List of definitions and notations

Set theory

$|M|$ is the cardinality of a set $M$ (if $G$ is a finite group, then $|G|$ is called its order).

$x \in M \ (x \notin M)$ means that $x$ is (is not) an element of a set $M$. $N \subseteq M \ (N \not\subseteq M)$ means that $N$ is (is not) a subset of the set $M$; moreover, if $M \neq N \subseteq M$ we write $N \subset M$.

$\emptyset$ is the empty set.

$N$ is called a nontrivial subset of $M$, if $N \neq \emptyset$ and $N \subset M$. If $N \subset M$ we say that $N$ is a proper subset of $M$.

$M \cap N$ is the intersection and $M \cup N$ is the union of sets $M$ and $N$. If $M, N$ are sets, then $N - M = \{x \in N \mid x \notin M\}$ is the difference of $N$ and $M$.

$\mathbb{Z}$ is the set (ring) of integers: \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \).

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{Q}$ is the set (field) of all rational numbers.

$\mathbb{R}$ is the set (field) of all real numbers.

$\mathbb{C}$ is the set (field) of all complex numbers.

Number theory and general algebra

$p$ is always a prime number.

$\pi$ is a set of primes; $\pi'$ is the set of all primes not contained in $\pi$.

$m, n, k, r, s$ are, as a rule, natural numbers.

$\pi(m)$ is the set of prime divisors of $m$; then $m$ is a $\pi$-number.

$n_p$ is the $p$-part of $n$, $n_\pi$ is the $\pi$-part of $n$.

$(m, n)$ is the greatest common divisor of $m$ and $n$.

$m \mid n$ should be read as: $m$ divides $n$.

$m \nmid n$ should be read as: $m$ does not divide $n$. 
GF($p^m$) is the finite field containing $p^m$ elements.

$\mathbb{F}^*$ is the multiplicative group of a field $\mathbb{F}$.

$L(G)$ is the lattice of all subgroups of a group $G$.

If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the standard prime decomposition of $n$, then $\lambda(n) = \sum_{i=1}^{k} \alpha_i$.

**Groups**

We consider only finite groups which are denoted, with a pair exceptions, by upper case Latin letters.

If $G$ is a group, then $\pi(G) = \pi(|G|)$.

$G$ is a $p$-group if $|G|$ is a power of $p$; $G$ is a $\pi$-group if $\pi(G) \subseteq \pi$.

$G$ is, as a rule, a finite $p$-group.

$H \leq G$ means that $H$ is a subgroup of $G$.

$H < G$ means that $H \leq G$ and $H \neq G$ (in that case $H$ is called a proper subgroup of $G$). $\{1\}$ denotes the group containing only one element.

$H$ is a nontrivial subgroup of $G$ if $\{1\} < H < G$.

$H$ is a maximal subgroup of $G$ if $H < G$ and it follows from $H \leq M < G$ that $H = M$.

$H \trianglelefteq G$ means that $H$ is a normal subgroup of $G$; moreover, if, in addition, $H \neq G$ we write $H \vartriangleleft G$ and say that $H$ is a proper normal subgroup of $G$. Expressions ‘normal subgroup of $G$’ and ‘$G$-invariant subgroup’ are synonyms.

$H \vartriangleleft G$ is called a nontrivial normal subgroup of $G$ provided $H > \{1\}$.

$H$ is a minimal normal subgroup of $G$ if (a) $H \trianglelefteq G$; (b) $H > \{1\}$; (c) $N \triangleleft G$ and $N < H$ implies $N = \{1\}$. Thus, the group $\{1\}$ has no minimal normal subgroup.

$G$ is simple if it is a minimal normal subgroup of $G$ (so $|G| > 1$).

$H$ is a maximal normal subgroup of $G$ if $H < G$ and $G/H$ is simple.

The subgroup generated by all minimal normal subgroups of $G$ is called the socle of $G$ and denoted by $\mathrm{Sc}(G)$. We put, by definition, $\mathrm{Sc}(\{1\}) = \{1\}$.

$N_G(M) = \{ x \in G \mid x^{-1} M x = M \}$ is the normalizer of a subset $M$ in $G$.

$C_G(x)$ is the centralizer of an element $x$ in $G$ : $C_G(x) = \{ z \in G \mid zx = xz \}$.

$C_G(M) = \bigcap_{x \in M} C_G(x)$ is the centralizer of a subset $M$ in $G$.

If $A \leq B$ and $A, B \trianglelefteq G$, then $C_G(B/A) = H$, where $H/A = C_{G/A}(B/A)$. 
$A$ wr $B$ is the wreath product of the ‘passive’ group $A$ and the transitive permutation group $B$ (in what follows we assume that $B$ is regular); $B$ is called the active factor of the wreath product). Then the order of that group is $|A|^{|B|}$. 

$\text{Aut}(G)$ is the group of automorphisms of $G$ (the automorphism group of $G$).

$\text{Inn}(G)$ is the group of all inner automorphisms of $G$.

$\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$, the outer automorphism group of $G$.

If $a, b \in G$, then $a^b = b^{-1}ab$.

An element $x \in G$ inverts a subgroup $H \leq G$ if $h^x = h^{-1}$ for all $h \in H$.

If $M \leq G$, then $\langle M \rangle = \langle x \mid x \in M \rangle$ is the subgroup of $G$ generated by $M$.

$M^x = x^{-1}Mx = \{y^x \mid y \in M\}$ for $x \in G$ and $M \leq G$.

$[x, y] = x^{-1}y^{-1}xy = x^{-1}y^x$ is the commutator of elements $x, y$ of $G$. If $M, N \leq G$ then $[M, N] = \langle [x, y] \mid x \in M, y \in N \rangle$ is a subgroup of $G$.

$o(x)$ is the order of an element $x$ of $G$.

An element $x \in G$ is a $\pi$-element if $\pi(o(x)) \subseteq \pi$.

$G$ is a $\pi$-group, if $\pi(G) \subseteq \pi$. Obviously, $G$ is a $\pi$-group if and only if all of its elements are $\pi$-elements.

$G'$ is the subgroup generated by all commutators $[x, y], x, y \in G$ (i.e., $G' = [G, G]$), $G^{(2)} = [G', G'] = G'' = (G')', G^{(3)} = [G'', G''] = (G'')'$ and so on. $G'$ is called the commutator (or derived) subgroup of $G$.

$Z(G) = \bigcap_{x \in G} C_G(x)$ is the center of $G$.

$Z_i(G)$ is the $i$-th member of the upper central series of $G$; in particular, $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$.

$K_i(G)$ is the $i$-th member of the lower central series of $G$; in particular, $K_2(G) = G'$. We have $K_i(G) = [G, \ldots, G]$ ($i \geq 1$ times). We set $K_1(G) = G$.

If $G$ is nonabelian, then $\eta(G)/K_3(G) = Z(G/K_3(G))$.

$\mathcal{M}(G) = \langle x \in G \mid C_G(x) = C_G(x^p) \rangle$ is the Mann subgroup of a $p$-group $G$.

$\text{Syl}_p(G)$ is the set of $p$-Sylow subgroups of an arbitrary finite group $G$.

$S_n$ is the symmetric group of degree $n$.

$A_n$ is the alternating group of degree $n$.

$\Sigma_p^n$ is a Sylow $p$-subgroup of $S_p^n$.

$\text{GL}(n, F)$ is the set of all nonsingular $n \times n$ matrices with entries in a field $F$, the $n$-dimensional general linear group over $F$, $\text{SL}(n, F) = \{A \in \text{GL}(n, F) \mid \det(A) = 1 \in F\}$, the $n$-dimensional special linear group over $F$. 
If $H \leq G$, then $H_G = \bigcap_{x \in G} x^{-1} H x$ is the core of the subgroup $H$ in $G$ and $H^G = \bigcap_{H \leq N \leq G} N$ is the normal closure or normal hull of $H$ in $G$. Obviously, $H_G \leq G$.

If $G$ is a $p$-group, then $p^{b(x)} = |G : C_G(x)|$; $b(x)$ is said to be the breadth of $x \in G$, where $G$ is a $p$-group; $b(G) = \max\{b(x) \mid x \in G\}$ is the breadth of $G$.

$\Phi(G)$ is the Frattini subgroup of $G$ (= the intersection of all maximal subgroups of $G$), $\Phi(\{1\}) = \{1\}$, $p^{d(G)} = |G : \Phi(G)|$.

$\Gamma_i = \{H < G \mid \Phi(G) \leq H, |G : H| = p^i\}, i = 1, \ldots, d(G)$, where $G > \{1\}$.

If $H < G$, then $\Gamma_1(H)$ is the set of all maximal subgroups of $H$.

$\exp(G)$ is the exponent of $G$ (the least common multiple of the orders of elements of $G$). If $G$ is a $p$-group, then $\exp(G) = \max\{o(x) \mid x \in G\}$.

$k(G)$ is the number of conjugacy classes of $G (= G$-classes$)$, the class number of $G$.

$K_x$ is the $G$-class containing an element $x$ (sometimes we also write $ccl_G(x)$).

$C_m$ is the cyclic group of order $m$.

$G^m$ is the direct product of $m$ copies of a group $G$.

$A \times B$ is the direct product of groups $A$ and $B$.

$A \ast B$ is a central product of groups $A$ and $B$, i.e., $A \ast B = AB$ with $[A, B] = \{1\}$.

$E_{p^m} = C_p^m$ is the elementary abelian group of order $p^m$. $G$ is an elementary abelian $p$-group if and only if it is a $p$-group $> \{1\}$ and $G$ coincides with its socle. Next, $\{1\}$ is elementary abelian for each prime $p$.

A group $G$ is said to be homocyclic if it is a direct product of isomorphic cyclic subgroups (obviously, elementary abelian $p$-groups are homocyclic).

$ES(m, p)$ is an extraspecial group of order $p^{1+2m}$ (a $p$-group $G$ is said to be extraspecial if $G' = \Phi(G) = Z(G)$ is of order $p$). Note that for each $m \in \mathbb{N}$, there are exactly two nonisomorphic extraspecial groups of order $p^{2m+1}$.

$S(p^3)$ is a nonabelian group of order $p^3$ and exponent $p > 2$.

A special $p$-group is a nonabelian $p$-group $G$ such that $G' = \Phi(G) = Z(G)$ is elementary abelian. Direct products of extraspecial $p$-groups are special.

$D_{2m}$ is the dihedral group of order $2m$, $m > 2$. Some authors consider $E_{2^2}$ as the dihedral group $D_4$.

$Q_{2^m}$ is the generalized quaternion group of order $2^m \geq 2^3$.

$SD_{2^m}$ is the semidihedral group of order $2^m \geq 2^4$.

$M_{p^m}$ is a nonabelian $p$-group containing exactly $p$ cyclic subgroups of index $p$. 
cl(G) is the \textit{nilpotence class} of a \( p \)-group \( G \).

dl(G) is the \textit{derived length} of a \( p \)-group \( G \).

CL(G) is the set of all \( G \)-classes.

A \( p \)-group of \textit{maximal class} is a nonabelian group \( G \) of order \( p^m \) with \( \text{cl}(G) = m - 1 \).

\( \Omega_m(G) = \{ x \in G \mid o(x) \leq p^m \} \), \( \Omega'_m(G) = \{ x \in G \mid o(x) = p^m \} \) and \( \mathcal{U}_m(G) = \langle x^{p^m} \mid x \in G \rangle \).

A \( p \)-group is absolutely regular if \( |G/\mathcal{U}_1(G)| < p^p \).

A \( p \)-group is \textit{thin} if it is either absolutely regular or of maximal class.

\( G = A \cdot B \) is a \textit{semidirect product} with kernel \( B \) and complement \( A \).

A group \( G \) is an extension of \( N \trianglelefteq G \) by a group \( H \) if \( G/N \cong H \). A group \( G \) splits over \( N \) if \( G = H \cdot N \) with \( H \leq G \) and \( H \cap N = \{1\} \) (in that case, \( G \) is a semidirect product of \( H \) and \( N \) with kernel \( N \)).

\( H^\# = H - \{ e_H \} \), where \( e_H \) is the identity element of the group \( H \). If \( M \subseteq G \), then \( M^\# = M - \{ e_G \} \).

An automorphism \( \alpha \) of \( G \) is \textit{regular} (= \textit{fixed-point-free}) if it induces a regular permutation on \( G^\# \) (a permutation is said to be \textit{regular} if it has no fixed points).

An \textit{involution} is an element of order 2 in a group.

A \textit{section} of a group \( G \) is an epimorphic image of some subgroup of \( G \).

If \( F = \text{GF}(p^n) \), then we write \( \text{GL}(m, p^n), \text{SL}(m, p^n), \ldots \) instead of \( \text{GL}(m, F), \text{SL}(m, F), \ldots \).

c\( _n(G) \) is the number of cyclic subgroups of order \( p^n \) in a \( p \)-group \( G \).

s\( _n(G) \) is the number of subgroups of order \( p^n \) in a \( p \)-group \( G \).

e\( _n(G) \) is the number of subgroups of order \( p^n \) and exponent \( p \) in \( G \).

\( \mathcal{A}_n \)-group is a \( p \)-group \( G \) all of whose subgroups of index \( p^n \) are abelian but \( G \) contains a nonabelian subgroup of index \( p^{n-1} \). In particular, \( \mathcal{A}_1 \)-group is a minimal nonabelian \( p \)-group for some \( p \).

\( \alpha_n(G) \) is the number of \( \mathcal{A}_n \)-subgroups in a \( p \)-group \( G \).

\textbf{Characters and representations}

\( \text{Irr}(G) \) is the set of all \textit{irreducible} characters of \( G \) over \( \mathbb{C} \).

A character of degree 1 is said to be \textit{linear}.

\( \text{Lin}(G) \) is the set of all \textit{linear} characters of \( G \) (obviously, \( \text{Lin}(G) \subseteq \text{Irr}(G) \)).
Irr_1(G) = Irr(G) - Lin(G) is the set of all *nonlinear* irreducible characters of G; 
n(G) = |Irr_1(G)|.

χ(1) is the *degree* of a character χ of G,

χ_H is the *restriction* of a character χ of G to H ≤ G.

χ_G is the character of G induced from the character χ of some subgroup of G.

̅χ is a character of G defined as follows: ̅χ(x) = ̅χ(x) (here ̅w is the complex conjugate of w ∈ C).

Irr(χ) is the set of irreducible constituents of a character χ of G.

If χ is a character of G, then ker(χ) = \{x ∈ G | χ(x) = χ(1)\} is the *kernel* of a character χ.

Z(χ) = \{x ∈ G | |χ(x)| = χ(1)\} is the *quasikernel* of χ.

If N ≤ G, then Irr(G | N) = {χ ∈ Irr(G) | N ∼ ker(χ)}.

⟨χ, τ⟩ = |G|^{-1} ∑_{x ∈ G} χ(x)τ(x^{-1}) is the *inner product* of characters χ and τ of G.

I_G(ϕ) = \{x ∈ G | ϕ^x = ϕ\} is the *inertia subgroup* of ϕ ∈ Irr(H) in G, where H ≤ G.

1_G is the *principal character* of G (1_G(x) = 1 for all x ∈ G).

M(G) is the *Schur multiplier* of G.

cd(G) = {χ(1) | χ ∈ Irr(G)}.

mc(G) = k(G)/|G| is the *measure of commutativity* of G.

T(G) = ∑_{χ ∈ Irr(G)} χ(1),  f(G) = T(G)/|G|. 