Supporting Proofs
for
A Macro Tool for Calculating Energy Consumption Changes from New Technologies

The proofs of these Appendices are developed using only 3 factors—capital, labor and fuel. They are also developed considering only one technology improvement factor, τ_i, instead of the technology vector $\boldsymbol{\tau}$. For simplicity, we suppress the subscript *i* and use the notation τ.

The extension of these proofs to n factors and an n-dimensional technology vector is very direct, even obvious. However, these simplifications make the proofs less notation-intensive and easier to follow.

General Approach

The first Proofs Appendix establishes that applying Shephard's Lemma to obtain the correct expression for $\partial F / \partial \tau$ when *K*, *L*, and p_F are fixed requires a form of Shephard's Lemma that is explicit in τ, $F(\tau)$, $p_K(\tau)$, and $p_L(\tau)$.

The second Proofs Appendix establishes that the assumption of fixed real prices for fuel and the other complementary factors leads to a simplification of Shephard's Lemma. This form of Shephard's Lemma is used in the Appendix A: Tool Specification of the main article, equation (A-3).

The third Proofs Appendix establishes that the odd-looking optimization that mixes both cost and production functions in the same problem is actually a correct approach, and is the result of a kind of duality. (That optimization is required to generate partials of p_K and p_L with respect to τ and leads, via the Implicit Function Theorem, to equations (A-9) of Appendix A of the main article.)

The fourth Proofs Appendix shows the formal equivalence between the functional forms for engineering technology used in the production function and those used in the cost function.

Proofs Appendix D1. *The Application of Shephard's Lemma with Changing Tau*

STANDARD PRIMAL SIDE SETUP

First consider the problem from the primal side:

$$\underset{K,L,F}{Max}\, \pi = P \cdot Y(K, L, F, \tau_0) - p_K^0 K - p_L^0 L - p_F^0 F \qquad \text{(D-1)}$$

and suppose the solution is K_0, L_0 and F_0.

For a CRS production function, $P = c$, where P is the price of output used in standard formulations and c is the unit cost function.[1] Thus, it will also be true that $P = c_0 = c(p_K^0, p_L^0, p_F^0, \tau_0)$, and that total cost $C = c_0 Y_0$.

With fixed K_0 and L_0, this problem can be restated as:

$$\underset{F}{Max}\, \pi = c_0 \cdot Y(K_0, L_0, F, \tau_0) - p_K^0 K_0 - p_L^0 L_0 - p_F^0 F \qquad \text{(D-2)}$$

The first order conditions associated with (D-2) will ensure the following relationships hold at $F = F_0$:

$$\begin{aligned} \frac{\partial Y(K_0, L_0, F_0, \tau_0)}{\partial K} &= p_K^0 / c_0 \\ \frac{\partial Y(K_0, L_0, F_0, \tau_0)}{\partial L} &= p_L^0 / c_0 \\ \frac{\partial Y(K_0, L_0, F_0, \tau_0)}{\partial F} &= p_F^0 / c_0 \end{aligned} \qquad \text{(D-3)}$$

Now we ask what happens to F when $\tau_0 \to \tau_1$ with K and L fixed and real fuel price fixed at p_F^0 / c_0.

Under these conditions, F_0 will change to F_1, which we can solve for from the fuel first-order condition in (D-3):

$$\frac{\partial Y(K_0, L_0, F_1, \tau_1)}{\partial F} = p_F^0 / c_0 \qquad \text{(D-4)}$$

This can be done because Y by assumption is a known production function over the domain of the inputs K, L, F, and τ. Then we can calculate the partial we ultimately need as

$$\frac{\partial F}{\partial \tau} = \lim_{\tau_1 \to \tau_0} \left(\frac{F_1 - F_0}{\tau_1 - \tau_0} \right) \qquad \text{(D-5)}$$

Alternatively, equation (D-4) can be solved analytically (or from the Implicit Function Theorem) for F as a function of τ. In any case we can derive the needed elasticity of F with respect to τ and we are done.

(Note that, because of fixed real fuel price in (D-4), we so far have not had to deal with the possibility that the unit cost function will change with changing τ.)

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[1] See, for example, Mas-Colell, Whinston, and Green (1995), page 141.

DUAL SIDE SETUP

However, we want to derive this relationship, F as a function of τ, from the dual side.

First note that the solution to the primal problem (D-2), F_0, will be a solution to the corresponding dual problem:

$$\begin{aligned} &\underset{F}{Min}\, C(\tau) = p_K^0 K_0 + p_L^0 L_0 + p_F^0 F \\ &s.t.\ Y\left(K_0, L_0, F, \tau_0\right) = Y_0 \end{aligned} \tag{D-6}$$

$$where\, Y_0 = Y\left(K_0, L_0, F_0, \tau_0\right)$$

We then proceed by noting that the other first-order conditions in (D-3) change when $\tau_0 \rightarrow \tau_1$ and $F_0 \rightarrow F_1$:

$$\begin{aligned} \frac{\partial Y(K_0, L_0, F_1, \tau_1)}{\partial K} &\neq p_K^0 / c_0 \\ \frac{\partial Y(K_0, L_0, F_1, \tau_1)}{\partial L} &\neq p_L^0 / c_0 \end{aligned} \tag{D-7}$$

In particular, the only way for these two conditions to hold is if p_K and p_L change, since in general:[2]

$$\frac{p_K^0}{c_0} \neq \frac{p_K^1}{c(p_K^1, p_L^1, p_F^1, \tau_1)} = \frac{p_K^1}{c_1} \tag{D-8}$$

and same holds for the price of labor. This therefore results in new first-order conditions:

$$\begin{aligned} \frac{\partial Y(K_0, L_0, F_1, \tau_1)}{\partial K} &= p_K^1 / c_1 \\ \frac{\partial Y(K_0, L_0, F_1, \tau_1)}{\partial L} &= p_L^1 / c_1 \end{aligned} \tag{D-9}$$

But notice that this reveals the solution to a new primal-side problem. That is, if Y is well-behaved in second order as usual, (D-4) and (D-9) are both necessary and sufficient conditions, and it will be true that F_1 is the unique solution to the profit maximization problem

$$\underset{F}{Max}\, \pi = c_1 \cdot Y(K_0, L_0, F, \tau_1) - p_K^1 K_0 - p_L^1 L_0 - p_F^1 F \tag{D-10}$$

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[2] Note that while the real fuel price is unchanged at p_F^0 / c_0, the nominal price will change to some p_F^1 as c_0 changes to some c_1.

just as F_0 is the unique solution to the profit maximization problem (D-1), above.

We then proceed by noting that (D-10) has a dual in the cost function space, just as does (D-1). These can be used to set up Shephard's Lemma as the means to derive an expression for $\partial F / \partial \tau$ using only cost function parameters.

The cost dual equivalent of (D-2) is (D-6), whose solution is F_0.

The cost dual equivalent of (D-10) is:

$$\underset{F}{Min}\, C = p_K^1 K_0 + p_L^1 L_0 + p_F^1 F$$
$$s.t.\;\; Y(K_0, L_0, F, \tau_1) = Y_1 \qquad \text{(D-11)}$$

$$where\; Y_1 = Y(K_0, L_0, F_1, \tau_1)$$

The solution to (D-11) is F_1. Note that both p_K and p_L (for this same production function, and therefore same cost function) are therefore functions of τ.

[*Note that by comparing* (D-10) *and* (D-2) *it is apparent that* F *is a function of* τ*, as are* p_K*,* p_L*, and* p_F*. Thus, the general form of the optimization problem that generates* $F(\tau)$ *is:*

$$\underset{F}{Max}\, \pi(\tau) = c\big(p_K(\tau), p_L(\tau), p_F(\tau), \tau\big) Y(K_0, L_0, F, \tau)$$
$$- p_K(\tau) K_0 - p_L(\tau) L_0 - p_F(\tau) F \qquad \textit{(D-12)}$$

where $$\frac{p_F(\tau)}{c\big(p_K(\tau), p_L(\tau), p_F(\tau), \tau\big)} = \frac{p_F^0}{c\big(p_K(\tau_0), p_L(\tau_0), p_F(\tau_0), \tau_0\big)} = \frac{p_F^0}{c_0}$$]

We are now ready for the unifying theorem:

THEOREM RELATING F *AND* τ *VIA SHEPHARD'S LEMMA*

Theorem 1: Given a well-behaved CRS production function $Y(K, L, F, \tau)$ and its dual unit cost function $c(p_K, p_L, p_F, \tau)$, and given that for changes in τ we hold fixed $K = K_0$, $L = L_0$, and $p_F / P = p_F^0 / c_0$, the partial derivative of F with respect to τ will be of the form:

$$\frac{\partial F}{\partial \tau} = \frac{\partial}{\partial \tau}\left[Y(K_0, L_0, F(\tau), \tau) \frac{\partial c(p_K(\tau), p_L(\tau), p_F(\tau), \tau)}{\partial p_F} \right] \qquad \text{(D-13)}$$

Proof: First note that for a CRS function, Shephard's Lemma will be of the form

$$\frac{\partial C}{\partial p_F} = Y\frac{\partial c}{\partial p_F} = F \tag{D-14}$$

Shephard's Lemma will hold for both dual problems, (D-6) and (D-11). Applying (D-14) to (D-6), we have that

$$F_0 = Y_0\left(K_0, L_0, F_0, \tau_0\right)\frac{\partial c\left(p_K(\tau_0), p_L(\tau_0), p_F(\tau_0), \tau_0\right)}{\partial p_F} \tag{D-15}$$

Applying (D-14) to (D-11), we have that

$$F_1 = Y_1\left(K_0, L_0, F_1, \tau_1\right)\frac{\partial c\left(p_K(\tau_1), p_L(\tau_1), p_F(\tau_1), \tau_1\right)}{\partial p_F} \tag{D-16}$$

From these two equations, we see that

$$F(\tau) = Y\left(K_0, L_0, F(\tau), \tau\right)\frac{\partial c\left(p_K(\tau), p_L(\tau), p(\tau), \tau\right)}{\partial p_F} \tag{D-17}$$

and equation (D-13) above follows immediately.

Proofs Appendix D2. *Simplified form of Shephard's Lemma*

Fortunately, the requirement that fuel price remain fixed in real terms allows a useful simplification. First consider the generalized primal problem that is variable in τ, from equation (D-12) in Appendix D1:

$$\begin{aligned} \underset{F}{Max}\ \pi(\tau) &= c(p_K(\tau), p_L(\tau), p_F(\tau), \tau)\cdot Y(K_0, L_0, F, \tau) \\ &\quad - p_K(\tau)K_0 - p_L(\tau)L_0 - p_F(\tau)F \end{aligned} \tag{D-18}$$

The corresponding first order conditions are:

$$\begin{aligned} \frac{\partial Y\left(K_0, L_0, F, \tau\right)}{\partial F} &= \frac{p_F(\tau)}{c\left(p_K(\tau), p_L(\tau), p_F(\tau), \tau\right)} \\ \frac{\partial Y\left(K_0, L_0, F, \tau\right)}{\partial K} &= \frac{p_K(\tau)}{c\left(p_K(\tau), p_L(\tau), p_F(\tau), \tau\right)} \\ \frac{\partial Y\left(K_0, L_0, F, \tau\right)}{\partial L} &= \frac{p_L(\tau)}{c\left(p_K(\tau), p_L(\tau), p_F(\tau), \tau\right)} \end{aligned} \tag{D-19}$$

And the corresponding form of the dual problem is:

$$\begin{aligned} &\underset{F}{Min}\ C(\tau) = p_K(\tau)K_0 + p_L(\tau)L_0 + p_F(\tau)F \\ &s.t.\ \ Y\left(K_0, L_0, F, \tau\right) = Y_0 \\ &where\ Y_0 = Y\left(K_0, L_0, F_0, \tau\right) \end{aligned} \tag{D-20}$$

But we can recast (D-19) using the assumption of fixed real fuel price:

$$\begin{aligned}
\frac{\partial Y\left(K_0,L_0,F,\tau\right)}{\partial F} &= \frac{p_F(\tau)}{c\left(p_K(\tau),p_L(\tau),p_F(\tau),\tau\right)} = \frac{p_F^0}{c_0} \\
\frac{\partial Y\left(K_0,L_0,F,\tau\right)}{\partial K} &= \frac{p_K(\tau)}{c\left(p_K(\tau),p_L(\tau),p_F(\tau),\tau\right)} = \frac{\hat{p}_K(\tau)}{c_0} \\
\frac{\partial Y\left(K_0,L_0,F,\tau\right)}{\partial L} &= \frac{p_L(\tau)}{c\left(p_K(\tau),p_L(\tau),p_F(\tau),\tau\right)} = \frac{\hat{p}_L(\tau)}{c_0}
\end{aligned} \tag{D-21}$$

Then we can substitute (D-21) into (D-20) to give:

$$\begin{aligned}
&\underset{F}{Min}\, C(\tau) = \frac{c(p_K(\tau),p_L(\tau),p_F(\tau),\tau)}{c_0}\left[\hat{p}_K(\tau)K_0+\hat{p}_L(\tau)L_0+p_F^0F\right] \\
&s.t.\;\; Y\left(K_0,L_0,F,\tau\right)=Y_0 \\
&where\, Y_0 = Y\left(K_0,L_0,F_0,\tau\right)
\end{aligned} \tag{D-22}$$

This will deliver the same solution for $F(\tau)$ as:

$$\begin{aligned}
&\underset{F}{Min}\, C(\tau) = \hat{p}_K(\tau)K_0+\hat{p}_L(\tau)L_0+p_F^0F \\
&s.t.\;\; Y\left(K_0,L_0,F,\tau\right)=Y_0 \\
&where\, Y_0 = Y\left(K_0,L_0,F_0,\tau\right)
\end{aligned} \tag{D-23}$$

since for any τ it differs only by a multiplicative constant in the objective function. Furthermore, since we know that for $\tau=\tau_0$, $F(\tau)=F_0$ is a solution for both (D-22) and (D-23), it must be that $\hat{p}_K(\tau_0)=p_K(\tau_0)$ and $\hat{p}_L(\tau_0)=p_L(\tau_0)$. Therefore, the optimization problem (D-23) with $p_i(\tau)$ replacing $\hat{p}_i(\tau)$ can be substituted for the optimization problem (D-22), and accordingly Shephard's Lemma (derived by applying the Implicit Function theorem to the cost minimization problem) can be restated as:

$$F(\tau)=Y\left(K_0,L_0,F(\tau),\tau\right)\frac{\partial c\left(p_K(\tau),p_L(\tau),p_{\mathrm{F}}^0,\tau\right)}{\partial p_F} \tag{D-24}$$

This is the form used in the article.

Proofs Appendix D3. *Mixed Cost and Production Function Optimization*

EQUIVALENCE OF THE STANDARD PRIMAL PROFIT MAXIMIZING FORMULATION AND THE COST-CONSTRAINED FORMULATION USED IN THE CECANT ALGORITHM

Theorem 2: Given a CRS production function $Y(K,L,F,\tau)$ and its dual unit cost function $c(p_K, p_L, p_F, \tau)$, the following optimization problems are dual to each other:

Standard Formulation

$$\pi_1 = \max_{K,L,F} c_0 Y(K,L,F,\tau_0) - p_K^0 K - p_L^0 L - p_F^0 F \qquad \text{(D-25)}$$

Cost-constrained Formulation

$$\begin{aligned} \pi_2 = \max_{p_K, p_L, F} \; & c_0 Y(K_0, L_0, F, \tau_0) - p_K K_0 - p_L L_0 - p_F^0 F \\ s.t. \; & c(p_K, p_L, p_F^0, \tau_0) = c_0 \end{aligned} \qquad \text{(D-26)}$$

where duality means that if the solution to (D-25) is K_0, L_0, F_0, then the solution to (D-26) must be p_K^0, p_L^0, F_0, (and $\pi_1 = \pi_2$).

Proof: Begin by looking at the standard cost-function dual to (D-25). Since we have assumed CRS, the cost function defined at the point $C(p_K^0, p_L^0, p_F^0, Y_0)$ is

$$\begin{aligned} C_0 = c(p_K^0, p_L^0, p_F^0, \tau_0) Y_0 = \min_{K,L,E} \; & p_K^0 K + p_L^0 L + p_F^0 F \\ s.t. \; & Y(K,L,F,\tau_0) = Y_0 \end{aligned} \qquad \text{(D-27)}$$

Since (D-27) and (D-25) are duals, if the solution to (D-25) is K_0, L_0, F_0, the solution to (D-27) is K_0, L_0, F_0. In both cases, the solution delivers $Y(K_0, L_0, F_0, \tau_0) = Y_0$. Furthermore, the solution corresponds in (D-27) to $c(p_K^0, p_L^0, p_F^0, \tau_0)$, which we designate c_0, so that at the solution point

$$\begin{aligned} Y(K_0, L_0, F_0, \tau_0) &= Y_0 \\ c(p_K^0, p_L^0, p_F^0, \tau_0) &= c_0 \end{aligned} \qquad \text{(D-28)}$$

Now observe that (D-26) can be rewritten as:

$$\begin{aligned} \pi_2 = \max_F \left[c_0 Y(K_0, L_0, F, \tau_0) - p_F^0 F \right] + \min_{p_K, p_L} & \left(p_K K_0 + p_L L_0 \right) \\ s.t. \; c(p_K, p_L, p_F^0, \tau_0) = c_0 & \end{aligned} \qquad \text{(D-29)}$$

We need to show that the solution to (D-29) is p_K^0, p_L^0, F_0.

Proof is by contradiction. Suppose there exists a solution to (D-29) that is p_K^1, p_L^1, F_1, where at least one of the following is true: $p_K^1 \neq p_K^0$, $p_L^1 \neq p_L^0$, or $F_1 \neq F_0$.

We see from the first-order conditions on the Lagrangian for F that

$$\left.\frac{\partial Y}{\partial F}\right|_{K_0,L_0} = p_F^0 \big/ c_0 \qquad \text{(D-30)}$$

By our assumption, this will occur at $F = F_1 \neq F_0$. But this contradicts the identical first-order condition for F from (D-25) (recall that the function Y in (D-25) is the same Y as in (D-26)), where

$$\left.\frac{\partial Y}{\partial F}\right|_{K_0,L_0,F_0} = p_F^0 \big/ c_0 \qquad \text{(D-31)}$$

Therefore, $F_1 = F_0$.

This being true, we can rewrite (D-29) as

$$\begin{aligned} \pi_2 &= c_0 Y_0 - p_F^0 F_0 + \min_{p_K, p_L}\left(p_K K_0 + p_L L_0 \right) \\ s.t.\ & c(p_K, p_L, p_F^0, \tau_0) = c_0 \end{aligned} \qquad \text{(D-32)}$$

According to our supposition, there is a solution to (D-32) that is p_K^1, p_L^1. Then it must be true that

$$p_K^1 K_0 + p_L^1 L_0 < p_K^0 K_0 + p_L^0 L_0 \qquad \text{(D-33)}$$

or

$$p_K^1 K_0 + p_L^1 L_0 + p_F^0 F_0 < p_K^0 K_0 + p_L^0 L_0 + p_F^0 F_0 \qquad \text{(D-34)}$$

Note from the constraint of (D-32) that it must also be true that

$$c(p_K^1, p_L^1, p_F^0, \tau_0) = c_0 \qquad \text{(D-35)}$$

meaning that from (D-28) we can conclude that

$$c(p_K^0, p_L^0, p_F^0, \tau_0) = c(p_K^1, p_L^1, p_F^0, \tau_0) = c_0 \qquad \text{(D-36)}$$

and we can therefore write (D-27) as

$$C_0 = c(p_K^0, p_L^0, p_F^0, \tau_0) Y_0 = c(p_K^1, p_L^1, p_F^0, \tau_0) Y_0 = C_1 \qquad \text{(D-37)}$$

or more precisely as

$$\begin{aligned} C_0 &= \left\{ \min_{K,L}\ p_K^0 K + p_L^0 L + p_F^0 F_0 \,\middle|\, Y(K, L, F_0, \tau_0) = Y_0 \right\} \\ &= \left\{ \min_{K,L}\ p_K^1 K + p_L^1 L + p_F^0 F_0 \,\middle|\, Y(K, L, F_0, \tau_0) = Y_0 \right\} = C_1 \end{aligned} \qquad \text{(D-38)}$$

We are left with two possibilities: In the first possibility, the solution to the right-hand-side of (D-38) is $K = K_0$, and $L = L_0$, just as it is for the left-hand side. However, this contradicts the equality in (D-38), given the inequality in (D-34).

In the second possibility, the solution to the right-hand-side of (D-38) is K_1, L_1, where either $K_1 \neq K_0$ or $L_1 \neq L_0$ or both. The equality in (D-38) means we would then have

$$p_K^0 K_0 + p_L^0 L_0 + p_F^0 F_0 = p_K^1 K_1 + p_L^1 L_1 + p_F^0 F_0 \tag{D-39}$$

However, the right-hand-side of (D-39) cannot be the minimum found on the right-hand-side of (D-38), since from (D-34) we can find a K_0 and L_0 that will deliver a lower value.

We are left with a contradiction. Accordingly, our original assumption is wrong, and it must be true that $p_K^1 = p_K^0$, $p_L^1 = p_L^0$, and $F_1 = F_0$. Therefore the solution to (D-26) is p_K^0, p_L^0, F_0.

CORRECTNESS OF USING THE COST-CONSTRAINED FORMULATION TO DETERMINE THE PARTIALS OF p_K *AND* p_L

Theorem 3: Given a CRS production function $Y(K,L,F,\tau)$ and its dual unit cost function $c(p_K, p_L, p_F, \tau)$, and given that for changes in τ we hold fixed $K = K_0$, $L = L_0$, and $p_F / c(p_K, p_L, p_F, \tau) = p_F^0 / c_0$, the profit-maximizing partial derivatives of p_K and p_L with respect to τ can be determined by applying the Implicit Function Theorem to the following cost-constrained formulation of the profit maximization problem:

$$\begin{aligned} \pi = \max_{p_K, p_L, F} \; & c_0 Y(K_0, L_0, F, \tau) - p_K K_0 - p_L L_0 - p_F^0 F \\ s.t. \; & c(p_K, p_L, p_F^0, \tau) = c_0 \end{aligned} \tag{D-40}$$

Proof: First note that the solution of the Lagrangian for the above problem yields the following system of equations:

$$\begin{aligned} & K_0 + \mu \frac{\partial c}{\partial p_K} = 0 \\ & L_0 + \mu \frac{\partial c}{\partial p_L} = 0 \\ & c - c_0 = 0 \\ & p_F^0 - c_0 \frac{\partial Y}{\partial F} = 0 \end{aligned} \tag{D-41}$$

where μ is the Lagrange multiplier.

The Implicit Function Theorem allows us to ask how some endogenous variables change (in our case, p_K and p_L) when an exogenous variable is changed (in our case, τ). In particular, it allows us to derive the partials $\partial p_K / \partial \tau$ and $\partial p_L / \partial \tau$. What remains is to show that (D-40) correctly replicates the solutions of the primal when τ changes.

Recall what happens when we take the standard primal profit-maximizing problem (with K and L fixed and real energy price fixed at p_F^0 / c_0), namely

$$\underset{K,L,F}{Max}\, \pi = c_0 \cdot Y(K, L, F, \tau_0) - p_K^0 K - p_L^0 L - p_F^0 F \qquad \text{(D-42)}$$

and change $\tau_0 \to \tau_1$. Then we have a new primal problem

$$\underset{K,L,F}{Max}\, \pi = c_0 \cdot Y(K, L, F, \tau_1) - p_K^1 K - p_L^1 L - p_F^0 F \qquad \text{(D-43)}$$

The solution to (D-42) is K_0, L_0, F_0; the solution to (D-43) is K_0, L_0, F_1. But from Theorem 2, we know that each of these has a cost-constrained dual. The cost-constrained dual to (D-42) is

$$\begin{aligned} &\max_{p_K, p_L, F} \pi = c_0 Y(K_0, L_0, F, \tau_0) - p_K K_0 - p_L L_0 - p_F^0 F \\ &s.t. \quad c(p_K, p_L, p_F^0, \tau_0) = c_0 \end{aligned} \qquad \text{(D-44)}$$

whose solution is p_K^0, p_L^0, F_0. Similarly, the cost-constrained dual to (D-43) is

$$\begin{aligned} &\max_{p_K, p_L, F} \pi = c_0 Y(K_0, L_0, F, \tau_1) - p_K K_0 - p_L L_0 - p_F^0 F \\ &s.t. \quad c(p_K, p_L, p_F^0, \tau_1) = c_0 \end{aligned} \qquad \text{(D-45)}$$

whose solution is p_K^1, p_L^1, F_1. Recall also that p_K^1 and p_L^1 are those prices that satisfy the first-order conditions (D-9) on the primal problem (D-43).

Therefore, the formulation in (D-40) gives the correct description of how p_K^1 and p_L^1 behave when τ changes. Accordingly, the Implicit Function Theorem that operates on the system of equations derived from (D-40) will generate the correct partials.

Proofs Appendix D4. *Technology Vector Transformation to Cost Form*

PROOF THAT p_i / τ_i *IS THE CORRECT FORM FOR ENGINERRING TECHNICAL CHANGE IN THE COST FUNCTION*

Other researchers have told us this is a well-known result. But we have been unable to find its source. For the sake of completeness, and with apologies to the researcher who first proved it, we offer one version, at least, of the proof for this Theorem.

Lemma #1: Given a CRS production function $Y = f(\mathbf{x})$, the partial $\frac{\partial f}{\partial x_i}$ will be of the form $g\left(\frac{h(\mathbf{x})}{x_i}\right)$, where $h(\mathbf{x})$ is CRS.

Proof: Given that Y is CRS,

$$kf(\mathbf{x}) = f(k \cdot \mathbf{x}) = f(kx_1, kx_1, \cdots, kx_1, \cdots kx_1) \quad \forall k$$

or

$$f(\mathbf{x}) = \frac{1}{k} f(kx_1, kx_1, \cdots, kx_1, \cdots kx_1) \quad \forall k$$

Let $\breve{x}_i = kx_i$. Then

$$\frac{\partial f}{\partial x_i} = \frac{1}{k} \frac{\partial f}{\partial \breve{x}_i} \frac{\partial \breve{x}_i}{\partial x_i}$$

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial \breve{x}_i} \qquad \forall k, \forall i$$

These partials will thus be of a form $g(x_i)$ such that $g(x_i) = g\left(kx_i\right)$, where the value of k makes no difference to the value of g. Leaving aside the trivial case where $g = constant$, this can only be true if, for every appearance of kx_i in g, it appears in ratio with a function $h(kx_i)$, where the ratio must satisfy

$$\frac{h(kx_i)}{kx_i} = constant \tag{D-46}$$

This condition must hold for each i. This can occur only if h is a function of each x_i, and so it has form $h(x_1, x_2, \cdots, x_2, \cdots, x_2)$. Since (D-46) must hold for all k, and since these k multiply all factors simultaneously, this gives us the more explicit condition $\frac{h(kx_1, kx_2, \cdots, kx_i, \cdots, kx_n)}{kx_i} = constant$ for any k. This in turn will hold only if $h(kx_1, kx_2, \cdots, kx_i, \cdots, kx_n) = k \cdot h(x_1, x_2, \cdots, x_i, \cdots, x_n)$. But this is the definition of constant returns to scale, so h must be CRS.

Lemma #2: The function $h(\mathbf{x})$ in Lemma #1 is the production function $Y = f(\mathbf{x})$.

Proof: Let P be the price of output Y (i.e., the unit cost). Then the first order condition from the producers' maximization problem gives

$$\frac{\partial f}{\partial x_i} = \frac{p_i}{P}$$

From Lemma #1, we then have that

$$\frac{p_i}{P} = g_i\left(\frac{h(\mathbf{x})}{x_i}\right)$$

Solving this for x_i yields

$$x_i = \frac{h(\mathbf{x})}{g^{-1}\left(\frac{p_i}{P}\right)}$$

Proceeding as if we are deriving the dual unit cost function, we substitute this result back into the production function, so that

$$Y = f\left[\frac{h(\mathbf{x})}{g_1^{-1}\left(\frac{p_1}{P}\right)}, \frac{h(\mathbf{x})}{g_2^{-1}\left(\frac{p_2}{P}\right)}, \cdots, \frac{h(\mathbf{x})}{g_n^{-1}\left(\frac{p_n}{P}\right)}\right]$$

But since f is CRS, we can write this as

$$Y = h(\mathbf{x})\left[\frac{1}{g_1^{-1}\left(\frac{p_1}{P}\right)}, \frac{1}{g_2^{-1}\left(\frac{p_2}{P}\right)}, \cdots, \frac{1}{g_n^{-1}\left(\frac{p_n}{P}\right)}\right]$$

This implies that

$$P = r\left(\frac{h(\mathbf{x})}{Y}\right) \tag{D-47}$$

But we know that for a CRS function, $C(Y, \mathbf{p}) = Y \cdot c(\mathbf{p})$,[3] so the unit cost function is independent of Y. The only way for both this to be true and for (D-47) to hold is for $Y = f(\mathbf{x}) = h(\mathbf{x})$.

Theorem 4: Given a CRS production function $Y = f(\boldsymbol{\tau}, \mathbf{x})$ where $Y = f(\tau_1 x_1, \tau_2 x_2, \cdots, \tau_n x_n)$ so that $\boldsymbol{\tau}$ is the vector of engineering efficiency

[3] See, for instance, Diewert (1972).

parameters, the corresponding dual unit production function is of the form $c = c\left(\frac{p_1}{\tau_1}, \frac{p_2}{\tau_2}, \cdots, \frac{p_n}{\tau_n}\right)$.

Proof: Letting $\hat{x}_i = \tau_i x_i$, we have from the first order condition that

$$\frac{\partial Y}{\partial x_i} = \frac{\partial Y}{\partial \hat{x}_i}\frac{\partial \hat{x}_i}{\partial x_i} = \frac{\partial Y}{\partial \hat{x}_i}\tau_i = \frac{p_i}{P}$$

But from Lemma #1 and Lemma #2, we can write

$$\frac{\partial Y(\hat{\mathbf{x}})}{\partial \hat{x}_i} = \frac{1}{P}\frac{p_i}{\tau_i} = g\left(\frac{Y(\hat{\mathbf{x}})}{\hat{x}_i}\right)$$

so that

$$\hat{x}_i = \frac{Y(\hat{\mathbf{x}})}{g_i^{-1}\left(\frac{1}{P}\frac{p_i}{\tau_i}\right)}$$

Substituting this back in the production function gives

$$Y(\hat{\mathbf{x}}) = f\left[\frac{Y(\hat{\mathbf{x}})}{g_1^{-1}\left(\frac{1}{P}\frac{p_1}{\tau_1}\right)}, \frac{Y(\hat{\mathbf{x}})}{g_2^{-1}\left(\frac{1}{P}\frac{p_2}{\tau_2}\right)}, \cdots, \frac{Y(\hat{\mathbf{x}})}{g_n^{-1}\left(\frac{1}{P}\frac{p_n}{\tau_n}\right)}\right]$$

.

But since Y is CRS, this becomes

$$1 = f\left[\frac{1}{g_1^{-1}\left(\frac{1}{P}\frac{p_1}{\tau_1}\right)}, \frac{1}{g_1^{-1}\left(\frac{1}{P}\frac{p_2}{\tau_2}\right)}, \cdots, \frac{1}{g_n^{-1}\left(\frac{1}{P}\frac{p_n}{\tau_n}\right)}\right]$$

The CRS assumption means any expression common to all terms can come out front as a multiplier of f. So solving for P gives

$$P = P\left(\frac{p_1}{\tau_1}, \frac{p_2}{\tau_2}, \cdots, \frac{p_n}{\tau_n}\right)$$

Since P is the unit cost, c, we have

$$c = c\left(\frac{p_1}{\tau_1}, \frac{p_2}{\tau_2}, \cdots, \frac{p_n}{\tau_n}\right) \quad .$$