*Journal of Industrial Organization
Education*

Manuscript 1037

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on Discrete and Continuum Differentiated
Object Models

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Abstract

Most lectures teach the relationship between the CES, Cobb-Douglas, and Leontief functions using the value of elasticity of substitution, namely, in the discrete object model. This lecture note aims at being a reference for algebraic computations of the Leontief and Cobb-Douglas functions by taking limits of CES functions both in discrete and continuum goods models. The argument on the discrete case uses l'Hôpital's rule as usually done. The argument on the continuum case also uses l'Hôpital's rule to show the convergence to the Cobb-Douglas function. To guarantee the convergence to the Leontief function, however, we rely on the squeeze principle.

KEYWORDS: Convergence of functions, CES, Linear, Cobb-Douglas, Leontief

Author Notes: I would like to express my deepest gratitude to the anonymous referee and Professor Chang who provided helpful comments and suggestions to improve the initial version of this lecture note. Special thanks also go to all students of State University of New York at Buffalo (Fall 2009 and Spring 2010) who attended my sessions.

1 Introduction

Amongst various functional forms in economics, it seems the *constant elasticity of substitution* (CES) form is the best used functional form. The history of the CES function goes back to the 1950s—see, for example, Solow (1956, p. 85), which still has no special naming. The naming “CES” was assigned initially by Arrow et al. (1961, p. 230). A detailed discussion about the “CES form” is done by Armington (1969), especially on the elasticity of substitution that is also referred to as the Armington elasticity.

During the 1970s, many economists tried to generalize the CES function to allow variations of the elasticity of substitution.¹ These attempts showed such generalization is possible under very limited circumstances. Aside from such generalizing attempts, the CES function obtained large attention after the invention of the taste-for-variety model: Spence (1976) for the continuum variety case and Dixit and Stiglitz (1977) for the discrete variety case. The taste-for-variety model is the basis of various fields of current economics such as industrial organization, international trade, urban economics & regional science, and open-macroeconomics, etc. Although it seems the larger benefit of using a multiple-variety model is in the variety of consumption models, it is also interesting to see the variety of inputs in production function. For example, varieties of skills can increase the quantity of production, as recently discussed by Kurokawa (2011).

In addition to the (general) CES form, the Cobb-Douglas form is another popular functional form in economics. The Cobb-Douglas (production) function was named after Cobb and Douglas (1928) but initially discussed by Wicksell (1893) in the late nineteenth century (translated by Frowein, 1954). In Cobb and Douglas op. cit., they examined the applicability of the functional form to the real macroeconomic data to describe the aggregate economy of the United States. The Cobb-Douglas form is now applied to both production and utility functions. As we will see, the elasticity of substitution of the Cobb-Douglas function is designed to be one (see Cobb and Douglas op. cit.).

Differentiability and strict concavity encourage economic students to use “ordinary” CES or Cobb-Douglas form.² However, in some cases, we apply linear and Leontief forms to study our models. The Leontief form, named after Wassily Leontief, is intensively discussed in the 1930s-1940s, however, it is still an important functional form in economics. The Leontief form describes production technologies and preference relations that are constrained by the minimum of respective inputs. In the classroom, one of the most important applications of the

¹Interested readers should refer to Mishra (2007), which gives a concise survey of these (production) functions.

²“Ordinary” CES here indicates the CES form neither linear nor Leontief.

Leontief (production) function is the Harrod-Domar economic growth model (Domar, 1946; Harrod, 1939). The elasticity of substitution of the Leontief function is known to be zero. In theoretical studies, we do not use linear production (utility) function so often if there are multiple input factors, as we typically obtain corner solutions or indeterminacy of equilibrium; if there is only one factor, Ricardian international trade model and risk-neutral utility function are examples of linear production function and utility function models, respectively. The linear form is especially important in empirical studies: for example, log-linear function is one of the linear form (cf. a generalization of linear form is considered by Diewert, 1971). The elasticity of substitution of linear form is known to be infinity (the elasticity of substitution of Diewert's generalized linear function is not infinity, which is two).

In this lecture note, we see how to obtain the Cobb-Douglas and the Leontief functions from the CES function. Usually, the exposition employs arguments about the elasticity of substitution. However, it is also interesting to see how the same result is obtained algebraically. A similar attempt of this lecture note seems to be done by Csontos and Ray (1992) for the convergence of CES to Leontief. For lectures, however, it seems better to present all critical cases (linear, Cobb-Douglas, and Leontief) as well as both discrete and continuous models. This lecture note uniquely provides all of these cases and models in one article.

After introducing the elasticity of substitution in Section 1.1 and mathematical tools in Section 1.2, we look at the discrete-variety model in Section 2 and continuum-variety model in Section 3. Some concluding comments are also made in Section 4. In these discussion, utility function is considered as if it is a production function of utility. In a sense, such understanding is not wrong. However, it should be noted that utility function is derived from axioms of preference while the production function is derived from axioms of production technology.

1.1 Elasticity of Substitution

In most cases, the relationship between the CES function and the Cobb-Douglas, Leontief, and linear functions are explained by the elasticity of substitution σ between two factors, X_1 and X_2 (input factors or consumption goods). For example, if function $F(X_1, X_2)$ represents a preference of consumer, the elasticity of substitution between two goods, X_1 and X_2 , is defined by

$$\sigma = \frac{\% \text{ change of "consumption ratio"}}{\% \text{ change of "marginal rate of substitution" (MRS}_{12})}, \quad (1)$$

where F is at least twice differentiable and MRS_{12} is the ratio of marginal utilities from the two goods:

$$MRS_{12} = \frac{\partial F / \partial X_1}{\partial F / \partial X_2}. \quad (2)$$

If F represents a technology of production technology, for now letting X_1 be labor input L and X_2 be capital input K , σ is defined in terms of marginal rate of *technical* substitution between labor and capital:

$$\sigma = \frac{\% \text{ change of "input capital-labor ratio"}}{\% \text{ change of "marginal rate of technical substitution" (MRTS}_{12})}, \quad (3)$$

where $MRTS_{12}$ is defined by

$$MRTS_{12} = \frac{\partial F / \partial L}{\partial F / \partial K}. \quad (4)$$

On the factors space (e.g. X_1 - X_2 plain), $MRTS_{12}$ (or alternatively MRS_{12}) is the slope of the level curve of F (iso-quant curve for production function and indifference curve for utility function) at a certain point for each fixed value $\mu \equiv F(X_1, X_2)$. This is easily verified by looking at the total derivative of F for a fixed μ .³

$$\begin{aligned} d\mu &\equiv \frac{\partial F}{\partial X_1} dX_1 + \frac{\partial F}{\partial X_2} dX_2 \quad (d\mu \equiv 0) \\ \Rightarrow \quad MRTS_{12} &\equiv -\frac{dX_2}{dX_1}. \end{aligned} \quad (5)$$

In the following discussion, we use MRTS instead of MRS. However, please be notified that MRS is usually used in discussions of utility and MRTS is usually used in discussions of production.

The MRTS (MRS) and factor-ratio are graphically represented by Figure 1: if the equilibrium point is given by E_1 , say *status-quo*, the MRTS (MRS) is given by $\angle OB_1E_1$, and the corresponding factor-input ratio (or consumption ratio) is given by $\angle B_1OE_1$. If there is a change from E_1 to E_2 , the new MRTS (MRS) and factor-input (consumption) ratio are respectively given by $\angle OB_2E_2$ and $\angle B_2OE_2$. In this figure, lines represented by A_1B_1 and A_2B_2 are cost (or budget) lines to produce the given μ by using a certain factor-input ratio X_2/X_1 represented by OC 's. From this figure, it is clear that the elasticity of substitution measures the percent change of factor-ratio on the iso-quant (or indifference) curve.

³If F is a utility function, μ is a given utility level, and if F is a production function, it is a given production quantity.

Figure 1: Elasticity of substitution σ measures the required change of factor-ratio to stay on the same level curve (iso-quant or indifference curve) when the equilibrium point moves, for example, from E_1 to E_2 (i.e., the change of relative prices). An attention should be paid that the mentioned changes are percent changes instead of nominal changes. As discussed in Cobb and Douglas (1928) and as we will see soon, the Cobb-Douglas function is defined (or designed) to be $\sigma = 1$.

Figure 2: Graphical representations of σ 's of the Leontief and linear functions (examine the movements of slopes of the level curves to see the values of σ 's)

Suppose F is Leontief, so that $F(X_1, X_2) = \min\{a_1X_1, a_2X_2\}$,⁴ where a 's are positive weight parameters representing the technology for production function and preference for utility function. In this case, a level curve of F is depicted as shown in Figure 2 (left). Alternatively, if F is linear, so that $F(X_1, X_2) = a_1X_1 + a_2X_2$, a level curve of F is depicted as shown in Figure 2 (right). In the Leontief case, a change in MRTS_{12} does not change the input-factor ratio; hence, $\sigma = 0$. In contrast, in the linear case, a change in MRTS_{12} drives the input-factor ratio to the corners; hence, $\sigma = +\infty$. In the two-factor case, the CES function is defined by

$$F(X_1, X_2) = \left(a_1X_1^\theta + a_2X_2^\theta\right)^{\frac{1}{\theta}}, \quad (6)$$

where θ is the technological (or preference) parameter associated with the elasticity of substitution σ (see below).

The MRTS_{12} of CES function (4) is calculated as follows.

$$\begin{aligned} \frac{\partial F}{\partial X_i} &= a_i X_i^{\theta-1} \left(a_1X_1^\theta + a_2X_2^\theta\right)^{\frac{1-\theta}{\theta}} \\ \Rightarrow \text{MRTS}_{12} &= \frac{a_1}{a_2} \left(\frac{X_2}{X_1}\right)^{1-\theta}. \end{aligned} \quad (7)$$

Because the percent change of a function is given by the derivative of its logarithm value, (7) implies

$$\% \text{ Change of } \text{MRTS}_{12} \text{ in } \frac{X_2}{X_1} \equiv \frac{d \ln \text{MRTS}_{12}}{d(X_2/X_1)} = \frac{1-\theta}{X_2/X_1}. \quad (8)$$

Because the percent change of X_2/X_1 in itself is

$$\frac{d \ln(X_2/X_1)}{d(X_2/X_1)} = \frac{1}{X_2/X_1}, \quad (9)$$

from (1), σ is given by

$$\sigma \equiv \frac{d \ln(X_2/X_1)}{d \ln \text{MRTS}_{12}} = \frac{1}{1-\theta}. \quad (10)$$

In the multiple-object model, as we can arbitrarily choose combinations of objects to compute the corresponding σ , the above relationship holds in the more general CES function that we will study in the later sections.

From our knowledge, $\sigma = 0$ if F is Leontief and $\sigma \mapsto +\infty$ if it is linear. The linear case is easily checked by substituting $\theta = 1$ ($\sigma \mapsto +\infty$). However, it is not

⁴See also Section 4 for the discussion about coefficients a 's.

straightforward to see if the above CES function actually converges to the Leontief function as $\theta \mapsto -\infty$ though it must be. Similarly, the convergence to the Cobb-Douglas function is not straightforward. The Cobb-Douglas function is defined by

$$F(X_1, X_2) = X_1^\alpha X_2^{1-\alpha}, \quad (11)$$

where $\alpha \in (0, 1)$ is known to be the share of consumption of X_1 (so that $1 - \alpha$ is that of X_2). From (11), if F is a production function, marginal products of X_1 and X_2 are

$$\frac{\partial F}{\partial X_1} = \alpha \times \left(\frac{X_2}{X_1}\right)^{1-\alpha} \quad \text{and} \quad \frac{\partial F}{\partial X_2} = (1 - \alpha) \times \left(\frac{X_2}{X_1}\right)^{-\alpha}. \quad (12)$$

Then we find

$$\text{MRTS}_{12} \equiv \frac{\partial F / \partial X_1}{\partial F / \partial X_2} = \frac{\alpha}{1 - \alpha} \times \frac{X_2}{X_1}, \quad (13)$$

and it implies

$$\% \text{ Change of } \text{MRTS}_{12} \text{ in } \frac{X_2}{X_1} \equiv \frac{d \ln \text{MRTS}_{12}}{d(X_2/X_1)} = \frac{1}{X_2/X_1}. \quad (14)$$

Therefore, from (1), the consumption share is fixed and so $\sigma = 1$. This means the CES function must converge to the Cobb-Douglas as $\theta \mapsto 0$, or $\sigma \equiv 1/(1 - \theta) \mapsto 1$, though it is not easy to see if it is actually the case.

1.2 Mathematical Methods

In the following argument, we see the formal algebraic manipulations to show the convergence of the CES function to the Leontief and the Cobb-Douglas functions in CES functions of multiple discrete goods and continuum goods. In most parts of the argument, we apply l'Hôpital's rule: for differentiable functions f and g , we find

$$\lim \frac{f(\theta)}{g(\theta)} = \lim \frac{f'(\theta)}{g'(\theta)}. \quad (15)$$

This famous rule was found in the sixteenth century by Guillaume de l'Hôpital. The intuitive meaning of this rule is that the ratio of the two functions converges to the ratio of slopes of these functions. In the continuum model, we also use the squeeze theorem (principle). For three functions from θ to $\mathbb{R}$, g , f , and ϕ , such that $g(\theta) \leq \phi(\theta) \leq f(\theta)$, the theorem tells

$$\lim g(\theta) = \lim f(\theta) = k \quad \Rightarrow \quad \lim \phi(\theta) = k. \quad (16)$$

Then we will find algebraically that the continuum CES function converges to the corresponding Leontief functions as $\theta \mapsto -\infty$ ($\sigma \mapsto 0$).

The use of l'Hôpital's rule is a well-known method (namely in the discrete model), but the use of the squeeze theorem is not common. That is because most lectures focus on the discrete model rather than the continuum model.

2 Discrete Case

We consider a discrete object, i.e. commodity or input factor, X indexed by i . Let $X_i > 0$ be the quantity of consumption of variety i and A_i be the weight on each input in the *constant-elasticity-of-substitution* (CES) function defined by

$$F(X) = \left(\sum_{i=1}^N A_i X_i^\theta \right)^{\frac{1}{\theta}}, \quad (17)$$

where $\theta \in [-\infty, 1]$ is the parameter associated with the elasticity of substitution $\sigma = 1/(1 - \theta) \in [0, +\infty]$.⁵ The difficulty of allowing non-constant elasticity of substitution is discussed in Uzawa (1962) (cf. *partial* elasticity of substitution: Allen, 1938, pp. 503-505). Mainly for computational convenience, A_i may be given by $A_i = a_i^{1-\theta}$ for the sake of simpler equilibrium value expression (see also the argument in the concluding section; Section 4). As discussed in the introductory section, it is well known that the CES function converges to the Cobb-Douglas function as σ approaches one (or $\theta \mapsto 0$), Leontief function as σ approaches zero (or $\theta \mapsto -\infty$), and linear function as σ approaches infinity. It is obvious that the above CES function converges to the linear function for $\sigma \mapsto +\infty$ as $\theta \mapsto 1$. However, the convergence to the Leontief and the Cobb-Douglas is not straightforward.

We consider the convergence of parameter θ as $\theta \mapsto 0$ (Cobb-Douglas function) and $\theta \mapsto -\infty$ (Leontief function) in $\ln F(X)$:

$$\lim_{\theta \mapsto 0, -\infty} \ln F(X) = \lim_{\theta \mapsto 0, -\infty} \frac{1}{\theta} \ln \left(\sum_{i=1}^N A_i X_i^\theta \right). \quad (18)$$

For each case of the convergence, we apply l'Hôpital's rule to θ to get

$$\lim_{\theta \mapsto 0, -\infty} \ln F(X) = \lim_{\theta \mapsto 0, -\infty} \frac{\sum_{i=1}^N A_i X_i^\theta \ln X_i}{\sum_{i=1}^N A_i X_i^\theta}. \quad (19)$$

For $\theta \mapsto 0$, it is straightforward to get the corresponding Cobb-Douglas function:

$$\lim_{\theta \mapsto 0} \ln F(X) = \sum_{i=1}^N \alpha_i \ln X_i. \quad (20)$$

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⁵As a special case of CES, Diewert's (1971, p. 505) generalized linear function is given by $\theta = 1/2$ (hence, $\sigma = 2$) and appropriately corrected a 's.

where α_i is defined by

$$\alpha_i \equiv \frac{A_i}{\mathcal{A}} \quad \text{so that} \quad \sum_{i=1}^N \alpha_i \equiv 1. \quad (21)$$

In the above equation, $\mathcal{A}$ is defined to be

$$\mathcal{A} \equiv \sum_{i=1}^N A_i. \quad (22)$$

For $\theta \mapsto -\infty$, we need some technical manipulations by dividing both the numerator and denominator by $X_m = \min \{X_1, \dots, X_N\}$ to get

$$\lim_{\theta \mapsto -\infty} \frac{\sum_{i=1}^N A_i X_i^\theta \ln X_i}{\sum_{i=1}^N A_i X_i^\theta} = \lim_{\theta \mapsto -\infty} \frac{\sum_{i=1}^N A_i (X_i/X_m)^\theta \ln X_i}{\sum_{i=1}^N A_i (X_i/X_m)^\theta} = \ln X_m, \quad (23)$$

We get $(X_i/X_m)^\theta \mapsto 0$ ($\theta \mapsto -\infty$) for $X_i > X_m$ and $(X_i/X_m)^\theta \equiv 1$ for $X_i = X_m$ for $i \neq m$. Subsequently, we obtain the corresponding Leontief function for $\theta \mapsto -\infty$:

$$\lim_{\theta \mapsto -\infty} F(X) = \min \{X_1, \dots, X_N\}. \quad (24)$$

Notice this result holds even if $\max \{X_1, \dots, X_N\} = +\infty$ as $(X_i/X_m)^\theta \mapsto 0$ by $\theta < 0$ for $X_i > X_m$ (if $X_i \equiv X_m$ for all i , it is trivial to find the above result).

3 Continuum Case

We consider a continuum object X indexed by x for each variety. Let ∞ and $-\infty$ respectively be the upper and lower boundaries of X . Let $\mathcal{F}(x)$ be, say, the *cumulative utility-weight function* of X such that $\mathcal{F}(x) = 0$ for $x \mapsto -\infty$, $\mathcal{F}(x) = 1$ for $x \mapsto \infty$. Let function f be associated with $\mathcal{F}$ by $\mathcal{F}'(x) = f(x)$. In addition, in order to focus on goods, assume $f(x) > 0$ for all x . In this sense, we can alternatively say $\mathcal{F}$ and f are the *cumulative distribution* and *density functions* of the weights of differentiated objects, respectively; hence, these distribution functions characterize the technology or the preference. Using Stieltjes integral form, then, the continuum version of the CES function is defined by

$$V(X) = \left(\int_X c(x)^\theta d\mathcal{F}(x) \right)^{\frac{1}{\theta}}, \quad (25)$$

where θ is the technology (preference) parameter as defined in the discrete model and $c(x) > 0$ is the quantity of input of each variety. Then we consider the convergence of $\ln V(X)$ for $\theta \mapsto 0$ and $\theta \mapsto -\infty$.

For $\theta \mapsto 0$, we can use the same method as the discrete model (l'Hôpital's rule in terms of θ) to obtain a similar result:

$$\lim_{\theta \mapsto 0} \ln V(X) = \lim_{\theta \mapsto 0} \frac{\int_X c(x)^\theta \ln c(x) d\mathcal{F}(x)}{\int_X c(x)^\theta d\mathcal{F}(x)} = \frac{\int_X \ln c(x) d\mathcal{F}(x)}{\int_X d\mathcal{F}(x)}. \quad (26)$$

In particular, if $\mathcal{F}(x)$ is not truncated by X , we can find

$$\int_X d\mathcal{F}(x) = 1 \quad \Rightarrow \quad \lim_{\theta \mapsto 0} \ln V(X) = \int_X \ln c(x) d\mathcal{F}(x).$$

If X truncates the distribution, we will get $\int_X d\mathcal{F}(x) \neq 1$. However, we can get an equivalent result by defining the truncated distribution function appropriately. For example, letting $G(x)$ be the truncated cumulative distribution function with $G'(x) = g(x)$, we define

$$G(x) \equiv \frac{G(x)}{\int_X d\mathcal{F}(y) dy} \quad \Rightarrow \quad g(x) \equiv \frac{f(x)}{\int_X d\mathcal{F}(y) dy}. \quad (27)$$

Whence, we obtain

$$\lim_{\theta \mapsto 0} \ln V(X) = \int_X \frac{f(x)}{\int_X d\mathcal{F}(y) dy} \ln c(x) dx \equiv \int_X \ln c(x) dG(x). \quad (28)$$

For $\theta \mapsto -\infty$, we need some technical manipulations. By definition, as $V(X) \equiv \bar{c}$ if $c(x) \equiv \bar{c}$ for all x , we have

$$\inf c(x) \leq \left(\int_X c(x)^\theta d\mathcal{F}(x) \right)^{\frac{1}{\theta}} \leq \sup c(x). \quad (29)$$

Keeping this inequality in mind, consider the first-order derivative of $V(X)$ with respect to θ :

$$\frac{\partial V}{\partial \theta} = \left(\int_X c(x)^{\theta-1} d\mathcal{F}(x) \right) \left(\int_X c(x)^\theta d\mathcal{F}(x) \right)^{\frac{1-\theta}{\theta}} > 0. \quad (30)$$

This inequality condition (30) indicates $V(X)$ reaches the infimum of $c(x)$ indicated by (29) for $\theta \mapsto -\infty$ (Leontief case).

To ensure the convergence of the CES to the Leontief for $\theta \mapsto -\infty$, we consider if $V(X) \mapsto \sup c(x) \in [\inf c(x), +\infty]$. Again by $V(X) \equiv \bar{c}$ if $c(x) \equiv \bar{c}$ for all x , $V(X) \mapsto \sup c(x)$ if and only if $\inf c(x) = \sup c(x)$, which means the value of F

is determined by the infimum of $c(x)$.⁶ Therefore, by (29), (30), and the squeezing principle, the CES function approaches the corresponding Leontief function:⁷

$$\lim_{\theta \rightarrow -\infty} V(X) = \inf c(x). \quad (31)$$

4 Concluding Comments

Including well-known facts in the discrete case, calculations in the previous sections are summarized as follows.

$$\left(\sum_{i=1}^N \alpha_i X_i^\theta \right)^{\frac{1}{\theta}} \mapsto \begin{cases} \prod_{i=0}^N X_i^{\alpha_i} & (\theta \mapsto 0), \\ \min \{X_1, \dots, X_N\} & (\theta \mapsto -\infty), \end{cases} \quad (32)$$

$$\left(\int_X c(x)^\theta d\mathcal{F}(x) \right)^{\frac{1}{\theta}} \mapsto \begin{cases} \exp \left(\int_X \ln c(x) d\mathcal{F}(x) \right) & (\theta \mapsto 0), \\ \inf c(x) & (\theta \mapsto -\infty). \end{cases} \quad (33)$$

For the Leontief cases, according to the results of our computations, we can see that we *cannot* obtain the form such as

$$\min \{\beta_1 X_1, \dots, \beta_N X_N\} \quad (34)$$

or the corresponding continuum expression as limits of CES functions unless we redefine the corresponding distribution functions: for example, in the form β_i^θ instead of α_i in the discrete case. This procedure corresponds to altering the measurement unit of each object and it is interesting to discuss the reason why we cannot obtain (34) without such a modification.

Now suppose we redefine the measurement of X to get expression (34). Then, in the discrete case, we have

$$F(\beta X) = \left(\sum_{i=1}^N \alpha_i (\beta_i X_i)^\theta \right)^{\frac{1}{\theta}}, \quad (35)$$

which is also a known variation form if $\beta_i = \alpha_i$. In this case, following the same procedure as we have seen in the previous sections, we will obtain

$$F(\beta X) \mapsto \begin{cases} \mathcal{B} \times \prod_{i=0}^N X_i^{\alpha_i} & (\theta \mapsto 0), \\ \min \{\beta_1 X_1, \dots, \beta_N X_N\} & (\theta \mapsto -\infty), \end{cases} \quad (36)$$

⁶This argument is not needed if $c(x)$ is finite, but it is not generally guaranteed.

⁷Notice inequalities (29) and (30) just tell $V(X) \rightarrow \sup c(x)$ for $\theta \rightarrow +\infty$, but θ must be less than or equal to one; therefore, it does not mean $V(X)$ reaches the supremum, $\sup c(x)$, when it is linear because the linear case is attained by $\theta = 1$.

where $\mathcal{B}$ is defined by $\mathcal{B} = \prod_{i=1}^N \beta_i^{\alpha_i}$ that is derived from $F(\beta X) \mapsto \prod_{i=1}^N (\beta_i X_i)^{\alpha_i} = \prod_{i=1}^N \beta_i^{\alpha_i} \times \prod_{i=1}^N X_i^{\alpha_i}$ for $\theta \mapsto 0$. Suppose $\beta_i \neq 1$ for some i . Then, as a benchmark case, we consider $\mathcal{B} = 1$. So long as there is no restriction on $\beta_i > 0$, $\mathcal{B}$ solves

$$\ln \mathcal{B} = \sum_{i=1}^N \alpha_i \ln \beta_i = 0, \quad (37)$$

which requires some β_i to be larger than one and the other β_i to be less than one as $\alpha_i > 0$. This result provides the following two final remarks.

Remark 1 Suppose $\mathcal{B} = 1$. Then $\prod_{i=0}^N X_i^{\alpha_i} \mapsto \min \{\beta_1 X_1, \dots, \beta_N X_N\}$ for $\theta \mapsto -\infty$ if and only if β_i solves (37). In such cases, we will find $\beta_i \geq 1$ for some i and $\beta_i \in (0, 1)$ for the other i .

Proof. It is obvious from (37) and $\alpha_i > 0$. ■

Remark 2 Suppose $\mathcal{B} = 1$. Then $\prod_{i=0}^N X_i^{\alpha_i} \mapsto \min \{\beta_1 X_1, \dots, \beta_N X_N\}$ implies $\beta_i \neq \alpha_i$ if the Cobb-Douglas form is homogeneous of degree one.

Proof. $\beta_i = \alpha_i$ implies $\beta_i \in (0, 1)$ and $\ln \mathcal{B} < 0$ (or $0 < \mathcal{B} < 1$) if the Cobb-Douglas function is homogeneous of degree one (contradiction to Remark 1). Therefore $\beta_i \neq \alpha_i$ must apply if the Cobb-Douglas function with $\mathcal{B} = 1$ is homogeneous of degree one. ■

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