Variational Approach to Impulsive Differential Equations Using the Semi-Inverse Method

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The semi-inverse method is used to establish a variational principle for the Dirichlet boundary value problem with impulses. All the boundary conditions can be obtained as natural conditions by making the variational principle stationary.

Key words: Impulsive; Variational Principle; Semi-Inverse Method.

1. Introduction

Many dynamical systems have an impulsive dynamical behaviour due to abrupt changes at certain instants during the evolution process [1 – 3]; in this paper, we will consider the following Dirichlet impulsive problem:

\begin{align}
- u''(t) + \lambda u(t) &= \sigma(t), \quad t \in [0, T], \\
\Delta u'(t_j) &= d_j, \quad j = 1, 2, 3, \ldots, p, \\
u(0) &= u(T) = 0,
\end{align}

where $0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$ and $\Delta u'(t_j)$ is defined as

\[ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-). \]

Nieto and his colleagues established variational principles for various impulsive problems [1 – 3]; in this paper we suggest an alternative approach to the establishment of the variational formulation for the above problem.

2. Semi-Inverse Method

The semi-inverse method [4] is a powerful tool to establish a variational formulation directly from governing equations and boundary/initial conditions. The basic idea of the semi-inverse method is to construct a trial-functional with an unknown function. For the present problem, we can construct a trial-functional in the form

\[ J(u) = \int_0^T \left\{ \frac{1}{2} u'^2 + F(u) \right\} dt, \]

where $F$ is an unknown function of $u$.

There are alternative approaches to construct trial-functionals, see [5 – 10].

Making the functional (5) stationary with respect to $u$, we have the following stationary condition (Euler–Lagrange equation):

\[ -u'' + \frac{\partial F}{\partial u} = 0. \]

Equation (6) should be equivalent to (1); to this end, we set

\[ \frac{\partial F}{\partial u} = \lambda u(t) - \sigma(t). \]

From (7), the unknown function $F$ can be identified as

\[ F = \frac{1}{2} \lambda u^2 - \sigma u. \]

We, therefore, obtain the following functional:

\[ J(u) = \int_0^T \left\{ \frac{1}{2} u'^2 + \frac{1}{2} \lambda u^2 - \sigma u \right\} dt. \]
In order to incorporate the impulsive condition (2) and the boundary condition (3) into the above variational formulation, we construct a trial-functional in the form

\[
J(u) = \int_0^T \left\{ \frac{1}{2} u'^2 + \frac{1}{2} \lambda u'^2 - \sigma u \right\} dt + \sum_{j=1}^p B_j \left. \left( \frac{\partial B_0}{\partial u} \right) \right|_{t_j}
\]

\[+ B_0 \big|_{t=0} + B_T \big|_{t=T},
\]

where \( B_j \) \( (j = 0, 1, 2, 3, \ldots, p, p + 1) \) is an unknown continuous function.

Making (10) stationary, we have

\[
\delta J(u) = \int_0^T \left\{ u' \delta u' + \lambda u \delta u - \sigma \delta u \right\} dt
\]

\[+ \sum_{j=1}^p \frac{\partial B_j}{\partial u} \left. \left( \frac{\partial B_0}{\partial u} \right) \right|_{t_j} \delta u |_{t_j} + \frac{\partial B_0}{\partial u} \delta u |_{t=0} + \frac{\partial B_T}{\partial u} \delta u |_{t=T}.
\]

\[
= \int_0^T \left\{ -u'' + \lambda u - \sigma \right\} \delta u dt + \sum_{j=1}^p u' \left. \delta u \right|_{t_j}
\]

\[+ u' \left. \delta u \right|_0 + \sum_{j=1}^p \frac{\partial B_j}{\partial u} \left. \left( \frac{\partial B_0}{\partial u} \right) \right|_{t_j} \delta u |_{t_j} + \frac{\partial B_0}{\partial u} \delta u |_{t=0} + \frac{\partial B_T}{\partial u} \delta u |_{t=T}.
\]

\[
= \int_0^T \left\{ -u'' + \lambda u - \sigma \right\} \delta u dt
\]

\[+ \sum_{j=1}^p \left( u' + \frac{\partial B_j}{\partial u} \right) \left. \delta u \right|_{t_j}
\]

\[+ \left( u' + \frac{\partial B_0}{\partial u} \right) \left. \delta u \right|_{t=0} + \left( u' + \frac{\partial B_T}{\partial u} \right) \left. \delta u \right|_{t=T} = 0.
\]

For any arbitrary \( \delta u \), we have (1) as Euler–Lagrange equation, and the following natural boundary/initial conditions:

\[
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\]

\[\text{at } t = t_0 = 0:\]

\[-u'(0) + \frac{\partial B_0}{\partial u} = 0,\]

\[\text{(12a)}\]

\[\frac{\partial B_0}{\partial u} = 0;\]

\[\text{(12b)}\]

\[\text{at } t = t_j:\]

\[u'(t_j^-) - u'(t_j^+) + \frac{\partial B_j}{\partial u} (t_j) = 0;\]

\[\text{(13)}\]

\[\text{at } t = t_{p+1} = T:\]

\[u'(T) + \frac{\partial B_T}{\partial u} = 0,\]

\[\text{(14a)}\]

\[\frac{\partial B_T}{\partial u} = 0.\]

\[\text{(14b)}\]

In (13), we set

\[
\frac{\partial B_j}{\partial u}(t_j) = d_j
\]

so that it turns out to be (2). From (15), we can identify \( B_j \) as follows:

\[
B_j(t_j) = \int_{t_0}^{u(t_j)} d_j dt.
\]

Equations (12) and (14) should satisfy the boundary condition (3); to this end, we set

\[
B_0 = u'(0)u(0)
\]

\[\text{(17)}\]

and

\[
B_T = -u'(T)u(T).
\]

\[\text{(18)}\]

Please note in above derivation we have used the property \( \int_0^T \sum_{j=0}^{p+1} \int_{t_j}^{t_{j+1}} \), where \( T_0 = 0 \) and \( T_{p+1} = T \).

We, therefore, obtain the following needed variational principle:

\[
J(u) = \int_0^T \left\{ \frac{1}{2} u'^2 + \frac{1}{2} \lambda u'^2 - \sigma u \right\} dt
\]

\[+ \sum_{j=1}^p \int_{t_0}^{u(t_j)} d_j dt + u'(0)u(0) - u'(T)u(T).
\]

\[\text{(19)}\]

It is easy to prove that the stationary conditions of the above functional satisfy (1) – (3).
3. Conclusions

In this paper the semi-inverse method is applied to establish a variational formulation for the Dirichlet boundary value problem with impulses. The method can be extended to other impulsive problems with ease.

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