Let $G$ be a finite group, let a prime $p$ divide $|G|$, $|G| = p^\alpha \cdot n$, where $p \nmid n$. By Sylow’s theorem, $G$ has a subgroup $P$ of order $p^\alpha$; $P$ is called a Sylow $p$-subgroup of $G$. By the same theorem, all maximal $p$-subgroups are conjugate in $G$ (and so they are Sylow subgroups) and their number is congruent to 1 (mod $p$). The structure of finite groups is closely related to the properties and the embedding their Sylow $p$-subgroups, and there are many problems which can be solved only by a detailed examination of the relevant $p$-groups. The history of finite group theory shows that the investigation of $p$-subgroups is one of its most powerful methods. Finite $p$-groups are ideal objects for combinatorial and cohomological investigations.

In the study of finite $p$-groups the main difficulty lies in the fact that the number of such groups is very large; for example, there are exactly 267 nonisomorphic groups of order $2^6$ (P. Hall and J. Senior; see [HS]), 2328 groups of order $2^7$, 56092 groups of order $2^8$ (E. A. O’Brien), 10494213 groups of order $2^9$ (B. Eick and E. A. O’Brien), 49487365422 groups of order $2^{10}$ (H. U. Besche, B. Eick and E. A. O’Brien), 504 groups of order $3^6$. Therefore, it is very difficult to find nontrivial properties of almost all $p$-groups. (The following results are most general properties of $p$-groups: nilpotence, monomiality, Burnside’s basis theorem, counting theorems of Sylow, Miller and Kulakoff.) So it is natural to seek common properties for sufficiently large sets of $p$-groups.

In this elementary book we will prove a number of deep theorems on finite $p$-groups.

Some basic properties of finite $p$-groups were proved by Sylow, Frobenius and Burnside. But namely Philip Hall (1904–1982) laid the foundations of modern $p$-group theory in three fundamental papers [Hal1, Hal2, Hal3] (note that these are the only his papers devoted to finite $p$-groups; at the time of publication of the last of these papers he was 36). I consider these papers as No’s 1, 2 and 3 in finite $p$-group theory. The first of these papers presents the detailed investigation of regular $p$-groups, a wide subclass of $p$-groups in the sense that for given $n$ there are a finite number of primes $p$ such that there exist irregular $p$-groups of order $p^n$. In the second paper Hall proves a number of deep properties of irregular $p$-groups and establishes some strong regularity criteria. All modern approaches to classification of finite $p$-groups are based on the third Hall’s paper. Thus, Hall transformed $p$-group theory from a collection of miscellaneous facts and results into an organized field and the very essential part of finite group theory.

Undoubtedly, Norman Blackburn’s papers [Bla3] and [Bla5] are the most outstanding achievements in $p$-group theory after Hall. In those papers Blackburn studied
very important $p$-groups of maximal class, which, as a rule, are irregular, and deduced from obtained results a number of deep properties of irregular $p$-groups. He also investigated and characterized a number of important classes of $p$-groups. Most of investigations in post Hall era of finite $p$-group theory develop Blackburn’s ideas.

Avinoam Mann in his important paper [Man6] simplified the proofs of a number of basic theorems concerning $p$-groups of maximal class and also contributed significantly in the theory of regular $p$-groups in his subsequent papers (see [Man2–Man5]) and obtained a lot of important results about characters of $p$-groups (see [Man12]). Next, in the seminal paper [LubM], he and Lubotzky introduced and studied so called powerful $p$-groups – the class groups which obtained wide applications in finite $p$-group theory and in the theory of analytic pro-$p$-groups.

Recently, after 2000, Zvonimir Janko wrote a long series important papers (see Bibliography) devoted, in most cases, to 2-groups which I consider as highest achievements in the theory of 2-groups, the most prosperous part of $p$-group theory. Exposition of main results of these authors is the main aim of the present book.

We prove a number of important properties of regular $p$-groups and $p$-groups of maximal class; our exposition is based on mentioned above papers of Hall, Blackburn, and Mann. These results are central. Next, we prove almost all significant counting theorems that are known up to date; see §§1, 5, 10, 12, 13, 17–19, 36, 37 (the book contains all the material which is necessary to understand the proofs of those theorems). In some places we use §§4.3, 4.4 of Suzuki’s brilliant book [Suz] essentially (see §§7, 29, and Appendix 1 and Appendix 18 in Volume 2). Most proofs are new. We assume that the reader is familiar with the basic facts of finite group theory and character theory. To do the book more or less intelligible even to non-specialists, we omitted the proof of such difficult result as Zassenhaus commutator identity; for its proof see [Hup, §3.9]. Apart of this, the book is self-contained. The Hall–Petrescu identity is presented in Appendix 1 and its proof is taken from [Suz, §4.3]. Since our book is entirely elementary, such notions as varieties and Lie algebras do not appear in what follows. Hence, we have to omit a number of important topics. We did not intend to give an encyclopedic exposition of the subject.

We omit or consider briefly a number of important topics such as Sylow $p$-subgroups of important groups, coclass and breadth of $p$-groups, the Burnside problem and so on. The forthcoming book “Finite $p$-Groups” by A. Mann must fill some of these gaps. Our consideration of the Schur multiplier is fairly fragmental (we recommend to the interested reader Karpilovsky’s book [Kar], giving the encyclopedic presentation of this matter; for more elementary exposition see [BZ, Chapter 6]). There is an excellent exposition of $p$-group theory in Chapters 3 and 8 of three volume book by Huppert and Blackburn. However all mentioned books have small intersections with our one and not so elementary.

To the reader who has some knowledge of the rudiments of combinatorics, finite groups, elementary algebra and number theory, this book should present no difficulty.

All groups in this book are finite and, as a rule, have prime power order.
Introduction and Sec. 1 contain some preparatory material. The starting point for many of our considerations is fundamental Frobenius’ theorem on the number of solutions to $x^n = 1$ in a group (the nice proof of this theorem, due to I.M. Isaacs and G.R. Robinson [IsR], is presented). The important Fitting’s lemma on the class of product of two normal nilpotent subgroups is proved. In Sec. 1 we prove a number of results which important in what follows (Lemma 1.1, Theorems 1.2, 1.10, 1.17 and so on). In Sec. 2, the nice proof of Hall’s expression for the class number of a $p$-group, due to Mann, is presented. In Sec. 4 the description of $p$-groups with cyclic Frattini subgroup is given. The proof is based essentially on counting theorems from Sec. 1.

In Sec. 5 a number of elementary counting theorems for $p$-groups is proved. As a rule, the proofs are based on Hall’s enumeration principle – Theorem 5.2 or his variants. We present enumeration principle free proofs of some strong counting theorems. We also present the new enumeration principle and prove with its help nice Y. Fan’s result [Fan] (see Theorem 5.17). Most of the material of this section appears in the book form at the first time. In Sec. 6, we prove theorems like Maschke’s. Namely, we show that if a $p'$-group $X$ acts on the abelian $p$-group $P$ and $R$ is an $X$-invariant subgroup of exponent $p$ in $P$, then $P = S \times S_1$, where $S, S_1$ are $X$-invariant and $\Omega_1(S) = R$. In Sec. 7 some basic facts on regular $p$-groups are proven. As we have noticed, our exposition follows closely to [Suz, Chapter 4] and Mann [Man2–5].

In Sec. 9 the basic properties of $p$-groups of maximal class are proved. A number of proofs is due to Mann. The main result of Sec. 10 is Theorem 10.1, a natural generalization of Alperin’s theorem on centralizers of normal abelian subgroups [Alp3]. As a consequence, we prove assertions on the number of elementary abelian subgroups of orders $p^3$ and $p^4$ in $p$-groups of odd order; for further results and another approach, see [KonJ, JonK]. The main results of Sec. 11 are due to Mann; the power structure of $p$-groups is investigated there in detail. The $p$-groups satisfying certain of the basic properties of regular groups, are studied. On this way, some interesting new criteria of regularity are proven. These results we do not use in what follows (however, groups introduced in that section, play important role in §88).

In Sec. 12 we prove a number of counting theorems for $p$-groups of maximal class. However, the most important result of that section is Blackburn’s fundamental Theorem 12.1 on $p$-groups without normal subgroups of order $p^p$ and exponent $p$. In Sec. 13 most important counting theorems are presented (Theorems 13.2, 13.5 and 13.6). In Sec. 13 we prove a number of counting theorems of new type (see, for example, Theorem 13.18).

In Sec. 15 we prove nice theorems of Thompson and Mann. For example, Thompson’s theorem asserts, that if $G$ is a $p$-group, $p > 2$, such that $\Omega_1(G) \leq Z(G)$, then $d(G) \leq d(\Omega_1(G))$. According to theorem of Mann, if $G$ is a 2-group such that $\Omega_2(G) \leq Z(G)$, then $d(G) \leq d(\Omega_2(G))$.

In Sec. 16 the $p$-groups all of whose nonnormal subgroups are cyclic, are classified. A number of counting theorems is proved in Sec. 17, 18.
In Sec. 24 we offer a new proof of Hall’s theorem on Hall chains in normal subgroups (further development of this theme see in Sec. 88 from Volume 2).

In Sec. 26 we prove basic properties of powerful \( p \)-groups.

In Sec. 31 we consider the \( p \)-groups with small \( p' \)-groups of operators. Sec. 32–34 are devoted to automorphisms of \( p \)-groups.

In Sec. 36 we offer short proofs of a number of classical characterization theorems (most of them were proved in previous sections). Part of these proofs is based on Blackburns characterization of metacyclic \( p \)-groups (see Theorem 36.1).

In Appendix 1, we prove the result of fundamental importance — the Hall–Petrescu formula. Other appendices supplement the main text.

More detailed information about topics considered in the book, see in Contents and Subject Index.

We inserted in the text numerous exercises of varied degrees of difficulty, and they constitute its essential part. Many of them are given with hints or full solutions. Some exercises are, in fact, theorems or open questions.

Both parts of the book are concluded by comprehensive lists of problems which I began to write more than 25 years ago. At least 50 problems from these lists were solved mainly, by Janko (almost all his solutions are presented in Volume 2). There are two comprehensive lists of unsolved problems with interesting comments published by Mann [Man20] and Shalev [Sha5]. These lists and our one do not overlap.

The bibliography includes a number of the most important papers devoted to topics considered in this book. For more comprehensive bibliography, see Internet (in particular, MathSciNet).

The list of most important notations and definitions follows Contents. Our definitions and notations, as a rule, are standard.

I am indebted to Avinoam Mann for numerous useful discussions and help. The correspondence with Martin Isaacs allowed me to acquaint the reader with a number of his old and new results. Moreover, Mann and Isaacs familiarized me with a number of their papers prior of publication. I am especially indebted to Zvonimir Janko and Noboru Ito. Janko helped me generously in checking the whole text (some places he read many times); he also wrote the Foreword, Sections 16, 27, 28, 35, subsection 2° in Section 26, Theorem 34.8 and Appendices 12, 14. Note that Janko is a coauthor of Volume 2 of this book. Ito carefully read the whole this part and all appendices from Volume 2 and made a lot of corrections and useful suggestions. Lev Kazarin helped me with Sec. 22 (and also with Sec. 46, 63 and 65 from Volume 2). The first hundred pages of this Volume were read by M. Y. Xu (Beijing University), and he sent me the lists of misprints and suggestions. I am also indebted to Gregory Freiman, Marcel Herzog (both at Tel-Aviv University), Moshe Roitman, who also wrote Appendix 7, and Izu Vaisman (both at University of Haifa) for help and support.

I dedicate this volume to the memory of my parents Sarah (1916–1983) and Gilya (1911–1999) and my friends Grisha Karpilovsky (1940–1997) and Emanuel Zhmud (1918–2007).