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INTEGRAL OF FUNCTION WITH VALUES IN COMPLETE MODULAR SPACE

Abstract. The theorem on existence of an integral of a function with values in a modular space and some fundamental properties of this integral are given.

Let $X$ be a real vector space. A functional $\rho : X \to \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+ = [0, +\infty]$, is called a convex pseudomodular on $X$ if $\rho(0) = 0$, $\rho(-u) = \rho(u)$ and $\rho(\alpha u + \beta v) \leq \alpha \rho(u) + \beta \rho(v)$ for all $u, v \in X$ and $\alpha, \beta \geq 0, \alpha + \beta = 1$. If, additionally, $\rho(u) = 0$ only for $u = 0$, then $\rho$ is called a convex modular on $X$. The vector space $X_\rho = \{u \in X : \rho(au) < \infty$ for some $a > 0\}$ is called the modular space generated by $\rho$. Examples of modular spaces, e.g. Orlicz spaces, may be found in [3].

A sequence $(u_k)$ of elements of $X_\rho$ is called modular convergent to $u$, $u \in X_\rho$, if there exists a $\lambda > 0$ such that $\rho(\lambda(u_k - u)) \to 0$ as $k \to \infty$. A sequence $(u_k)$ is called a Cauchy sequence in $X_\rho$, if $\rho(\lambda(u_k - u_l)) \to 0$ as $k, l \to \infty$, for some $\lambda > 0$.

The modular space $X_\rho$ is called complete if every Cauchy sequence in $X_\rho$ is convergent in $X_\rho$. In the following by $X_\rho$ we shall mean a complete modular space.

We assume henceforth that $\Omega$ is a non-empty abstract set and let $(\Omega, \Sigma, \mu)$ be a finite measure space with a complete and positive measure on $\Sigma$.

Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of mutually disjoint, $\Sigma$ - measurable subsets of $\Omega$ and let $\{c_1, c_2, \ldots, c_k\}$ be a corresponding collection of points of $X_\rho$. The mapping $f$ on $\Omega$ into $X_\rho$ defined by

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where $\chi_B$ is the characteristic function of $B$, is called a simple function.

An arbitrary function $f$ defined almost everywhere on $\Omega$ into $X_\rho$, is said to be a $\rho - \Sigma$-measurable function (briefly $\rho$-measurable) if there exists a sequence $(f_n)$ of simple functions such that

(1) $\rho(\lambda_1(f_n(x) - f(x))) \to 0$ as $n \to \infty$ for some $\lambda_1 > 0$

and for almost all $x \in \Omega$ and

(2) $\rho(\lambda_2 f_n(x)) = \rho(\lambda_2 f(x))$ as $n \to \infty$ for some $\lambda_2 > 0$

and for almost all $x \in \Omega$.

Let us remark that if $\lambda_2 \geq \lambda_1$, then the constants $\lambda_1$ and $\lambda_2$ may be taken identical.

Lemma 1. Let $\rho$ be a convex pseudomodular in a real vector space $X$. If $u, v \in X_\rho$ and $\rho(v) < \infty$, then for arbitrary $\alpha$ such that $0 < \alpha \leq 1$ the inequality

$$\rho(u) - \rho(v) \leq \frac{1}{2} \alpha \rho \left( \frac{2}{\alpha} (u - v) \right) + \frac{1}{2} \alpha \rho(2v)$$

holds.

Proof. Let $0 < \alpha \leq 1$ and $\beta = 1 - \alpha$. By convexity of $\rho$ we have

$$\rho(u) \leq \alpha \rho \left( \frac{u}{\alpha} - \frac{\beta}{\alpha} v \right) + \beta \rho(v).$$

Hence

$$\rho(u) - \rho(v) \leq \alpha \rho \left( \frac{1}{\alpha} (u - v) + v \right) \leq \frac{1}{2} \alpha \rho \left( \frac{2}{\alpha} (u - v) \right) + \frac{1}{2} \alpha \rho(2v).$$

The following statement shows that if the condition (1) holds for every $\lambda > 0$, then (2) follows from (1).

Proposition. If $\rho(cu) < \infty$ for some $c > 0$ and $\rho(a(u_n - u)) \to 0$ as $n \to \infty$ for every $a > 0$, then there exists a constant $a_1 > 0$ such that $\rho(a_1 u_n) \to \rho(a_1 u)$ as $n \to \infty$.

Proof. Let $\epsilon > 0$ be arbitrary and let $\rho \left( \frac{1}{4} cu_n \right) \geq \rho \left( \frac{1}{4} cu \right)$. Let $\alpha \in (0,1]$ be so small that $\frac{1}{4} \alpha \rho(cu) < \frac{1}{2} \epsilon$. For given $\alpha$

$$\frac{1}{2} \alpha \rho \left( \frac{c}{2\alpha}(u_n - u) \right) < \frac{1}{2} \epsilon.$$
holds for sufficiently large \( n \). Thus, by Lemma 1, we have
\[
0 \leq \rho \left( \frac{1}{4} cu_n \right) - \rho \left( \frac{1}{4} cu \right) \leq \frac{1}{2} \alpha \rho \left( \frac{c}{2\alpha} (u_n - u) \right) + \frac{1}{4} \alpha \rho (cu) < \epsilon
\]
for sufficiently large \( n \) and \( \alpha \in (0, 1] \).

Let now \( \rho \left( \frac{1}{4} cu_n \right) < \rho \left( \frac{1}{4} cu \right) \). Then
\[
0 < \rho \left( \frac{1}{4} cu \right) - \rho \left( \frac{1}{4} cu_n \right) \leq \frac{1}{2} \alpha \rho \left( \frac{c}{2\alpha} (u - u_n) \right) + \frac{1}{2} \alpha \rho (cu_n).
\]

For the sequence \( \rho \left( \frac{1}{2} cu_n \right) \) there exists a constant \( M > 0 \) such that
\[
\rho \left( \frac{1}{2} cu_n \right) \leq \rho (c(u_n - u)) + \rho (cu) \leq M
\]
for every \( n \). Hence, we obtain \( 0 < \rho \left( \frac{1}{4} cu \right) - \rho \left( \frac{1}{2} cu_n \right) < \epsilon \) for sufficiently small \( \alpha \in (0, 1] \) and sufficiently large \( n \). Finally, we have \( |\rho \left( \frac{1}{4} cu_n \right) - \rho \left( \frac{1}{4} cu \right)| < \epsilon \) for sufficiently large \( n \). \( \blacksquare \)

Let \( f : \Omega \rightarrow X_\rho \) be a simple function. The \( \rho - \text{Bochner integral of } f \) is defined by
\[
\int_{\Omega} f(x) d\mu = \sum_{i=1}^{k} c_i \mu (B_i).
\]

Immediately from above definition it follows

**Lemma 2.** (i) Let \( f : \Omega \rightarrow X_\rho \) be a simple function. Then
\[
\rho \left( \int_{\Omega} f(x) d\mu \right) \leq \frac{1}{c} \int_{\Omega} \rho (cf(x)) d\mu,
\]
where \( c = \mu (\Omega) \).

(ii) Let \( f, g \) be two simple functions, \( \alpha, \beta \in R \). Then \( \alpha f + \beta g \) is also a simple function and
\[
\int_{\Omega} (\alpha f(x) + \beta g(x)) d\mu = \alpha \int_{\Omega} f(x) d\mu + \beta \int_{\Omega} g(x) d\mu.
\]

A function \( f : \Omega \rightarrow X_\rho \) is said to be \( \rho - \text{Bochner integrable} \) if there exists a sequence \( (f_n) \) of simple functions satisfying (1) and (2) such that

\[
\lim_{n \rightarrow \infty} \int_{\Omega} \rho (\lambda_3 (f_n(x) - f(x))) d\mu = 0 \quad \text{for some} \quad \lambda_3 > 0,
\]

\[
\lim_{n,m \rightarrow \infty} \int_{\Omega} |\rho (\lambda_2 f_n(x)) - \rho (\lambda_2 f_m(x))| d\mu = 0,
\]

where the constant \( \lambda_2 \) is the same as in (2).
For any set $B \in \Sigma$ and for any $\rho$-measurable and integrable $f$ we define the \textit{$\rho$ – Bochner integral of $f$ over $B$} by

$$\int_B f(x)d\mu = \lim_{n \to \infty} \int \chi_B(x)f_n(x)d\mu.$$ 

The above limit exists and its value is independent of the approximating sequence of simple functions $(f_n)$.

Let us denote

$$s_n = \int \chi_B(x)f_n(x)d\mu.$$ 

The existence of the limit follows from the inequality

$$\rho \left( \frac{\lambda_3}{2c}(s_n - s_m) \right) \leq \frac{1}{2c} \int_B \rho(\lambda_3(f_n(x) - f(x)))d\mu$$

$$+ \frac{1}{2c} \int_B \rho(\lambda_3(f_m(x) - f(x)))d\mu$$

where $c = \mu(\Omega)$, and from the completeness of $X_\rho$. The constant $\lambda_3$ is chosen as in (3).

\textbf{Theorem 1.} \textit{Let $\rho$ be a convex modular on $X$. A $\rho$-measurable function $f : \Omega \to X_\rho$ is $\rho$-Bochner integrable if and only if the function $\rho(cf(x))$ is $\mu$-integrable with the constant $c$ from (2).}

\textbf{Proof.} Since $f$ is $\rho$-measurable there exists a sequence $(f_n)$ of simple functions satisfying (1) and (2). Define functions $y_n, n = 1, 2, \ldots,$ on $\Omega$ by

$$y_n(x) = \begin{cases} f_n(x) & \text{if } x \in A_n \\ 0 & \text{otherwise} \end{cases}$$

where $A_n = \{x \in \Omega: \rho(\lambda_2 f_n(x)) \leq 2\rho(\lambda_2 f(x))\}$. Obviously every $y_n$ is simple. We put $B = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A'_k$, where $A'_k = \Omega - A_k$. Then we have $\mu(B) = 0$, (see [2]). Hence $\rho(\lambda_1(y_n(x) - f(x))) \to 0$ as $n \to \infty$ almost everywhere on $\Omega$ and $\rho(\lambda_2 y_n(x)) \to \rho(\lambda_2 f(x))$ as $n \to \infty$ almost everywhere on $\Omega$.

Moreover, if $\lambda_1 \leq \lambda_2$ then

$$\rho \left( \frac{1}{2} \lambda_1(y_n(x) - f(x)) \right) \leq \rho(\lambda_2 y_n(x)) + \rho(\lambda_2 f(x)) \leq 3\rho(\lambda_2 f(x))$$

for almost all $x \in \Omega$. The dominated convergence theorem yields

$$\lim_{n \to \infty} \int \rho \left( \frac{1}{2} \lambda_1(y_n(x) - f(x)) \right) d\mu = 0.$$
Integral of function

In the case $\lambda_1 > \lambda_2$ the constants in (1) and (2) may be taken identical putting the smaller of them. Then

$$\rho \left( \frac{1}{2} \lambda_2 (y_n(x) - f(x)) \right) \leq 3 \rho (\lambda_2 f(x))$$

and

$$\lim_{n \to \infty} \int_{\Omega} \rho \left( \frac{1}{2} \lambda_2 y_n(x) - f(x) \right) d\mu = 0.$$ 

Arguing in a similar manner we may easily show that

$$\lim_{m,n \to \infty} \int_{\Omega} |\rho(\lambda_2 y_n(x)) - \rho(\lambda_2 y_m(x))| d\mu = 0.$$ 

Thus $f$ is $\rho$-Bochner integrable.

Let the function $f$ be $\rho$-Bochner integrable and let $(f_n)$ be a sequence of simple functions with properties (1) - (4). Let us denote $z_n(x) = \rho(\lambda_2 f_n(x))$. The functions $z_n$ are simple and the sequence $(z_n)$ is convergent to $\rho(\lambda_2 f(x))$ for almost all $x \in \Omega$. Applying (4) we obtain for any fixed $\epsilon > 0$

$$\left| \int_{\Omega} z_n(x) d\mu - \int_{\Omega} z_m(x) d\mu \right| < \epsilon$$

for sufficiently large $m$, $n$. Hence $\rho(\lambda_2 f(x))$ is $\mu$-integrable over $\Omega$.

Let us consider the modular space $X_\rho$ with Luxemburg norm $|| \cdot ||_\rho$ generated by the convex modular $\rho$. We shall investigate the connection between Bochner integral of $f$ and $\rho$-Bochner integral of $f$. We shall show that if the function $f : \Omega \to X_\rho$ is strongly measurable and Bochner integrable, then $f$ is also $\rho$-measurable and $\rho$-Bochner integrable on $\Omega$.

Let us suppose that the function $f$ is Bochner integrable. Then there exists a sequence $(f_n)$ of simple functions, $f_n : \Omega \to X_\rho$, such that

(5) $$\lim_{n \to \infty} ||f_n(x) - f(x)||_\rho = 0 \quad \text{for almost all} \quad x \in \Omega,$$

and

(6) $$\lim_{n \to \infty} \int_{\Omega} ||f_n(x) - f(x)||_\rho d\mu = 0.$$

In virtue of (5),

(7) $$\rho(\lambda(f_n(x) - f(x))) \to 0 \quad \text{almost everywhere in} \ \Omega \ \text{for every} \ \lambda > 0.$$ 

Let $0 < \epsilon \leq 1$. By (5) we have $||\lambda(f_n(x) - f(x))||_\rho < \epsilon \leq 1$ almost everywhere in $\Omega$ for every $\lambda > 0$ and $n > N(\lambda, x)$. Hence

$$\rho(\lambda(f_n(x) - f(x))) \leq ||\lambda(f_n(x) - f(x))||_\rho$$
almost everywhere in $\Omega$ for sufficiently large $n$. Integrating above inequality over $\Omega$ and applying (6) we obtain

$$\lim_{n \to \infty} \int_{\Omega} \rho(\lambda(f_n - f(x))) \, d\mu = 0 \quad \text{for every} \quad \lambda > 0.$$ 

In order to prove (2) let us first show that for every strongly measurable function $f$ there exists a constant $d > 0$ such that $\rho(df(x)) < \infty$ for almost all $x \in \Omega$. Let $g : \Omega \to X_\rho$ be a simple function, $g(x) = \sum_{i=1}^{k} c_i \chi_{B_i}(x)$, where $\{c_1, c_2, \ldots, c_k\} \subset X_\rho$. For every $i$, $i = 1, 2, \ldots, k$, there exists $\lambda_i > 0$ such that $\rho(\lambda_i c_i) < \infty$. Let $a = \min \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Then

$$\rho(ac_i) < \infty \quad \text{for} \quad i = 1, 2, \ldots, k.$$ 

Hence $\int_{\Omega} \rho(\lambda g(x)) \, d\mu = \sum_{i=1}^{k} \rho(\lambda_i c_i) \mu(B_i) < \infty$. We conclude: there exists a constant $a > 0$, independent on $x$, such that the real function $\rho(\lambda g(x))$ is $\mu$-integrable on $\Omega$.

For arbitrary $n_0 \in N$, by (9), we have that

$$\rho(cf_{n_0}(x)) < \infty \quad \text{for some} \quad c > 0 \quad \text{and every} \quad x \in \Omega.$$ 

Thus, by (7) and (10), we obtain

$$\rho \left(\frac{1}{2} cf(x)\right) \leq \frac{1}{2} \rho(\lambda f(x) - f_{n_0}(x)) + \frac{1}{2} \rho(cf_{n_0}(x)) < \infty$$

for almost all $x \in \Omega$. Therefore, putting $d = \frac{1}{2} c$, we have

$$\rho(df(x)) < \infty \quad \text{for almost all} \quad x \in \Omega.$$ 

In virtue of Proposition, by (7) and (12), we obtain

$$\rho \left(\frac{1}{8} cf_n(x)\right) \to \rho \left(\frac{1}{8} cf(x)\right) \quad \text{as} \quad n \to \infty, \quad \text{almost everywhere in} \quad \Omega.$$ 

Thus $f$ is $\rho$-measurable.

Replacing the constant $\frac{1}{2} c$ by $\frac{1}{8} c$ in (11) and integrating this inequality over $\Omega$, we have by (8)

$$\int_{\Omega} \rho \left(\frac{1}{8} cf(x)\right) \, d\mu \leq \int_{\Omega} \rho \left(\frac{1}{4} c f(x) - f_{n_0}(x)\right) \, d\mu + \int_{\Omega} \rho \left(\frac{1}{4} cf_{n_0}(x)\right) \, d\mu < \infty.$$ 

Thus the real function $\rho \left(\frac{1}{8} cf(x)\right)$ is $\mu$-integrable on $\Omega$. Hence, by Theorem 1, $f$ is $\rho$-Bochner integrable on $\Omega$.

Now, let us take into account the following example. Let us consider the Lebesgue space $L^p(0, 1)$, $p \geq 1$ as the Orlicz space $L^\phi$ where $\phi(s) = \frac{1}{p} s^p$, $p \geq 1$. Then obviously $L^\phi(0, 1)$, $p \geq 1$ is a modular space generated by the
modular \( \rho(u) = \frac{1}{p} \int_0^1 |u(t)|^p dt \). Let \( \| \cdot \|_\rho \) be a Luxemburg norm generated by \( \rho \). Then we can express the norm \( \| \cdot \|_\rho \) in the following form \( \|u\|_\rho = \left( \frac{1}{p} \right)^{\frac{1}{p}} \|u\|_p \), where \( \|u\|_p = \left( \int_0^1 |u(t)|^p dt \right)^{\frac{1}{p}} \). If \( f : \Omega \to L^p(0,1) \) is \( \rho \)-integrable, then \( \rho(f_n(x) - f(x)) \to 0 \) and \( \int_\Omega \rho(f_n(x) - f(x))d\mu \to 0 \) as \( n \to \infty \) almost everywhere in \( \Omega \) for some sequence \( (f_n) \) of simple functions. The first condition is equivalent to \( \|f_n(x) - f(x)\|_\rho \to 0 \) as \( n \to \infty \) a.e. in \( \Omega \) and for the second of them we have

\[
\int_\Omega \rho(f_n(x) - f(x))d\mu = \int_\Omega \|f_n(x) - f(x)\|_\rho^p d\mu.
\]

This equality implies that

\[
\int_\Omega \|f_n(x) - f(x)\|_\rho d\mu \to 0 \quad \text{as} \quad n \to \infty \quad \text{a.e. in} \quad \Omega.
\]

Thus \( f \) is a strongly measurable and Bochner integrable.

Further elementary facts about the \( \rho \)-Bochner integral are collected below.

**Theorem 2.** If \( f \) is a \( \rho \)-Bochner integrable function, then

(i) \( \lim_{\mu(A) \to 0} \int_A f(x)d\mu = 0 \),

(ii) if \( (A_i) \) is a sequence of pairwise disjoint members of \( \Sigma \) and \( A = \bigcup_{i=1}^\infty A_i \), then

\[
\int_A f(x)d\mu = \sum_{i=1}^\infty \int_{A_i} f(x)d\mu.
\]

**Proof.** (i) Let \( (f_n) \) be a sequence of simple functions. Let us denote

\[
\nu(A) = \int_A f(x)d\mu \quad \text{and} \quad \nu_n(A) = \int_A f_n(x)d\mu \quad \text{for any} \quad A \in \Sigma.
\]

There exists the constant \( a_1 > 0 \) such that for every \( \epsilon > 0 \)

\[
\int_\Omega \left( \frac{1}{2} a_1 (f_n(x) - f(x)) \right) d\mu < \epsilon \quad \text{for} \quad n > n_0.
\]

We can find the constant \( a_2 > 0 \) such that the real functions \( \rho(2a_2 f_n(x)) \), \( n = 1, 2, \ldots, n_0 \) are \( \mu \)-integrable. Thus, we can choose \( \delta > 0 \), that

\[
\int_A \rho(2a_2 f_n(x))d\mu < \epsilon \quad \text{for every} \quad A \in \Sigma \quad \text{with} \quad \mu(A) < \delta.
\]
and $n = 1, 2, \ldots, n_0$. Taking $a = \min\{a_1, a_2\}$, we have that for $n > n_0$

$$
\int_A \left( \frac{1}{4} a f_n(x) \right) \, d\mu \leq \frac{1}{2} \int_A \rho \left( \frac{1}{2} a (f_n(x) - f_{n_0}(x)) \right) \, d\mu
$$

$$
+ \frac{1}{2} \int_A \rho \left( \frac{1}{2} a f_{n_0}(x) \right) \, d\mu < \epsilon
$$

provided $\mu(A) < \delta$. Combining (14) and (15) we obtain that for every $n$

$$
\int \rho \left( \frac{1}{4} a f_n(x) \right) \, d\mu < \epsilon \quad \text{provided} \quad \mu(A) < \delta.
$$

Since, for $c = \frac{a}{4\mu(\Omega)}$, the following inequality

$$
\rho(c \nu_n(A)) \leq \frac{1}{\mu(\Omega)} \int_A \rho \left( \frac{1}{4} a f_n(x) \right) \, d\mu
$$

holds. Then, in virtue of (16), the functions $\nu_n$ are uniformly absolutely continuous.

From the definition of $\rho$-Bochner integral and the inequality

$$
\rho(\nu(A)) \leq \frac{1}{2} \rho \left( 2 \left( \int_A f(x) \, d\mu - \int_A f_n(x) \, d\mu \right) \right) + \frac{1}{2} \rho(2\nu_n(A))
$$

it follows that there exists a constant $b > 0$ such that $\rho(b \nu(A)) < \epsilon$ provided $\mu(A) < \delta$.

(ii) Let us show first that for any fixed $n$, we have

$$
\nu_n \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \nu_n(A_i).
$$

For arbitrary set $B \in \Sigma$ and $\epsilon > 0$ by Lemma 2 and (3) we have

$$
\rho \left( \frac{\lambda_3}{\mu(\Omega)} (\nu_n(B) - \nu_m(B)) \right) \leq \frac{1}{\mu(\Omega)} \int_B \rho(3(f_n(x) - f_m(x))) \, d\mu < \epsilon
$$

for $n, m > n_0$. Hence, by completeness of the space $X_\rho$, the sequence $(\nu_n(B))$ is uniformly convergent to $\nu(B)$ with respect to $B \in \Sigma$. Thus, there exists a constant $a > 0$ such that

$$
\rho(a(\nu(A) - \nu_n(A))) < \epsilon \quad \text{and} \quad \rho \left( a \left( \nu_n \left( \bigcup_{i=1}^{k} A_i \right) - \nu \left( \bigcup_{i=1}^{k} A_i \right) \right) \right) < \epsilon
$$

for sufficiently large $n$. 
Arguing in a similar manner as in the first part of the proof of the thesis (i), we obtain that there exists a constant $c > 0$ such that

$$\int_A \rho(cf_n(x))d\mu < \infty$$

for every $n$. Therefore the series $\sum_{i=1}^{\infty} \int_{A_i} \rho(cf_n(x))d\mu$ is convergent and consequently

$$\sum_{i=k+1}^{\infty} \int_{A_i} \rho(cf_n(x))d\mu \to 0 \quad \text{as} \quad k \to \infty. \quad (19)$$

It follows from (17) that

$$\rho\left(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\right) \leq \frac{1}{\mu(\Omega)} \sum_{i=k+1}^{\infty} \int_{A_i} \rho(\mu(\Omega)f_n(x))d\mu.\quad (20)$$

Hence, by (19), we get that for $c_1 = \frac{c}{\mu(\Omega)}$

$$\rho\left(c_1\left(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\right)\right) < \epsilon \quad \text{for sufficiently large } k.\quad (21)$$

By convexity of $\rho$, we have that for every pair of positive integers $k$ and $n$

$$\rho\left(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\right) \leq \frac{1}{3}\rho(3(\nu(A) - \nu_n(A)))$$

$$+ \frac{1}{3}\rho\left(3\left(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\right)\right)$$

$$+ \frac{1}{3}\rho\left(3\left(\sum_{i=1}^{k} \nu_n(A_i) - \sum_{i=1}^{k} \nu(A_i)\right)\right).$$

Finally, let $b = \frac{1}{3} \min\{a, c_1\}$. Then, by (18) and (20), we have, for sufficiently large positive integers $k$

$$\rho\left(b\left(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\right)\right) < \epsilon.$$

This completes the proof of the theorem. \qed

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References


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