Remarks on Hodge numbers and invariant complex structures of compact nilmanifolds

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Abstract: If $N$ is a simply connected real nilpotent Lie group with a $\Gamma$-rational complex structure, where $\Gamma$ is a lattice in $N$, then $H^{s,t}(\Gamma \backslash N) \simeq H^{s,t}(N^\mathbb{C})$ for each $s$, $t$. We study relations between invariant complex structures and Hodge numbers of compact nilmanifolds from a viewpoint of Lie algebras.

Keywords: nilmanifold; Dolbeault cohomology group; complex structure

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1 Introduction

Deformations of compact complex solvmanifolds have been considered by many mathematicians. Nakamura [3] has computed Hodge numbers of small deformations of compact complex parallelizable solvmanifolds. On the other hand, invariant complex structures on generalized flag manifolds can be classified by using $t$-root systems, which is an algebraic method (cf. [1]). However, it seems that studies of invariant complex structures on compact complex solvmanifolds which are not small deformations are not so systematic, and not so much. In this paper, we study relations between invariant complex structures and Hodge numbers of compact nilmanifolds from a viewpoint of Lie algebras. Recall that a Euclidean space $\mathbb{R}^n \times \mathbb{R}^n$ has a symplectic form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

where $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is a global natural coordinate on $\mathbb{R}^n \times \mathbb{R}^n$. Let us consider the complexification $(\mathbb{C}^n \times \mathbb{C}^n, \omega^\mathbb{C})$ of $(\mathbb{R}^n \times \mathbb{R}^n, \omega)$. Let $L$, $L'$ be lattices in $\mathbb{C}^n$. Then, we have a holomorphic (Lagrangian) fibration

$$\pi_1 : L \backslash \mathbb{C}^n \times L' \backslash \mathbb{C}^n \longrightarrow L \backslash \mathbb{C}^n.$$

Let $S = \{(\mathbb{C}^n \times \mathbb{C}^n, \text{id}_{\mathbb{C}^n} \times \text{id}_{\mathbb{C}^n})\}$, $\mathcal{T} = \{(\mathbb{C}^n \times \mathbb{C}^n, \text{id}_{\mathbb{C}^n} \times c)\}$, where $c(q_1, \ldots, q_n) = (\bar{q}_1, \ldots, \bar{q}_n)$. Then the holomorphic symplectic form $\omega^\mathbb{C} = \sum_{i=1}^{n} dp_i \wedge dq_i$ on $(\mathbb{C}^n \times \mathbb{C}^n, S)$ can be considered as a non-degenerate closed (1, 1)-form on $(\mathbb{C}^n \times \mathbb{C}^n, \mathcal{T})$. Hence, we have a compact pseudo-Kähler manifold endowed with $(\mathbb{C}^n \times \mathbb{C}^n, \mathcal{T})$ and $\omega^\mathbb{C}$. Let $N$ be a simply connected nilpotent Lie group, and $n$ its Lie algebra. The exponential map $\exp : n \longrightarrow N$ is a diffeomorphism. Since $n \cong \mathbb{R}^n$, we can consider an analogous construction of different invariant complex structures on a compact nilmanifold (see Section 2, and [7–9] for details).

Let $M$ be a compact Kählerian manifold, and $h^{s,t} = h^{s,t}(M)$ the Hodge number of $M$. Then, $M$ satisfies $h^{s,t} = h^{t,s}$ for each $s$, $t$ by the Hodge theory. In general, a compact complex manifold does not satisfy the relations. In the paper [8], we have that there exist compact 4-dimensional complex manifolds $M_1$ and $M_2$ which satisfy that $M_1$ and $M_2$ are diffeomorphic, and $h^{s,t}(M_1) = h^{t,s}(M_2)$ for each $s$, $t$, namely, we consider
two invariant complex structures on a compact real nilmanifold. In the paper [9], we consider a general case of [8].

Let \((h, J)\) be a Lie algebra with a complex structure, and \(h^\perp_i\) the vector spaces of the \(\pm \sqrt{\text{−}1}\) eigenvectors of the complex structure \(J\), respectively. We denote by \(H^*_{\partial_j} (h^\perp)\) the cohomology ring of the differential bigraded algebra \(\bigwedge^* (h^\perp)^*\), associated to \(h^\perp\) with respect to the operator \(\partial_j\) in the canonical decomposition \(d = \partial_j + \bar{\partial}_j\) on \(\bigwedge^* (h^\perp)^*\). Let \(g\) be a real Lie algebra, and \(g = a \ltimes b\) a decomposition such that \(a\) is a Lie subalgebra of \(g\) and \(b\) is an ideal of \(g\). Then, we can construct complex structures \(J\) and \(\bar{J}\) on \(\mathbb{R} (g^\perp)^*\) from the decomposition (for the details of \(J\) and \(\bar{J}\), see Section 3), where \(g^\perp\) is the complexification of \(g\), and \(\mathbb{R} (g^\perp)^*\) is a real Lie algebra obtained from \(g^\perp\) by the scalar restriction. We denote \(H^k_{\partial_j} (\mathbb{R} (g^\perp)^*)\) by \(h^{k, t}(g_j)\). Then, we have the following results.

**Theorem 1.1.** (1) \(h^{1, 0} (g_j) - h^{1, 0} (g_j) = \dim [a, b]\).
(2) If \(h^{1, 0} (g_j) = h^{0, 1} (g_j)\), then \(g\) is solvable of derived length at most 2, and \(h^{s, 0} (g_j) = h^{0, s} (g_j)\) for each \(s\).
(3) Assume that \(b\) is abelian. Then,
\[
\sum_{s + t = r} h^{s, t} (g_j) = \sum_{s + t = r} h^{s, t} (g_j)
\]
for each \(r\).

In the case where \(b\) is abelian, we have several more results. We can consider two graded algebras \(\{A_i (\lambda)\}, \{A_m (\bar{\lambda})\}\) such that \(A_i (\lambda) \subset \bigwedge^* (\mathbb{R} (g^\perp)^*) \otimes \mathbb{C}\) and \(A_m (\bar{\lambda}) \subset \bigwedge^* (\mathbb{R} (g^\perp)^*) \otimes \mathbb{C}\) with respect to \(J\). Then, we have the following results on decompositions (see Section 3 for the definitions of \(H^k_{\partial_j} (A_i (\lambda))\) and \(H^0, h (A_m (\bar{\lambda}))\)).

**Theorem 3.11** Assume that \(b\) is abelian. Then
\[
H^k_{\partial} (g_j^\perp) \approx \left( \bigoplus_i H^k_{\partial_j} (A_i (\lambda)) \right) \otimes \left( \bigoplus_m H^0, h (A_m (\bar{\lambda})) \right).
\]

**Theorem 3.12** Assume that \(b\) is abelian. Then
\[
h^{s, t}(g_j) = \sum \dim H^k_{\partial_j} (A_i (\lambda)) \cdot \dim H^0, h (A_m (\bar{\lambda})).
\]

Let \(N\) be a simply connected real nilpotent Lie group, \(\Gamma\) a lattice in \(N\). Note that if a left-invariant complex structure \(J\) on \(N\) is \(\Gamma\)-rational, then \(H^k_{\partial_j} (\Gamma \setminus N) \cong H^k_{\partial_j} (\mathbb{R} (g^\perp)^*)\) for each \(s, t\). Thus, results on \(H^k_{\partial_j} (\mathbb{R} (g^\perp)^*)\) of the nilpotent Lie algebra with rational complex structures yield results on \(H^k_{\partial_j} (\Gamma \setminus N)\) of a compact nilmanifold with invariant rational complex structures.

## 2 Preliminaries

Let \(H\) be a Lie group, and \(h\) its Lie algebra. We denote by \(H^*(h) = H^*(h, \mathbb{R})\) the cohomology of the complex \(\bigwedge^* (h^\perp)^*\) of left-invariant differential forms on the Lie group \(H\). A left-invariant almost complex structure on \(H\) can be identified with a linear mapping \(J : h \rightarrow h\) such that \(J^2 = -\text{id}\). The almost complex structure \(J\) is said to be integrable if
\[
\]
for all \(X, Y \in h\). We shall refer to a pair \((h, J)\) consisting of a Lie algebra and an integrable almost complex structure simply as a Lie algebra with a complex structure (cf.,[6]).

Let \(g = (g, J)\) be a Lie algebra with a complex structure, and \(g^\perp_i\) the vector spaces of the \(\pm \sqrt{\text{−}1}\) eigenvectors of the complex structure \(J\), respectively. We denote by \(H^*(g^\perp)^*\), the cohomology ring of the differential bigraded algebra \(\bigwedge^* (g^\perp)^*\), associated to \(g^\perp\) with respect to the operator \(\partial_j\) in the canonical decomposition
$d = \partial_j + \bar{\partial}_j$ on $\bigwedge^s \alpha^*(g^C)^\ast$. We also define subspaces $Z_{\partial_j}^s(g^C)$ and $B_{\partial_j}^s(g^C)$ of $\bigwedge^s \alpha^*(g^C)^\ast$ by a natural manner:

\[
Z_{\partial_j}^s(g^C) = \text{Ker } \partial_j \cap \bigwedge^s \alpha^*(g^C)^\ast,
\]

\[
B_{\partial_j}^s(g^C) = \text{Im } \bar{\partial}_j \cap \bigwedge^s \alpha^*(g^C)^\ast.
\]

We write $h^{s, t}(g_J) = \dim H_{\partial_j}^{s, t}(g^C)$.

From now on, when there exist no possibility of confusion, we omit the subscript $J$.

**Theorem 2.1** (Sakane[5]). Let $N$ be a simply connected complex nilpotent Lie group, and $\Gamma$ a lattice in $N$. Then,

\[
H^{s, t}_\partial(\Gamma \backslash N) \cong H^{0, t}_\partial(n^-) \otimes \bigwedge^s (n^+) = H^t(n^-) \otimes \bigwedge^s (n^+) = \Gamma^\ast_t \otimes \bigwedge^s (n^+) \ast
\]

for each $s, t$.

The decompositions $H^{s, t}_\partial(n^C) \cong H^{0, t}_\partial(n^-) \otimes \bigwedge^s (n^+) \ast$ play an important role in Section 3.

Let $N$ be a simply connected real nilpotent Lie group. It is well known that there exists a lattice in $N$ if and only if there exists a rational Lie subalgebra $n_Q$ such that $n \cong n_Q \otimes \mathbb{R}$. Let $\Gamma$ be a lattice in $N$, and $n_Q$ the $\mathbb{Q}$-span of $\exp^{-1}(\Gamma)$ in $n$, where $\exp : n \rightarrow N$ is the exponential map (cf.[4, Chapter 2]). Then, a complex structure on $N$ is said to be $\Gamma$-rational if $\text{Ker } \exp$ is a lattice in $N$ (cf.[4, Chapter 2]). Moreover, $\Gamma$ is $\Gamma$-rational. We say a complex structure $J$ on a Lie algebra $\mathfrak{h}$ is rational if $J(n_Q) \subset \mathfrak{h}_Q$ for some rational Lie subalgebra $\mathfrak{h}_Q$ such that $\mathfrak{h} \cong \mathfrak{h}_Q \otimes \mathbb{R}$.

**Theorem 2.2** (Console-Fino[2]). Let $N$ be a simply connected nilpotent Lie group, and $\Gamma$ a lattice in $N$. Let $J$ be a $\Gamma$-rational complex structure on $n$. Then,

\[
H^{s, t}_\partial(\Gamma \backslash N) \cong H^{s, t}_\partial(n^C)
\]

for each $s, t$.

### 3 Main results

We consider the following Lie algebra $\mathfrak{g}$ over $\mathbb{R}$:

\[
\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b},
\]

where $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Take bases of the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$:

\[
\mathfrak{a} = \text{span}_\mathbb{R} \{U_1^{\mathfrak{a}}, \ldots, U_p^{\mathfrak{a}}\},
\]

\[
\mathfrak{b} = \text{span}_\mathbb{R} \{V_1^{\mathfrak{b}}, \ldots, V_q^{\mathfrak{b}}\}.
\]

Consider the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. Since $\mathfrak{g}^C = \mathfrak{g} \oplus \sqrt{-1} \mathfrak{g}$, $\mathfrak{g}^C$ has the following basis:

\[
\{U_1^{\mathfrak{g}}, \ldots, U_p^{\mathfrak{g}}, V_1^{\mathfrak{g}}, \ldots, V_q^{\mathfrak{g}}, U_1^{\mathfrak{g}}, \ldots, U_p^{\mathfrak{g}}, V_1^{\mathfrak{g}}, \ldots, V_q^{\mathfrak{g}}\},
\]

where $U_i^{\mathfrak{g}} = \sqrt{-1} U_i^{\mathfrak{g}}, V_i^{\mathfrak{g}} = \sqrt{-1} V_i^{\mathfrak{g}}$. Let

\[
\{\alpha_1^{\mathfrak{g}}, \ldots, \alpha_p^{\mathfrak{g}}, \beta_1^{\mathfrak{g}}, \ldots, \beta_q^{\mathfrak{g}}, \alpha_1^{\mathfrak{g}}, \ldots, \alpha_p^{\mathfrak{g}}, \beta_1^{\mathfrak{g}}, \ldots, \beta_q^{\mathfrak{g}}\}
\]

be the dual basis of

\[
\{U_1^{\mathfrak{g}}, \ldots, U_p^{\mathfrak{g}}, V_1^{\mathfrak{g}}, \ldots, V_q^{\mathfrak{g}}, U_1^{\mathfrak{g}}, \ldots, U_p^{\mathfrak{g}}, V_1^{\mathfrak{g}}, \ldots, V_q^{\mathfrak{g}}\}.
\]
Let $f$ be the (almost) complex structure on $\mathbb{R}(g^C)$ defined by

$$JU^1_i = U^2_i (JU^2_i = -U^1_i), \quad JV^1_j = V^2_j (JV^2_j = -V^1_j)$$

for each $i, j$. Note that $(\mathbb{R}(g^C), f)$ is a complex Lie algebra. We consider other almost complex structure $\tilde{f}$ on $\mathbb{R}(g^C)$ defined by

$$JU^1_i = -U^2_i (JU^2_i = U^1_i), \quad JV^1_j = V^2_j (JV^2_j = -V^1_j)$$

for each $i, j$. Let $\mathbb{R}(G^C)$ be the simply connected real Lie group corresponding to $\mathbb{R}(g^C)$.

Proposition 3.1. $\tilde{f}$ is integrable on $\mathbb{R}(G^C)$. If $f$ is a rational complex structure, then $\tilde{f}$ is also a rational complex structure.

From now on, we denote $(\mathbb{R}(g^C), f')$ by $g_f$, where $f'$ is a complex structure on $\mathbb{R}(g^C)$.

Assume that

$$[U^1_i, U^2_j] = \sum_{k=1}^p C^k_{ij} U^k, \quad [U^1_i, V^1_j] = \sum_{t=1}^q D^t_{is} V^t, \quad [V^2_j, V^1_j] = \sum_{h=1}^q E^h_{js} V^h$$

for each $i, j, s$ and $t$. Let $g_s$ be the Lie algebra defined by

$$g_s = \text{span}(U_1, \ldots, U_p, V_1, \ldots, V_q)$$

which satisfies

$$[U_i, U_j] = \sum_{k=1}^p C^k_{ij} U_k, \quad [U_i, V_s] = \sum_{t=1}^q D^t_{is} V_t (i, j = 1, \ldots, p, s = 1, \ldots, q)$$

with other brackets vanishing.

Theorem 3.2 ([9]). For each $r$,

$$h^{0,r}(g_f) = \dim H^r(a \times b).$$

Theorem 3.3 ([9]). For each $r$,

$$\sum_{s+t=r} h^{s,t}(g_f) = \dim H^r(g_s \times b \times \mathbb{R}^{\dim a}).$$

The following proposition can be considered as a special case of Theorem 3.2. However, this proposition is important.

Proposition 3.4.

$$h^{0,1}(g_f) = \dim g - \dim[a, a] - \dim[b, b].$$

Proof. By Theorem 3.2,

$$h^{0,1}(g_f) = \dim H^1(a \times b) = \dim H^1(a) + \dim H^1(b)$$

$$= \dim a - \dim[a, a] + \dim b - \dim[b, b]$$

$$= \dim g - (\dim[a, a] + \dim[b, b]).$$
Corollary 3.5.

\[ h^{1,0}(\mathfrak{g}_1) - h^{0,1}(\mathfrak{g}_1) = \dim[\mathfrak{a}, \mathfrak{a}] + \dim[\mathfrak{b}, \mathfrak{b}]. \]

**Proof.** Since \( h^{1,0}(\mathfrak{g}_1) = \dim \mathfrak{g} \), we obtain our claim. \( \square \)

Corollary 3.6. If \( h^{1,0}(\mathfrak{g}_1) = h^{0,1}(\mathfrak{g}_1) \), then \( \mathfrak{g} \) is solvable of derived length at most 2, and \( h^{s,0}(\mathfrak{g}_1) = h^{0,s}(\mathfrak{g}_1) \) for each \( s \).

**Proof.** By Corollary 3.5, \( \mathfrak{a} \) and \( \mathfrak{b} \) are abelian. Hence, \( \mathfrak{g} \) is solvable of derived length at most 2. Moreover, by Theorem 3.2,

\[ h^{0,s}(\mathfrak{g}_1) = \dim H^s(\mathfrak{a} \times \mathfrak{b}) = \dim H^s(\mathfrak{R}^{\dim \mathfrak{g}}) = h^{s,0}(\mathfrak{g}_1) \]

for each \( s \). \( \square \)

Corollary 3.7.

\[ h^{1,0}(\mathfrak{g}_1) = \dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{b}]. \]

**Proof.** Let \( \mathfrak{h} = \mathfrak{g}_s \times \mathfrak{b} \times \mathfrak{R}^{\dim \mathfrak{a}} \). By Theorem 3.3,

\[ h^{1,0}(\mathfrak{g}_1) + h^{0,1}(\mathfrak{g}_1) = \dim H^1(\mathfrak{h}) = \dim \mathfrak{h} - \dim[\mathfrak{h}, \mathfrak{h}] = 2 \dim \mathfrak{g} - (\dim[\mathfrak{a}, \mathfrak{a}] + \dim[\mathfrak{a}, \mathfrak{b}] + \dim[\mathfrak{b}, \mathfrak{b}]) = (\dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{a}] - \dim[\mathfrak{b}, \mathfrak{b}]) + (\dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{a}]). \]

Since \( h^{0,1}(\mathfrak{g}_1) = \dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{a}] - \dim[\mathfrak{b}, \mathfrak{b}] \), \( h^{1,0}(\mathfrak{g}_1) = \dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{b}] \). \( \square \)

In the case where \( \mathfrak{b} \) is abelian, we have the following results.

Proposition 3.8. Assume that \( \mathfrak{b} \) is abelian. Then,

\[ \sum_{s+t=r} h^{s,t}(\mathfrak{g}_1) = \sum_{s+t=r} h^{s,t}(\mathfrak{g}_1) \]

for each \( r \).

**Proof.** Note that \( \mathfrak{g} \) is a real Lie algebra, and \( (\mathfrak{g}_1)^\mathbb{C} = \mathfrak{g}_1^* \oplus \mathfrak{g}_1 \). Then, \( \mathfrak{g}^\mathbb{C} = \mathfrak{g}_1^* \cong \mathfrak{g}_1 \) as complex Lie algebras. Recall that \( h^{s,t}(\mathfrak{g}_1) = \dim \Lambda^s(\mathfrak{g}_1)^* \cdot \dim H^t(\mathfrak{g}_1) \) (Theorem 2.1). If \( \mathfrak{b} \) is abelian, then \( \mathfrak{g}_s \cong \mathfrak{g} \). Thus,

\[ \sum_{s+t=r} h^{s,t}(\mathfrak{g}_1) = \dim_{\mathbb{R}} H^t(\mathfrak{g}_s \times \mathfrak{b} \times \mathfrak{R}^{\dim \mathfrak{a}}) = \dim_{\mathbb{R}} H^t(\mathfrak{g} \times \mathfrak{R}^{\dim \mathfrak{g}}) = \sum_{s+t=r} \dim_{\mathbb{R}} H^s(\mathfrak{R}^{\dim \mathfrak{g}}) \cdot \dim_{\mathbb{R}} H^t(\mathfrak{g}) = \sum_{s+t=r} \dim_{\mathbb{R}} H^s(\mathfrak{R}^{\dim \mathfrak{g}}) \cdot \dim_{\mathbb{C}} H^t(\mathfrak{g}^\mathbb{C}, \mathbb{C}) = \sum_{s+t=r} \dim_{\mathbb{C}} \Lambda^s(\mathfrak{g}_1)^* \cdot \dim_{\mathbb{C}} H^t(\mathfrak{g}_1) = \sum_{s+t=r} h^{s,t}(\mathfrak{g}_1) \]

by Theorem 3.3. \( \square \)

Corollary 3.9. Assume that \( \mathfrak{b} \) is abelian. Then

\[ h^{1,0}(\mathfrak{g}_1) - h^{0,1}(\mathfrak{g}_1) = h^{1,0}(\mathfrak{g}_1) - h^{0,1}(\mathfrak{g}_1) = \dim[\mathfrak{a}, \mathfrak{a}]. \]
Proof. We obtain that
\[ h^{1,0}(g_J) - h^{0,1}(g_J) = h^{0,1}(g_J) - h^{1,0}(g_J) \]
by Proposition 3.8.

From now on, we assume that \( b \) is abelian. Put
\[ \lambda_i = a_i^1 + \sqrt{-1}a_i^2, \quad \mu_j = \beta_j^1 + \sqrt{-1}\beta_j^2 \]
for each \( i, j \). Then, we can take a basis of the space of the left-invariant \((1, 0)\)-forms of \((\mathfrak{g}(G^C), J)\) as follows:
\[ \{ \lambda_i, \mu_j \} = 1, \ldots, p, j = 1, \ldots, q \]
with the equations
\[ d\lambda_i = - \sum_{k,h} C_{k\bar{h}}^i \lambda_k \wedge \lambda_h, \quad d\mu_j = - \sum_{k,h} D_{k\bar{h}}^j \lambda_k \wedge \mu_h \]
for \( i = 1, \ldots, p, j = 1, \ldots, q \). Thus, we have
\[ \bar{\partial}\lambda_i = \bar{\partial}\mu_j = 0, \quad (3.1) \]
where \( i = 1, \ldots, p, j = 1, \ldots, q \). On the other hand, put
\[ \xi_i = a_i^1 - \sqrt{-1}a_i^2, \quad \eta_j = \beta_j^1 + \sqrt{-1}\beta_j^2 \]
for each \( i, j \). Then, we can take a basis of the space of the left-invariant \((1, 0)\)-forms of \((\mathfrak{g}(G^C), J)\) as follows:
\[ \{ \xi_i, \eta_j \} = 1, \ldots, p, j = 1, \ldots, q \]
with the equations
\[ d\xi_i = - \sum_{k,h} C_{k\bar{h}}^i \xi_k \wedge \xi_h, \quad d\eta_j = - \sum_{k,h} D_{k\bar{h}}^j \xi_k \wedge \eta_h \]
for \( i = 1, \ldots, p, j = 1, \ldots, q \). Thus, we have
\[ \bar{\partial}\xi_i = 0, \quad \bar{\partial}\eta_j = - \sum_{k,h} D_{k\bar{h}}^j \xi_k \wedge \eta_h \quad (3.2) \]
where \( i = 1, \ldots, p, j = 1, \ldots, q \). Let \( F \) be the homomorphism
\[ F : \bigoplus_{r \leq t} \bigl( \bigoplus_{s \leq t} (g_J^C)^* \bigr) \longrightarrow \bigoplus_{r \leq t} \bigl( \bigoplus_{s \leq t} (g_J^C)^* \bigr) \]
induced by the linear isomorphism \((g_J^C)^* \longrightarrow (g_J^C)^* \) defined by
\[ \lambda_i \mapsto \xi_i, \mu_j \mapsto \eta_j, \quad \bar{\lambda}_i \mapsto \bar{\xi}_i, \quad \bar{\mu}_j \mapsto \bar{\eta}_j \quad (i = 1, \ldots, p, j = 1, \ldots, q) \]
Then, by the equations of (1), (2), \( F \) is an isomorphism of differential graded algebras. In particular, by the cohomology map induced by \( F \), we have
\[ \sum_{s+t=r} H^{s,t}_\partial (g_J^C) \cong \sum_{s+t=r} H^{s,t}_\partial (g_J^C) \]
for each \( r \).
Let $\lambda_i = \lambda_{i1} \land \ldots \land \lambda_{il}$ and $gI = I$ for $I = (i_1, \ldots, i_l)$. Put

$$A^{k,0}_I(\lambda) = \left\{ \sum_{tI = l, 2k = k-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l, 2k = k-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{k,0} \right\},$$

$$A^{0,h}_I(\lambda) = \left\{ \sum_{tI = l, 2k = h-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l, 2k = h-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{0,h} \right\}.$$

Moreover, we define $A_I(\lambda)$, $A_I(\bar{\lambda})$ by $A_I(\lambda) = \bigoplus_k A^{k,0}_I(\lambda)$, $A_I(\bar{\lambda}) = \bigoplus_k A^{0,h}_I(\lambda)$, respectively. We consider the following subspaces of $H^{k,0}_\delta(g_I^C)$ and $H^{0,h}_\delta(g_I^C)$:

$$H^{k,0}_\delta(A_I(\lambda)) = \left\{ \sum_{tI = l, 2k = k-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l, 2k = k-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{k,0} \right\},$$

$$H^{0,h}_\delta(A_I(\lambda)) = \left\{ \sum_{tI = l, 2k = h-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l, 2k = h-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{0,h} \right\}.$$

**Lemma 3.10.** Assume that $b$ is abelian. Then

1. If $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-closed, then $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-closed for each $l$.
2. If $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-exact, then $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-exact for each $l$.

**Proof.** Since

$$\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K \rightarrow \sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$$

by the isomorphism $F$ of differential graded algebras, if $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-closed, then $\sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$ is also $d$-closed. Thus, each $(h-l, l)$-part of $\sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$ is $d$-closed because a $(h-l, l)$-form becomes a $(h-l, l+1)$-form by the operator $d$. Hence, $\sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$ is $d$-closed for each $l$. Similarly, if $\sum_{tI = l+1, 2k = k-l} c_{IK}\lambda_I \land \mu_K$ is $d$-exact, then $\sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$ is also $d$-exact. Thus, each $(h-l, l)$-part of $\sum_{tI = l+1, 2k = k-l} c_{IK}\xi_I \land \eta_K$ is $d$-exact. $\square$

Let

$$Z^{0,h}_\delta(A_I(\lambda)) = \left\{ \sum_{tI = l+1, 2k = h-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l+1, 2k = h-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{0,h} \right\},$$

$$B^{0,h}_\delta(A_I(\lambda)) = \left\{ \sum_{tI = l+1, 2k = h-l} c_{IK}\lambda_I \land \mu_K \mid \sum_{tI = l+1, 2k = h-l} c_{IK}\lambda_I \land \mu_K \in \left( g^{I}_{\delta} \right)^{0,h} \right\}.$$

Then, we have

$$Z^{0,h}_\delta(g_I^C) = \bigoplus_{l=0}^h Z^{0,h}_\delta(A_I(\lambda)), \quad B^{0,h}_\delta(g_I^C) = \bigoplus_{l=0}^h B^{0,h}_\delta(A_I(\lambda))$$

by Lemma 3.10. Thus, we obtain the following theorem.

**Theorem 3.11.** Assume that $b$ is abelian. Then

$$H^{k,0}_\delta(g_I^C) \cong \left( \bigoplus_l H^{k,0}_\delta(A_I(\lambda)) \right) \otimes \left( \bigoplus_m H^{0,h}_\delta(A_m(\lambda)) \right).$$

**Proof.** Since $H^{k,0}_\delta(g_I^C) = H^{k,0}_\delta(g_I^C) \otimes H^{0,h}_\delta(g_I^C) = A^{0}(g_I^C)^* \otimes H^{0,h}_\delta(g_I^C)$, we obtain our claim by Lemma 3.10. $\square$

We define that $\dim H^{k,0}_\delta(A_I(\lambda)) = 0$, $\dim H^{0,h}_\delta(A_m(\lambda)) = 0$ for $l > k$, $m > h$, respectively. Then, we have the following.
Theorem 3.12. Assume that $b$ is abelian. Then
\[ h^{s,l}(g_j) = \sum_{l+h-m=s} \dim H^{k,0}_\partial(A_i(\lambda)) \cdot \dim H^0_{\partial}(A_m(\lambda)). \]

**Proof.** Since
\[ H^{k,0}_\partial(A_i(\lambda)) \to H^{k-1}_\partial(g_j^C), \quad H^0_{\partial}(A_m(\lambda)) \to H^{h-m,m}_\partial(g_j^C) \]
by the cohomology map induced by $F$, we have
\[ H^{k,0}_\partial(A_i(\lambda)) \otimes H^0_{\partial}(A_m(\lambda)) \xrightarrow{F} H^{k+h-m,k-l+m}_\partial(g_j^C). \]
Thus, we obtain our claim by Theorem 3.11. \qed

**Corollary 3.13.** Assume that $b$ is abelian. Then
\[ h^{s,0}(g_j) = \sum_{k=0}^s \dim H^{k,0}_\partial(A_k(\lambda)) \cdot \dim H^0_{\partial}(A_0(\lambda)) \]
\[ = \sum_{k=0}^s \binom{p}{k} \cdot \dim H^0_{\partial}(A_0(\lambda)), \]
where $p = \dim a$.

**Proof.** If $t = 0$, then the conditions $k - l + m = t$ and $k + h = s + t$ become $l = k + m$ and $k + h = s$, respectively. Thus, $\dim H^{k,0}_\partial(A_t(\lambda)) = \dim H^{k,0}_\partial(A_k(A_m(\lambda))) = 0$ except the case of $m = 0$, and $\dim H^0_{\partial}(A_m(\lambda)) = \dim H^0_{\partial}(A_0(\lambda))$. \qed

**Corollary 3.14.** Assume that $b$ is abelian. Then
\[ h^{1,1}(g_j) = \sum_{k=0}^1 \dim H^{k,0}_\partial(A_k(\lambda)) \cdot \dim H^0_{\partial}(A_1(\lambda)) + \dim H^0_{\partial}(A_1(\lambda)) + \dim H^0_{\partial}(A_2(\lambda)) \]
\[ = h^{1,1}(g_j) + p(2p - 2 \dim[a, a] - h^{0,1}(g_j) - \dim g) + \dim g \cdot \dim[a, a] \]
\[ + \dim H^0_{\partial}(A_1(\lambda)) + \dim H^0_{\partial}(A_1(\lambda)), \]
where $p = \dim a$.

**Proof.** Note that $g_j^C \cong g_j^+ \cong g_j$ as complex Lie algebras, and $H^{0,1}_\partial(g_j^C) = H^{0,1}_\partial(A_0(\lambda)) \oplus H^{0,1}_\partial(A_1(\lambda))$. Since $H^{k,0}_\partial(g_j^C) = H^{k,0}_\partial(g_j) \otimes H^0_{\partial}(g_j^C) = \Lambda^k(g_j^+) \otimes H^0(g_j)$, we obtain
\[ h^{1,0}(g_j) = \dim g_j^+ = \dim g, \]
\[ h^{1,1}(g_j) = \dim g \cdot h^{0,1}(g_j), \]
\[ h^{0,1}(g_j) = \dim g/[g, g]. \]
Then, we have
\[ \dim h^{1,0}_\partial(A_0(\lambda)) = \dim g - \dim a, \]
\[ \dim h^{0,1}_\partial(A_0(\lambda)) = h^{0,1}(g_j) - \dim h^0_{\partial}(A_1(\lambda)), \]
\[ \dim h^{0,1}_\partial(A_1(\lambda)) = \dim a - \dim[a, a]. \]
Hence, we obtain our claim by a straightforward computation. \qed

**Remark 3.15.** (i) The decomposition in Theorem 3.11 is not true in general. Indeed, let $g$ be the 5-dimensional Heisenberg algebra, i.e.,
\[ g = \text{span}\{X_1, X_2, X_3, X_4, X_5\} \]
satisfying \([X_1, X_2] = [X_2, X_4] = X_5\) with other brackets vanishing. Let \(\{Z_1, Z_2, Z_3, Z_4, Z_5\}\) be a basis of the space of the left invariant holomorphic vector fields on \((\mathfrak{g}(\mathbb{C}), J)\) which satisfies \([Z_1, Z_2] = [Z_2, Z_4] = Z_5\), and \(\{\omega_i\}_{i=1,\ldots,5}\) its dual basis. Then, we have the following equations on anti-holomorphic left-invariant 1-forms on \((\mathfrak{g}(\mathbb{C}), J)\):

\[
\begin{align*}
\bar{\partial}\bar{\omega}_1 &= \bar{\partial}\bar{\omega}_2 = \bar{\partial}\bar{\omega}_3 = \bar{\partial}\bar{\omega}_4 = 0, \\
\bar{\partial}\bar{\omega}_5 &= -\bar{\omega}_1 \wedge \bar{\omega}_3 - \bar{\omega}_2 \wedge \bar{\omega}_4.
\end{align*}
\]

Thus,

\[
[\bar{\omega}_1 \wedge \bar{\omega}_3] = -[\bar{\omega}_2 \wedge \bar{\omega}_4] \in H^{0,2}_0(\mathfrak{g}(\mathbb{C})).
\]

Let \(a = \text{span}\{X_1\}\), \(b = \text{span}\{X_2, X_3, X_4, X_5\}\). Then, \(b\) is a non-abelian ideal, and

(a) \([\bar{\omega}_1 \wedge \bar{\omega}_3] \in H^{0,2}_0(A_1(\Lambda)), [-\bar{\omega}_2 \wedge \bar{\omega}_4] \in H^{0,2}_0(A_0(\Lambda)),\)

(b) \(\bar{\omega}_1 \wedge \bar{\omega}_3 \wedge \bar{\omega}_5 - \bar{\omega}_2 \wedge \bar{\omega}_4 \wedge \bar{\omega}_5\) is \(\bar{\partial}\)-closed, however, \(\bar{\omega}_1 \wedge \bar{\omega}_3 \wedge \bar{\omega}_5\) and \(\bar{\omega}_2 \wedge \bar{\omega}_4 \wedge \bar{\omega}_5\) are not \(\bar{\partial}\)-closed.

Moreover, \([\bar{\omega}_1 \wedge \bar{\omega}_3 \wedge \bar{\omega}_5 - \bar{\omega}_2 \wedge \bar{\omega}_4 \wedge \bar{\omega}_5] \notin H^{0,3}_0(A_m(\Lambda))\) for each \(m\).

Hence, we see

\[
H^{0,2}_0(A_1(\Lambda)) \cap H^{0,2}_0(A_0(\Lambda)) = \emptyset, \quad H^{0,3}_0(\mathfrak{g}(\mathbb{C})) = \bigoplus_m H^{0,3}_0(A_m(\Lambda)).
\]

See Section 5 in [9] for details of \(h^{0,\ell}(\mathfrak{g})\) when \(\mathfrak{g}\) is a Heisenberg algebra.

(ii) By Theorem 3.12, we can give a proof of Theorem 3.2 for the case where \(b\) is abelian. Indeed,

\[
h^{0,\ell}(\mathfrak{g}) = \sum_{h=0}^{t} \dim H^{0,0}_h(A_0(\Lambda)) \cdot \dim H^{0,h}_0(A_h(\Lambda))
\]

\[
= \sum_{h=0}^{t} \dim \bigwedge^{t-h}(b^*)^\ast \cdot \dim H^{0,h}_0(a^*)
\]

\[
= \sum_{h=0}^{t} \left( \frac{q}{t-h} \right) \cdot \dim H^h(b^*) \cdot \dim H^h(a^*).
\]

(iii) If \(\Gamma \backslash N\) has a holomorphic symplectic structure, then \(h^{1,1}(\Gamma \backslash N)\) has a relation to the dimension of the deformation space of complex structures. Because a holomorphic symplectic structure \(\Omega\) on a compact complex manifold \((M, J)\) induces an isomorphism of sheaves \(\Theta \cong \Omega^\ast\), where \(\Theta\) is the sheaf of germs of holomorphic vector fields and \(\Omega^\ast\) is the sheaf of germs of holomorphic 1-forms.

## 4 Examples

**Example 4.1.** Let \(N\) be the 5-dimensional Heisenberg group:

\[
N = H_\mathbb{C}(2) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\
0 & 1 & 0 & x_4 \\
0 & 0 & 1 & x_5 \\
0 & 0 & 0 & 1 \end{pmatrix} \middle| x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}.
\]

Let \(\{X_1, X_2, X_3, X_4, X_5\}\) be a basis of left-invariant vector fields on \(N\) which satisfies

\([X_1, X_3] = [X_2, X_4] = X_5.\)

Put

\[
a = \text{span}\{X_1, X_2\}, \quad b = \text{span}\{X_3, X_4, X_5\}.
\]
Then, we have the following nilpotent Lie group with rational complex structures:

\[
N_1 = (g(N^C), J) = \begin{pmatrix}
1 & z_1 & z_2 & z_5 \\
0 & 1 & 0 & z_3 \\
0 & 0 & 1 & z_4 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
z_1, z_2, z_3, z_4, z_5 \in \mathbb{C}
\end{pmatrix},
\]

\[
N_2 = (g(N^C), J) = \begin{pmatrix}
1 & \bar{z}_1 & \bar{z}_2 & z_5 \\
0 & 1 & 0 & z_3 \\
0 & 0 & 1 & z_4 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
z_1, z_2, z_3, z_4, z_5 \in \mathbb{C}
\end{pmatrix}.
\]

Since \([a, a] = [b, b] = 0\), we have \(h^{1,0}(\Gamma \setminus N_1) = h^{0,1}(\Gamma \setminus N_2) = 5\) by Corollary 3.5, where \(\Gamma\) is a lattice in \(N_1 \cong N_2\) such that \(\bar{J}\) is \(\Gamma\)-rational (By the arguments before Theorem 2.2 and Proposition 3.1, there exists such a lattice). Hence, \(h^{s,0}(\Gamma \setminus N_1) = h^{0,s}(\Gamma \setminus N_2)\) for each \(s\) by Corollary 3.6.

**Example 4.2.** Let us consider the nilpotent Lie group defined by

\[
N = \begin{pmatrix}
1 & x_1 & x_3 & x_6 \\
0 & 1 & x_2 & x_5 \\
0 & 0 & 1 & x_4 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}
\end{pmatrix}.
\]

Let \(\{X_1, X_2, X_3, X_4, X_5, X_6\}\) be a basis of left-invariant vector fields on \(N\) which satisfies

\([X_1, X_2] = X_3, [X_2, X_4] = X_5, [X_1, X_5] = [X_3, X_4] = X_6\).

Put

\[a = \text{span}\{X_1, X_2, X_3\}, \quad b = \text{span}\{X_4, X_5, X_6\}.$

Then, we have the following nilpotent Lie group with rational complex structures:

\[
N_1 = (g(N^C), J) = \begin{pmatrix}
1 & z_1 & z_3 & z_6 \\
0 & 1 & z_2 & z_5 \\
0 & 0 & 1 & z_4 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
z_1, z_2, z_3, z_4, z_5, z_6 \in \mathbb{C}
\end{pmatrix},
\]

\[
N_2 = (g(N^C), J) = \begin{pmatrix}
1 & \bar{z}_1 & \bar{z}_3 & \bar{z}_6 \\
0 & 1 & \bar{z}_2 & \bar{z}_5 \\
0 & 0 & 1 & \bar{z}_4 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
z_1, z_2, z_3, z_4, z_5, z_6 \in \mathbb{C}
\end{pmatrix}.
\]

Then,

\[
\lambda_1 = dz_1, \quad \lambda_2 = dz_2, \quad \lambda_3 = dz_3 - z_1 dz_2, \\
\mu_1 = dz_4, \quad \mu_2 = dz_5 - z_2 dz_4, \quad \mu_3 = dz_6 - z_3 dz_5 - z_1 dz_2.
\]

Thus,

\[
h^{1,0}(\Gamma \setminus N_1) = 6, \quad h^{0,1}(\Gamma \setminus N_1) = 3, \\
h^{1,0}(\Gamma \setminus N_2) = 4, \quad h^{0,1}(\Gamma \setminus N_2) = 5,
\]

where \(\Gamma\) is a lattice in \(N_1 \cong N_2\) such that \(\bar{J}\) is \(\Gamma\)-rational. Thus,

\[
h^{1,0}(\Gamma \setminus N_1) - h^{0,1}(\Gamma \setminus N_2) = 1 = \dim[a, a], \\
h^{1,0}(\Gamma \setminus N_2) - 2 = \dim g - \dim[a, b].
\]
Let $h^{k,0}(A_1(\lambda)) = \dim H^{k,0}_\bar{\partial}(A_1(\lambda))$ and $h^{0,k}(A_m(\lambda)) = \dim H^{0,k}_\bar{\partial}(A_m(\lambda))$. By a straightforward computation, we have

$$h^{0,0}(\tilde{A}_0(\lambda)) = 1,$$
$$h^{0,1}(\tilde{A}_0(\lambda)) = 1, h^{0,1}(A_1(\lambda)) = 2,$$
$$h^{0,2}(\tilde{A}_0(\lambda)) = 1, h^{0,2}(A_2(\lambda)) = 2,$$
$$h^{0,3}(\tilde{A}_0(\lambda)) = 1, h^{0,3}(A_3(\lambda)) = 2, h^{0,3}(A_2(\lambda)) = 2, h^{0,3}(A_3(\lambda)) = 1,$$
$$h^{0,4}(A_2(\lambda)) = 2, h^{0,4}(A_2(\lambda)) = 2, h^{0,4}(A_3(\lambda)) = 1,$$
$$h^{0,5}(A_2(\lambda)) = 2, h^{0,5}(A_3(\lambda)) = 1,$$
$$h^{0,6}(A_3(\lambda)) = 1,$$
$$h^{0,0}(\tilde{A}_0(\lambda)) = 1,$$
$$h^{1,0}(\tilde{A}_0(\lambda)) = 3, h^{1,0}(A_1(\lambda)) = 3,$$
$$h^{2,0}(\tilde{A}_0(\lambda)) = 3, h^{2,0}(A_1(\lambda)) = 9, h^{2,0}(A_2(\lambda)) = 3,$$
$$h^{3,0}(\tilde{A}_0(\lambda)) = 1, h^{3,0}(A_1(\lambda)) = 9, h^{3,0}(A_2(\lambda)) = 9, h^{3,0}(A_3(\lambda)) = 1,$$
$$h^{4,0}(A_1(\lambda)) = 3, h^{4,0}(A_2(\lambda)) = 9, h^{4,0}(A_3(\lambda)) = 3,$$
$$h^{5,0}(A_2(\lambda)) = 3, h^{5,0}(A_3(\lambda)) = 3,$$
$$h^{6,0}(A_3(\lambda)) = 1.$$

In particular, we see

$$h^{0,h}(\tilde{A}_m(\lambda)) = h^{0,6-h}(A_{p-m}(\lambda)), h^{k,0}(A_1(\lambda)) = h^{6-k,0}(A_{p-1}(\lambda))$$

for each $k, h, l$ and $m$. By Theorem 3.12, we have

$$h^{2,0}(\Gamma \backslash N_1) = 15, \quad h^{1,1}(\Gamma \backslash N_1) = 18, \quad h^{0,2}(\Gamma \backslash N_1) = 5,$$
$$h^{2,0}(\Gamma \backslash N_2) = 7, \quad h^{1,1}(\Gamma \backslash N_2) = 20, \quad h^{0,2}(\Gamma \backslash N_2) = 11.$$

Moreover, by Proposition 3.8, we have

$$\sum_{s+t=r} h^{s,t}(\Gamma \backslash N_1) = \sum_{s+t=r} h^{s,t}(\Gamma \backslash N_2)$$

for each $r$.

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