Some new remarks on MHD Jeffery-Hamel fluid flow problem

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Abstract: A Hamilton-Poisson realization of the MHD Jeffery-Hamel fluid flow problem is proposed. The nonlinear stability of the equilibrium states is discussed. A comparison between the analytic solutions obtained using the OHAM method and the exact solutions provided by the Hamilton-Poisson realization are presented.

Keywords: nonlinear stability, analytic solution

1 Introduction

The well known Jeffery-Hamel problem deals with the flow of an incompressible viscous fluid between the nonparallel walls. This flow situation was initially formulated by Jeffery [1] and Hamel [2]. Jeffery-Hamel flows are exact similarity solutions of the Navier-Stokes equations in the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex, and with a source or sink at the vertex. A lot of papers propose different methods to solve the nonlinear magnetohydrodynamics (MHD) Jeffery-Hamel blood flow problem: numerical solutions [3], analytical solutions [4–8], or solutions obtained via stochastic numerical methods based on computational intelligence techniques [9].

Recently, several techniques have been used for solving different nonlinear differential equations, such as: stochastic numerical methods [10], spectral analysis based on continuous wavelet transform [11], wavelet analysis [12], and the fractional derivative technique [13].

The challenge of this paper is to find some new properties of the MHD Jeffery-Hamel fluid flow problem which can gives us a different point of view from the classical ones. The main goals of our work are to find a Hamilton-Poisson realization (see [14]) of the MHD Jeffery-Hamel fluid flow problem and to point out some of its geometrical and dynamical properties from a mechanical geometry point of view. In addition, once the Casimir functions of the Hamilton-Poisson structure are found, the exact solution of the equation is the intersection between the surfaces $H = \text{const}$ and $C = \text{const}$. As a consequence, we can sketch a comparison with the analytic solutions proposed in [15].

The structure of this paper is as follows. In the second section of this work we prepare the framework of our study by writing the nonlinear differential equation as a Hamilton-Poisson one. The Poisson structure of the system, the corresponding Casimirs and the phase portrait are presented here.

The spectral stability and the nonlinear stability of the equilibrium states are the subjects of the third section. In the last section a comparison of the exact solution provided by the Hamilton-Poisson realization and the analytic solution given in [15] is proposed.

For the beginning, let us recall very briefly the definitions of general Poisson manifolds and the Hamilton-Poisson systems.

Definition: Let $M$ be a smooth manifold and let $C^\infty(M)$ denote the set of the smooth real functions on $M$. A Poisson bracket on $M$ is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, denoted as:

$$\{F, G\} \mapsto \{F, G\} \in C^\infty(M), F, G \in C^\infty(M)$$

which verifies the following properties:

- skew-symmetry:

$$\{F, G\} = - \{G, F\} ;$$

- Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0;$$

- Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\} .$$
The Poisson realization of the nonlinear equation Eq. (1) becomes:

\[ \eta^2 = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \]

Let the matrix given by:

\[ \Pi = \{ (x_i, x_j) \} \]

**Proposition:** Any Poisson structure \{ , \} on \( \mathbb{R}^n \) is completely determined by the matrix \( \Pi \) via the relation:

\[ \{f, g\} = (\mathcal{L}_f \nabla g) \]

**Definition:** A Hamilton-Poisson system on \( \mathbb{R}^n \) is the triple \((\mathbb{R}^n, \{ , \}, H)\), where \{ , \} is a Poisson bracket on \( \mathbb{R}^n \) and \( H \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) is the energy (Hamiltonian). Its dynamics is described by the following differential equations system:

\[ \dot{x} = \Pi \cdot \nabla H \]

where \( x = (x_1, x_2, \ldots, x_n)^t \).

**Definition:** Let \{ , \} a Poisson structure on \( \mathbb{R}^n \), A Casimir of the configuration \((\mathbb{R}^n, \{ , \})\) is a smooth function \( C \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) which satisfy:

\[ \{f, C\} = 0, \forall f \in C^\infty(\mathbb{R}^n, \mathbb{R}). \]

\section{The MHD Jeffery-Hamel fluid flow problem}

The MHD Jeffery-Hamel fluid flow problem can be written as [15]:

\[ F'(\eta) + 2 a Re F(\eta) F'(\eta) + (4 - Ha) a^2 F'(\eta) = 0, \quad (1) \]

where \( \eta > 0, Re \) is the Reynolds number, \( Ha \geq 0 \) is the Hartmann number, \( 0 < a \ll 1 \) is the flow angle and prime denotes derivative with respect to \( \eta \). Also, the physical model is presented in [15].

Using the notations:

\[ F(\eta) = f_1(\eta), \quad F'(\eta) = f_2(\eta), \quad F''(\eta) = f_3(\eta), \]

the nonlinear equation Eq. (1) becomes:

\[
\begin{cases}
  f_1'(\eta) = f_2(\eta) \\
  f_2'(\eta) = f_3(\eta) \\
  f_3'(\eta) = -2 a Re f_1(\eta) f_2(\eta) - (4 - Ha) a^2 f_1(\eta)
\end{cases}, \quad \eta > 0, \quad (2)
\]

**Proposition:** The system Eq. (2) has the Hamilton-Poisson realization

\[(\mathbb{R}^3, \Pi_-, H),\]

where

\[
\Pi_- = \begin{bmatrix}
  0 & 1 & 0 \\
  -1 & 0 & 2 a Re f_1 + (4 - Ha) a^2 \\
  0 & -2 a Re f_1 - (4 - Ha) a^2
\end{bmatrix}
\]

is the minus Lie-Poisson structure and

\[
H(f_1, f_2, f_3) = \frac{1}{2} f_2^2 - \frac{2}{3} a Re f_1^3 - \frac{4 - Ha}{2} a^2 f_1^2 - f_1 f_3 - a Re f_1^2 - (4 - Ha) a^2 f_1
\]

is the Hamiltonian.

Proof: Indeed, we have:

\[
\Pi_- \cdot \nabla H = \begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

and the matrix \( \Pi_- \) is a Poisson matrix, see [14].

The next step is to find the Casimirs of the configuration described by the above Proposition. Since the Poisson structure is degenerate, there exist Casimir functions. The defining equations for the Casimir functions, denoted by \( C \), are

\[
\Pi_-^0 \partial_C C = 0.
\]

It is easy to see that there exists only one functionally independent Casimir of our Poisson configuration, given by \( C : \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[
C(f_1, f_2, f_3) = f_3 + a Re f_1^2 + (4 - Ha) a^2 f_1.
\]

Consequently, the phase curves of the dynamics Eq. (2) are the intersections of the surfaces \( H(f_1, f_2, f_3) = \text{const} \) and \( C(f_1, f_2, f_3) = \text{const} \). see the Figures.

\section{Nonlinear stability problem}

The concept of stability is an important issue for any differential equation. The nonlinear stability of the equilibrium point of a dynamical system can be studied using the tools of mechanical geometry, so this is another good reason to find a Hamilton-Poisson realization. For more details, see [14]. We start this section with a short review of the most important notions.

**Definition:** An equilibrium state \( x_e \) is said to be **nonlinear stable** if for each neighborhood \( U \) of \( x_e \) in \( D \) there is a neighborhood \( V \) of \( x_e \) in \( U \) such that trajectory \( x(t) \) initially in \( V \) never leaves \( U \).
This definition supposes well-defined dynamics and a specified topology. In terms of a norm \( \| \cdot \| \), nonlinear stability means that for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if
\[
\| x(0) - x_0 \| < \delta
\]
then
\[
\| x(t) - x_0 \| < \varepsilon, \quad (\forall) \quad t > 0.
\]
It is clear that nonlinear stability implies spectral stability; the converse is not always true.

The equilibrium states of the dynamics Eq. (2) are
\[
e_1^M = (M, 0, 0), \quad e_2 = \left( -\frac{(4 - Ha)a}{2Rey}, 0, 0 \right), \quad M \in \mathbb{R}.
\]

**Proposition 1:** The equilibrium states \( e_1^M = (M, 0, 0) \) are nonlinearly stable for any \( M \in \mathbb{R} \).

**Proof:** We will use energy-Casimir method, see [14] for details. Let
\[
F_\varphi(f_1, f_2, f_3) = H(f_1, f_2, f_3) + \varphi[C(f_1, f_2, f_3)] = \frac{1}{2} f_2^2 - \frac{3}{2} aRey f_3^2 - \frac{4 - Ha}{2} a^2 f_2^2 - f_1 f_3 - f_3 - aRey f_1^2 - \frac{2}{3} aRey \delta f_1 + \varphi(f_3 + aRey f_3^2 + (4 - Ha)a^2 f_1)
\]
be the energy-Casimir function, where \( \varphi : \mathbb{R} \to \mathbb{R} \) is a smooth real valued function.

Now, the first variation of \( F_\varphi \) is given by
\[
\delta F_\varphi(f_1, f_2, f_3) = f_2 \delta f_2 - f_3 \delta f_1 + [-f_1 - 1 + \varphi(f_3 + aRey f_3^2 + (4 - Ha)a^2 f_1)] \times \left( 2 aRey \delta f_1 + (4 - Ha)a^2 \delta f_1 + \delta f_3 \right)
\]
so we obtain
\[
\delta F_\varphi(e_1^M) = \left[ -M - 1 + \varphi(aRey M^2 + (4 - Ha)a^2 M) \right] \times \left( 2 aRey \delta f_1 + (4 - Ha)a^2 \delta f_1 + \delta f_3 \right)
\]
that is equals zero for any \( M \in \mathbb{R} \) if and only if
\[
\varphi \left( aRey M^2 + (4 - Ha)a^2 M \right) = M + 1.
\]

The second variation of \( F_\varphi \) at the equilibrium of interest is given by
\[
\delta^2 F_\varphi(e_1^M) = \left[ (2 aRey M + (4 - Ha)a^2 M - (4 - Ha)a^2 M) \cdot (\delta f_1)^2 + (\delta f_2)^2 \cdot (1 - (2 aRey M + (4 - Ha)a^2)) \cdot \delta f_3 + \right. \] 
\[
\left. \varphi(aRey M^2 + (4 - Ha)a^2 M) \right] \cdot \delta f_1 \cdot \delta f_2 + \varphi(aRey M^2 + (4 - Ha)a^2 M) \cdot \delta f_3^2.
\]

If we choose now \( \varphi \) such that the relation (3) is valid and
\[
(2 aRey M + (4 - Ha)a^2 M - (4 - Ha)a^2 M) - 1 > 0,
\]
then the second variation of \( F_\varphi \) at the equilibrium of interest is positive defined and so our equilibrium states \( e_1^M \) are nonlinearly stable.

**Proposition 2:** The equilibrium state \( e_2 = \left( -\frac{(4 - Ha)a}{2Rey}, 0, 0 \right) \) is nonlinearly stable for \( 0 \leq Ha < 4 \).

**Proof:** We will use energy-Casimir method, see [14] for details. Let
\[
F_\varphi(f_1, f_2, f_3) = H(f_1, f_2, f_3) + \varphi[C(f_1, f_2, f_3)] = \frac{1}{2} f_2^2 - \frac{3}{2} aRey f_3^2 - \frac{4 - Ha}{2} a^2 f_2^2 - f_1 f_3 - f_3 - aRey f_1^2 - \frac{2}{3} aRey \delta f_1 + \varphi(f_3 + aRey f_3^2 + (4 - Ha)a^2 f_1)
\]
be the energy-Casimir function, where \( \varphi : \mathbb{R} \to \mathbb{R} \) is a smooth real valued function.

Now, the first variation of \( F_\varphi \) is given by
\[
\delta F_\varphi(f_1, f_2, f_3) = f_2 \delta f_2 - f_3 \delta f_1 + [-f_1 - 1 + \varphi(f_3 + aRey f_3^2 + (4 - Ha)a^2 f_1)] \times \left( 2 aRey f_1 \delta f_1 + (4 - Ha)a^2 \delta f_1 + \delta f_3 \right)
\]
so we obtain
\[
\delta F_\varphi(e_2) = \left[ \frac{(4 - Ha)a^2}{2Rey} - 1 + \varphi \left( \frac{(4 - Ha)a^4}{2Rey} \right) \right] \times \left( 2 aRey f_1 \delta f_1 + (4 - Ha)a^2 \delta f_1 + \delta f_3 \right)
\]
that is equals zero if and only if
\[
\varphi \left( \frac{(4 - Ha)a^4}{2Rey} \right) = 1 - \frac{(4 - Ha)a^2}{2Rey}.
\]

The second variation of \( F_\varphi \) at the equilibrium of interest is given by
\[
\delta^2 F_\varphi(e_2) = \left[ (1 - a) a^4 (4 - Ha)^2 \cdot \varphi \left( -\frac{(4 - Ha)a^4}{2Rey} \right) \right] \times (2 - a) + (a - 1)a^4 (4 - Ha) \cdot (\delta f_1)^2 + (\delta f_2)^2 + 2 \left[ -1 - (a - 1)a^4 (4 - Ha) \right] \cdot \delta f_1 \cdot \delta f_2 + \varphi \left( -\frac{(4 - Ha)a^4}{2Rey} \right) \cdot (2 - a) \cdot (\delta f_3)^2.
\]
\[
+\bar{\phi} \left( -\frac{(4 - Ha)^2 a^4}{2 \text{Rey}} \cdot (2 - \alpha) \right) \cdot (\delta f_3)^2.
\]

If we choose now \( \phi \) such that the relation (4) is valid and
\[
(1 - \alpha) a^2 (4 - Ha) \cdot \bar{\phi} \left( -\frac{(4 - Ha)^2 a^4}{2 \text{Rey}} \cdot (2 - \alpha) \right) - 1 < 0
\]
for \( 4 - Ha > 0 \), then the second variation of \( F_\phi \) at the equilibrium state \( e_2 \) is nonlinearly stable.

\[\Box\]

4 Comparison of the exact solution and analytical solution

Consequently we have derived the following result:

**Remark:** The phase curves of the dynamics (2) are the intersections of the surfaces \( H(f_1, f_2, f_3) = \text{const} \) and \( C(f_1, f_2, f_3) = \text{const} \).

\[
\begin{align*}
H(f_1, f_2, f_3) &= \frac{1}{2} f_1^2 - \frac{2}{3} a \text{Rey} f_1^3 - \frac{4 - Ha}{2} a^2 f_1^2 - f_1 f_3 - f_3 - a \text{Rey} f_2^2 - (4 - Ha) a^2 f_1 = C_{st1} \\
C(f_1, f_2, f_3) &= f_1 + a \text{Rey} f_1^2 + (4 - Ha) a^2 f_1 = C_{st2},
\end{align*}
\]

where
\[
C_{st1} = \frac{1}{2} f_1^2(0) - \frac{2}{3} a \text{Rey} f_1^3(0) - \frac{4 - Ha}{2} a^2 f_1^2(0) - f_1(0)f_3(0) - f_3(0) - a \text{Rey} f_2^2(0) - (4 - Ha) a^2 f_1(0) =
\]
\[
= \frac{1}{2} f''(0)^2 - \frac{2}{3} a \text{Rey} F(0)^3 - \frac{4 - Ha}{2} a^2 F(0)^2 - F(0) f''(0) - f''(0) - a \text{Rey} F(0)^2 - (4 - Ha) a^2 F(0),
\]

\[
C_{st2} = f_3(0) + a \text{Rey} f_1^2(0) + (4 - Ha) a^2 f_1(0) =
\]
\[
= f''(0) + a \text{Rey} F(0)^2 + (4 - Ha) a^2 F(0).
\]

Using the physical conditions as [15]:
\[
f_1(0) = 1, \quad f_2(0) = 0, \quad f_1(1) = 0,
\]

the numerical values of the second-order derivative \( f_3(0) = F''(0) \) were obtained via Optimal Homotopy Perturbation Method [15] for some values of the physical parameters, i.e. \( \text{Rey} = 50, (\alpha = \pi/24, Ha = 250), (\alpha = \pi/24, Ha = 500), (\alpha = \pi/24, Ha = 1000), (\alpha = \pi/36, Ha = 250), (\alpha = \pi/36, Ha = 500) \) and \( (\alpha = \pi/36, Ha = 1000), \) respectively. These values are presented in Table 1.

Finally, we compare the phase curves given by Eq. (5) of the dynamics (2) with the corresponding approximate analytic solutions from [15] for the physical values presented in Table 1.

**Observation:** If \( \tilde{f}(\eta) \) is the approximate analytic solution obtained via Optimal Homotopy Perturbation Method [15], then the corresponding residual functions are:
\[
\begin{align*}
R_H &= \frac{1}{2} (\tilde{f})^2 - \frac{2}{3} a \text{Rey} (\tilde{f})^3 - \frac{4 - Ha}{2} a^2 (\tilde{f})^2 - (4 - Ha) a^2 \tilde{f} - C_{st1} \\
R_C &= \tilde{f}'' + a \text{Rey} (\tilde{f})^2 + (4 - Ha) a^2 \tilde{f} - C_{st2}.
\end{align*}
\]

**Example 1:** \( \text{Rey} = 50, \alpha = \pi/24, Ha = 250 \). The phase curve and the corresponding residual are presented in Figs 1, 2 and 3 respectively.
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Values of the integrals:

\[ \int_0^1 R_H^2(\eta) \, d\eta \approx 6.26536 \times 10^{-8}; \]
\[ \int_0^1 R_C^2(\eta) \, d\eta \approx 5.62805 \times 10^{-8}. \]

**Example 2:** \( \text{Rey} = 50, \alpha = \pi/24, Ha = 500. \) The phase curve and the corresponding residual are presented in Figs 4, 5 and 6 respectively.

Values of the integrals:

\[ \int_0^1 R_H^2(\eta) \, d\eta \approx 3.37046 \times 10^{-6}; \]
\[ \int_0^1 R_C^2(\eta) \, d\eta \approx 1.48946 \times 10^{-6}. \]

**Example 3:** \( \text{Rey} = 50, \alpha = \pi/24, Ha = 1000. \) The phase curve and the corresponding residual are presented in Figs 7, 8 and 9 respectively.

Values of the integrals:

\[ \int_0^1 R_H^2(\eta) \, d\eta \approx 1.27702 \times 10^{-6}; \]
\[ \int_0^1 R_C^2(\eta) \, d\eta \approx 5.45545 \times 10^{-7}. \]
Example 4: $Re_y = 50$, $\alpha = \pi/36$, $Ha = 250$. The phase curve and the corresponding residual are presented in Figs 10, 11 and 12 respectively.

Values of the integrals:
\[
\int_0^1 R_H^2(\eta) \, d\eta \approx 7.0543 \cdot 10^{-8};
\]
\[
\int_0^1 R_C^2(\eta) \, d\eta \approx 6.16148 \cdot 10^{-8}.
\]

Example 5: $Re_y = 50$, $\alpha = \pi/36$, $Ha = 500$. The phase curve and the corresponding residual are presented in Figs 13, 14 and 15 respectively.

Example 6: $Re_y = 50$, $\alpha = \pi/36$, $Ha = 1000$. The phase curve and the corresponding residual are presented in Figs 16, 17 and 18 respectively.

Values of the integrals:
\[
\int_0^1 R_H^2(\eta) \, d\eta \approx 5.09766 \cdot 10^{-7};
\]
\[
\int_0^1 R_C^2(\eta) \, d\eta \approx 4.60622 \cdot 10^{-7}.
\]
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Values of the integrals:
\[ \int_0^1 R_H^2(\eta) \, d\eta \approx 2.86153 \cdot 10^{-6} ; \]
\[ \int_0^1 R_C^2(\eta) \, d\eta \approx 1.56449 \cdot 10^{-7} . \]

5 Conclusions

The stability of a nonlinear differential problem governing the MHD Jeffery-Hamel fluid flow is investigated. Due to the existence of a Poisson formulation, the results were obtained using specific tools, such as the energy-Casimir method.

Finally, the analytical integration of the nonlinear system (obtained via the Optimal Homotopy Asymptotic Method and presented in [15]) is compared with the exact solution (obtained as intersections of the surfaces \( H(f_1, f_2, f_3) = \text{const} \) and \( C(f_1, f_2, f_3) = \text{const} \)).
Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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