), it obtains the form:

\[
\hat{B} = \sum_{k=0}^{M} e^{-Ah_k} B_k = \sum_{k=0}^{M} T e^{-J(A)h_k} T^{-1} B_k.
\]  

(16)

Next we use Chen’s Theorem 4 to calculate the \( T^{-1} \hat{B} \) matrix term. From (16) we have directly:

\[
T^{-1} \hat{B} = \sum_{k=0}^{M} e^{-J(A)h_k} T^{-1} B_k.
\]  

(17)

The \( A_0 \) matrix eigenvalues (13) are expressed by (18): [4]

\[
\mu_i = -2 + 2 \cos \left( \frac{\pi i}{N} \right), \quad i = 1, 2, \ldots, N.
\]  

(18)

The \( A_0 \) (13) Jordan transformation matrix is symmetrical, equal to its own inverse and expressed by equation: [4]

\[
T = T^T = T^{-1} = \begin{bmatrix}
\sin \frac{\pi}{N} & \sin \frac{2\pi}{N} & \cdots & \sin \frac{(N-1)\pi}{N} \\
\sin \frac{2\pi}{N} & \sin \frac{2\pi}{N} & \cdots & \sin \frac{(N-1)\pi}{N} \\
\vdots & \vdots & \ddots & \vdots \\
\sin \frac{(N-1)\pi}{N} & \sin \frac{(N-1)\pi}{N} & \cdots & \sin \frac{(N-1)^2\pi}{N}
\end{bmatrix}.
\]  

(19)

The eigenvalues of the state matrix:

\[
A = \frac{1}{h^2} A_0
\]  

(20)

expressed by the equation:

\[
\lambda_i = \frac{\mu_i}{h^2}
\]

where \( \mu_i \) is given by (18), and the \( A \) (20) matrix eigenvectors \( A \) have the same form as the \( A_0 \) matrix (formula (19)). The eigenvalues are distinct (formula (18)), thus the Jordan form of the \( A \) matrix is as follows:

\[
J(A) = \text{diag} \left[ \lambda_1 \cdots \lambda_{N-1} \right].
\]  

(21)

The \( T \) matrix (19) is symmetrical, thus the \( T^{-1} \hat{B} \) matrix term (17) can be rewritten in the form:

\[
T^{-1} \hat{B} = \sum_{k=0}^{M} e^{-J(A)h_k} TB_k.
\]  

(22)

The exponent \( e^{-J(A)h_k} \) from (21) becomes:

\[
e^{-J(A)h_k} = \text{diag} \left[ e^{-\lambda_1 h_k} \cdots e^{-\lambda_{N-1} h_k} \right], \quad k = 0, 1, \ldots, M.
\]  

(23)
Next multiplying $e^{-J(A)h_k}$ (23) by $T$ (19) we have:

$$e^{-J(A)h_k}T = \begin{bmatrix}
e^{-\lambda_1h_k} \sin \frac{\pi}{N} & \ldots & e^{-\lambda_1h_k} \sin \left(\frac{N-1}{N} \frac{2\pi}{N}\right) \\
e^{-\lambda_2h_k} \sin \frac{2\pi}{N} & \ldots & e^{-\lambda_2h_k} \sin \left(\frac{N-1}{N} \frac{2\pi}{N}\right) \\
\vdots & \ddots & \vdots \\
e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{1}{N}\right) & \ldots & e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{2\pi}{N}\right)
\end{bmatrix},$$

\[(24)\]

$k = 0, 1, \ldots, M$.

Combining $e^{-J(A)h_k}T$ (24) with the input matrix $B_k$ (14) we have:

$$e^{-J(A)h_k}TB_k = \begin{bmatrix}
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_1h_k} \sin \frac{i\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_1h_k} \sin \frac{i\pi}{N} \\
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_2h_k} \sin \frac{2\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_2h_k} \sin \frac{2\pi}{N} \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{1}{N}\right) & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{2\pi}{N}\right)
\end{bmatrix},$$

$k = 0, 1, \ldots, M$.

\[(25)\]

Finally combining the $e^{-J(A)h_k}TB_k$ (25) term with the sum (22), we have the following $T^{-1}\hat{B}$ term:

$$T^{-1}\hat{B} = \sum_{k=0}^{M} \begin{bmatrix}
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_1h_k} \sin \frac{i\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_1h_k} \sin \frac{i\pi}{N} \\
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_2h_k} \sin \frac{2\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_2h_k} \sin \frac{2\pi}{N} \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N-1} b_{k1}(z_i)e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{1}{N}\right) & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i)e^{-\lambda_{N-1}h_k} \sin \left(\frac{N-1}{N} \frac{2\pi}{N}\right)
\end{bmatrix}.$$

\[(26)\]

As follows from Chen’s Theorem 4, each row of the matrix $T^{-1}\hat{B}$ must not be zero, because all the eigenvalues (18) are different. Considering the form (26) of the $T^{-1}\hat{B}$ term we have the final condition for the absolute controllability of the system (15)

$$\sum_{l=1}^{p} \left[ \sum_{k=0}^{M} \sum_{i=1}^{N-1} b_{kl}(z_i)e^{-\lambda_jh_k} \sin \frac{j\pi}{N} \right]^2 \neq 0, \quad j = 1, 2, \ldots, N - 1.$$
3.4. The relative controllability analysis.

The relative controllability analysis is performed based on Theorems 2 and 4. Without loss of the generality, for the simplicity of notation, we may assume that there exists an index $k_0 \leq M$, such that $t_1 - h_{k_0} = 0$. If such $k_0$ does not exist, then we introduce the additional delay $h_{k_0}$ with the control matrix $B_{k_0}$ \[3\]. The index $k_0$ plays an important role in Theorem 2. The relative controllability is defined for an arbitrary time interval $[t_0, t_1]$, $t_1 > t_0$ \[3\]. Now let us calculate the \(\tilde{B}\) matrix (formula (6): \[\tilde{B} = [B_0 \ B_1 \ \ldots \ B_{k_0-1}]\).

The $T^{-1}\tilde{B}$ term is of the form:

$$T^{-1}\tilde{B} = \begin{bmatrix} \sum_{i=1}^{N-1} b_{k1}(z_i) \sin \frac{i\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i) \sin \frac{i\pi}{N} \\ \sum_{i=1}^{N-1} b_{k1}(z_i) \sin \frac{2\pi i}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i) \sin \frac{2\pi i}{N} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{N-1} b_{k1}(z_i) \sin \frac{i(N-1)\pi}{N} & \ldots & \sum_{i=1}^{N-1} b_{kp}(z_i) \sin \frac{i(N-1)\pi}{N} \end{bmatrix},$$ \[28\]

$k = 0, 1, \ldots, k_0 - 1$.

From Chen's theorem 4 and the equations (27), (28) there follows the condition of the relative controllability of the dynamical system (15):

$$\sum_{k=0}^{k_0-1} \sum_{l=1}^{p} \left[ \sum_{i=1}^{N-1} b_{kl}(z_i) \sin \frac{j\pi i}{N} \right]^2 \neq 0, \quad j = 1, 2, \ldots, N - 1.$$ \[29\]

4. Conclusions

The results in this article can be summarised as follows:

- The infinite dimensional parabolic dynamical system (9), presented in the finite dimensional form (15), is controllable absolutely in the time interval $t_1 > t_0 + h_M$ if and only if the equation series (29) holds true:

$$\sum_{i=1}^{p} \sum_{k=0}^{M} \sum_{l=1}^{N-1} b_{kl}(z_i) e^{-\lambda_j h_k} \sin \frac{j\pi i}{N} \neq 0, \quad j = 1, 2, \ldots, N - 1.$$ \[29\]

- The infinite dimensional parabolic dynamical system (9), presented in the finite dimensional form (15), is controllable relatively in any time interval $t_1 > t_0$.
if and only if the equation series (30) holds true:

\[
\sum_{k=0}^{k_0-1} \sum_{l=1}^{p} \left[ \sum_{i=1}^{N-1} b_{kl}(z_i) \sin \frac{j \pi}{N} \right]^2 \neq 0, \quad j = 1, 2, \ldots, N - 1. \quad (30)
\]

References