List of frequently met concepts and notations

Set theory

- $|M|$ is the cardinality of a set $M$ (if $G$ is a group, then $|G|$ is called the order of $G$).
- $x \in M$ means that $x$ is an element of $M$. $N \subseteq M$ means that $N$ is a subset of $M$; if $N \neq M$, we write $N \subset M$.
- $\emptyset$ is the empty set.
- $N$ is called a nontrivial subset of $M$, if $N \neq \emptyset$ and $N \subseteq M$. If $N \subset M$, we say that $N$ is a proper subset of $M$.
- $M \cap N$ is the intersection and $M \cup N$ is the union of sets $M$ and $N$. If $M, N$ are sets, then $N - M$ is the difference of $N$ and $M$.
- $\mathbb{C}$ is the set (field) of complex numbers.
- $\mathbb{R}$ is the set (field) of real numbers.
- $\mathbb{Q}$ is the set (field) of rational numbers.
- $\mathbb{Z}$ is the set (ring) of integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$.
- $\mathbb{N}$ is the set of natural numbers.

Number theory and general algebra

- $p$ is always a prime number.
- $m, n$ are always natural numbers.
- $\text{GCD}(m, n)$ is the greatest common divisor of $m$ and $n$.
- $m \mid n$ should be read as follows: $m$ divides $n$.
- $\pi(m)$ is the set of all prime divisors of $m$.
- $\pi$ is a set of primes (it may be the empty set).
- $\pi'$ is the set of primes not contained in $\pi$.
- $m_{\pi}$ is the number satisfying the following three conditions: $\pi(m_{\pi}) \subset \pi$, $m_{\pi} \mid m$, $\pi(m/m_{\pi}) \subset \pi'$.
- We write $m_p, p'$ instead of $m_{\{p\}}, \{p'\}$, respectively.
- $m$ is a $\pi$-number if $\pi(m) \subseteq \pi$ (or $m_{\pi} = m$).
- $\mathbb{GF}(p^m)$ is the finite field containing $p^m$ elements.
- $\mathbb{F}^*$ is the multiplicative group of a field $\mathbb{F}$.
- $\mathbb{F}^n$ is the $n$-dimensional vector space over $\mathbb{F}$.
- $\mathbb{F}_n$ is the set of all $n \times n$ matrices over $\mathbb{F}$.
- If $A$ is a square matrix, then det $A$ and tr $A$ are the determinant and the trace of $A$ (that is, the sum of elements on its principal diagonal), respectively.
- $I_n$ is the $n \times n$ identity matrix.
- $\overline{a}$ is the number conjugate to $a \in \mathbb{C}$.
- $[x]$ is the integer part of $x \in \mathbb{R}$.

https://doi.org/10.1515/9783110224092-202
Groups

- $G$ is always a finite group.
- $H \leq G$ means that $H$ is a subgroup of $G$.
- $H < G$ means that $H \subseteq G$ and $H \neq G$ (in this case $H$ is called a proper subgroup of $G$); 
  \{1\} denotes the group of order 1; $H$ is a nontrivial subgroup of $G$ if $\{1\} < H < G$.
- $H$ is a maximal subgroup of $G$ if $H < G$ and $H \leq M < G$ imply that $H = M$.
- $H \trianglelefteq G$ means that $H$ is a normal subgroup of $G$; moreover, if $H \neq G$, we write $H \lhd G$ and say that $H$ is a proper normal subgroup of $G$. $H \lhd G$ is called a nontrivial normal subgroup of $G$ if $|H| > 1$.
- $H$ is a minimal normal subgroup of $G$ if (a) $H \subseteq G$, (b) $H > \{1\}$, (c) $N \trianglelefteq G$ and $N < H$ imply $N = \{e\}$. Thus, $\{1\}$ has no minimal normal subgroups.
- $G$ is simple if it is a minimal normal subgroup of $G$ (in particular, $|G| > 1$).
- $H$ is a maximal normal subgroup of $G$ if $G/H$ is simple.
- $G$ is a monolith if $G = \{1\}$ or if $G$ contains only one minimal normal subgroup.
- The subgroup generated by all minimal normal subgroups of $G$ is called the socle of $G$ and is denoted by $\text{Sc}(G)$. One can represent $\text{Sc}(G)$ as the direct product of certain minimal normal subgroups of $G$. We put $\text{Sc}(\{1\}) = \{1\}$. Obviously, $\text{Sc}(G)$ is a characteristic subgroup of $G$.
- $N_G(M) = \{x \in G \mid x^{-1}Mx = M\}$ is the normalizer of a subset $M$ in $G$.
- $C_G(x)$ is the centralizer of an element $x$ in $G$: $C_G(x) = \{z \in G \mid zx = xz\}$.
- $C_G(M) = \bigcap_{x \in M} C_G(x)$ is the centralizer of a subset $M$ in $G$.
- $\text{Aut } G$ is the group of all automorphisms of $G$ (the automorphism group of $G$).
- $\text{Inn } G$ is the group of all inner automorphisms of $G$.
- $\text{Out } G = \text{Aut } G/\text{Inn } G$.
- $[x, y] = x^{-1}y^{-1}xy$ is the commutator of elements $x, y$ of $G$. If $M, N \subseteq G$, then $[M, N] = \langle [x, y] \mid x \in M, y \in N \rangle$. (However, we use $[M, N] = [[x, y] \mid x \in M, y \in N]$ in Chapter XI.)
- If $M \subseteq G$, then $\langle M \rangle$ is the subgroup of $G$ generated by $M$.
- $G'$ is the subgroup generated by all commutators $[x, y], x, y \in G$ (i.e., $G' = [G, G]$), $G'' = (G')'$, $G''' = (G'')'$ and so on.
- $Z(G) = \bigcap_{x \in G} C_G(x)$ is the center of $G$.
- $\Phi(G)$ is the Frattini subgroup of $G$ (the intersection of all maximal subgroups of $G$).
- $F(G)$ is the Fitting subgroup of $G$ (the maximal normal nilpotent subgroup of $G$).
- $S(G)$ is the solvable radical of $G$ (the maximal solvable normal subgroup of $G$).
- $\exp G$ is the exponent of $G$ (the least common multiple of the orders of the elements of $G$).
- $o(x)$ is the order of an element $x$ of $G$.
- $k(G)$ is the number of conjugacy classes of $G$ (i.e., $G = G$-classes), the class number of $G$.
- If $M \subseteq G$, then $k_G(M)$ is the number of $G$-classes containing elements of $M$.
- $\pi(G) = \pi(|G|)$. 
O_π(G) is the maximal normal π-subgroup of G, O(G) = O_{2'}(G) (obviously, one has O_p(G) ∈ Syl_p(F(G))).
O^π(G) is the subgroup generated by all π'-elements of G.
C_m is the cyclic group of order m.
A × B is the direct product of groups A and B.
A * B is a central product of groups A and B.
G^0 = \{1\}; G^m is the direct product of m copies of G.
E_{p^m} = (C_p)^m is the elementary abelian group of order p^m.
A group G is said to be homocyclic if it is a direct product of isomorphic cyclic subgroups (obviously, elementary abelian p-groups are homocyclic).
ES(m, p) is an extraspecial group of order p^{1+2m} (a p-group G is said to be extraspecial if G' = Φ(G) = Z(G) is of order p).
A special p-group is a nonabelian p-group G such that G' = Φ(G) = Z(G) is elementary abelian.
(A, B) is a Frobenius group with kernel B and Frobenius complement A (A and B do not determine (A, B) up to isomorphism).
D_{2m} is the dihedral group of order 2m, m > 2.
Q_{2m} is the generalized quaternion group of order 2^m > 2, m > 3.
SD_{2m} is the semidihedral group of order 2^m ≥ 2^4.
cl G is the nilpotency class of a p-group G.
CL G is the set of all G-classes.
A p-group of maximal class is a nonabelian group G of order p^m with cl G = m - 1.
If G is a p-group, then Ω_m(G) = \langle x ∈ G | x^{p^m} = 1 \rangle, and m.G = \langle x^m | x ∈ G \rangle.
Syl(G) is the set of all Sylow subgroups of G.
Syl_p(G) is the set of all Sylow p-subgroups of G.
H is a Hall subgroup of G if (|H|, |G : H|) = 1.
H is a π-Hall subgroup of G if |H| = |G|_π.
S_π is the symmetric group of degree n.
A_π is the alternating group of degree n.
GL(n, F) is the set of all nonsingular n × n matrices with entries in a field F, the general linear group over F.
SL(n, F) = \{A ∈ GL(n, F) | det A = 1 ∈ F\}, the special linear group over F.
PGL(m, F) = GL(n, F)/Z(GL(n, F)).
PSL(n, F) = SL(n, F)/Z(SL(n, F)).
AGL(n, F) is the natural linear group of F^n by GL(n, F), the affine general linear group.
Sz(2^m) is the simple Suzuki group, m > 1 being odd.
For H < G, H_G = \bigcap_{x ∈ G} x^{-1}Hx is called the core of the subgroup H in G. Obviously, H_G ⊆ G.
An element x ∈ G is a π-element if π(ο(x)) ⊆ π.
G is a π-group if π(G) ⊆ π. Obviously, G is a π-group if and only if all its elements are π-elements.
O^π(G) = \langle x ∈ G | π(ο(x)) ⊆ π' \rangle.
O^n(G) = O^n(O^n(G)).

A group $G$ is an extension of $N \subseteq G$ by a group $H$ if $G/N = H$. A group $G$ splits over $N$ if $G = HN$ with $H < G$ and $H \cap N = 1$ (in that case, $G$ is a semidirect product of $H$ and $N$ with kernel $N$).

A group $G$ is $p$-solvable if all indices of its composition series are equal to $p$ or are $p'$-numbers. A group $G$ is $\pi$-solvable if it is $p$-solvable for all $p \in \pi$.

A group $G$ is said to be $\pi$-separable if all indices of its composition series are $\pi$- or $\pi'$-numbers.

If $M \subseteq G$, $x \in G$, then $M^x = x^{-1}Mx = \{x^{-1}ax \mid a \in M\}$.

$H$ is a TI-subgroup of $G$ if $H \cap H^x = 1$ for all $x \in G - N_G(H)$; $M$ is a TI-subset of $G$ if $M \cap M^x \subseteq \{1\}$ for all $x \in G - N_G(M)$.

$H^\# = H - \{e_H\}$, where $e_H$ is the identity element of the group $H$. If $M \subseteq G$, then $M^\# = M - \{e_G\}$.

A permutation $\sigma$ of a set $M$ is regular if $\sigma(x) \neq x$ for all $x \in M$. An automorphism $a$ of $G$ is regular (= fixed-point free) if it induces a regular permutation on $G^\#$.

If $x, y \in G$, then the expression “$x \sim y$ in $G$” means that $x, y$ are conjugate in $G$. Similarly, “$M \sim N$ in $G$” means that the subsets $M, N$ are conjugate in $G$.

An involution is an element of order 2 in a group.

An element $x \in G$ is real if $x \sim x^{-1}$ in $G$. An element $x$ is rational if all generators of the subgroup $\langle x \rangle$ are conjugate in $G$. An involution is a real and rational element.

A section of a group $G$ is an epimorphic image of some subgroup of $G$.

A group $G$ is $p$-closed if $|\text{Syl}_p(G)| = 1$ (i.e., $O_p(G) \in \text{Syl}_p(G)$).

A group $G$ is $p$-nilpotent if it has a normal $p$-complement, i.e., a normal subgroup $H$ of order $|G|/p^r$.

An $S(p^a, q^b, q^c)$-group is a $q$-closed minimal non-nilpotent group $G$ of order $p^a q^{b+c}$ with $|Z(G)| = p^{a-1} q^c$ (see Chapter XI).

If $F = GF(p^a)$, then we write $GL(m, p^a)$, etc., instead of $GL(m, F)$, etc.

If $M \subseteq G$, then $M^G$ is the normal closure of $M$ in $G$.

### Characters and representations

- $F[G]$ is the set of all functions from $G$ to $\mathbb{C}$.
- $CF[G]$ is the set of all central (= class) functions from $G$ to $\mathbb{C}$.
- $\text{Char}(G)$ is the set of all complex characters of $G$. It is convenient to consider the zero function $O_{G \to \mathbb{C}}$ as an element of the set $\text{Char}(G)$.
- $\text{Irr}(G)$ is the set of all irreducible characters of $G$.
- A character of degree 1 is said to be linear.
- $\text{Lin}(G)$ is the set of all linear characters of $G$ (obviously, $\text{Irr}(G) \subseteq \text{Irr}(G)$).
- $\text{Irr}_1(G) = \text{Irr}(G) - \text{Lin}(G)$ is the set of all nonlinear irreducible characters of $G$; $n(G) = |\text{Irr}_1(G)|$ is the number of nonlinear irreducible characters of $G$. 

VIII Characters of Finite Groups 2
• A class function $\theta$ is said to be a generalized character of $G$ if $\theta = \chi_1 - \chi_2$, where $\chi_1, \chi_2 \in \text{Char}(G)$.
• $\text{Ch}(G)$ is the set of all generalized characters of $G$.
• If $\theta, \lambda \in F[G], x \in G$, then $(\theta \lambda)(x) = \theta(x)\lambda(x)$.
• $FG$ is the group algebra of $G$ over the field $F$.
• $\chi(1)$ is the degree of a character $\chi$ of $G$; $\deg T$ is the degree of a representation $T$ of $G$.
• If $\chi \in \text{Char}(G)$, $\phi \in \text{Char}(H)$, $H < G$, then $\chi_H$ is the restriction of $\chi$ to $H$, and $\phi^G$ is the induced character ($\phi^G \in \text{Char}(G)$).
• If $\vartheta, \psi \in \text{CF}[G]$, then $\langle \vartheta, \psi \rangle = |G|^{-1} \sum_{x \in G} \vartheta(x)\overline{\psi(x)}$ is the scalar (or inner) product of $\vartheta$ and $\psi$.
• If $H \leq G$, $\phi \in \text{Irr}(H)$, then $I_G(\phi) = \{ x \in G \mid \phi^x = \phi \}$ is the inertia group of $\phi$ in $G$ (where $\phi^x(h) = \phi(xhx^{-1})$ for $h \in H$).
• If $H \leq G$ and $\phi \in \text{CF}[H]$, then $\phi$ is the function in $\text{CF}[G]$ that coincides with $\phi$ on $H$ and vanishes on $G - H$.
• $1_G$ is the principal character of $G$ ($1_G(x) = 1$ for all $x \in G$).
• $\rho_G$ is the regular character of $G$.
• $\text{Irr}(\chi)$ is the set of all irreducible constituents of a character $\chi$ of $G$. Furthermore, $\text{Irr}_1(\chi) = \text{Irr}(\chi) \cap \text{Irr}_1(G)$. (The expression $\psi \in \text{Irr}(\chi)$ means that the character $\psi$ is a constituent of $\chi$.)
• $X(G)$ is the character table of $G$, and $X_1(G)$ is its first column (consisting of the degrees of irreducible characters, counting multiplicities).
• $M(G)$ is the Schur multiplier of $G$.
• If $M$ is a set, the Kronecker symbol $\delta : M \times M \to \{0, 1\}$ is defined as follows: if $a = b$, then $\delta_{a,b} = 1$, and if $a \neq b$, then $\delta_{a,b} = 0$.
• $\text{cd} G = \{ \chi(1) \mid \chi \in \text{Irr}(G) \}$.
• $\text{cd}_1 G = \{ \chi(1) \mid \chi \in \text{Irr}_1(G) \} = \text{cd} G - \{ 1 \}$.
• $b(G) = \max \{ n \mid n \in \text{cd} G \}$.
• $\ker T$ is the kernel of a representation $T$.
• $\ker \chi$ is the kernel of a character $\chi$.
• $Z(\chi) = \{ x \in G \mid |\chi(x)| = \chi(1) \}$ is the quasikernel of $\chi \in \text{Char}(G)$.
• $T_\chi = \{ x \in G \mid \chi(x) = 0 \}$ is the set of zeros of $\chi \in \text{Ch}(G)$.
• $U_\chi = \{ x \in G \mid |\chi(x)| = 1 \}$ is the set of $\chi$-unitary elements of $G$ (where $\chi \in \text{Ch}(G)$).
• Let $N \trianglelefteq G$. Then $\text{Irr}_N(G) = \{ \chi \in \text{Irr}(G) \mid N < \ker \chi \}$. We often identify the sets $\text{Irr}_N(G)$ and $\text{Irr}(G/N)$. Next, $\text{Irr}(G, N) = \text{Irr}(G) - \text{Irr}(G/N)$; $\text{Lin}_N(G) = \text{Lin}(G) \cap \text{Irr}_N(G)$.
• $\text{Irr}_\phi(G) = \{ \chi \in \text{Irr}(G) \mid \langle \chi_N, \phi \rangle > 0 \}$, where $N \trianglelefteq G$, $\phi \in \text{Irr}(N)$.
• Let $H < G$, $\phi \in \text{Irr}(H), \chi \in \text{Irr}(G)$. Then $\chi$ is an extension of $\phi$ to $G$ if $\chi_H = \phi$.
• $\nu_2(\chi)$ is the Frobenius–Schur indicator of $\chi \in \text{Irr}(G)$ (see Chapter IV).
• $\text{mc}(G) = k[G]/|G|$ is the measure of commutativity of $G$.
• $T(G) = \sum_{\chi \in \text{Irr}(G)} \chi(1)$, and $f(G) = T(G)/|G|$.
• Let $T$ be a representation, affording the character $\chi$ of $G$. Then the function $\text{det} \chi : G \to \mathbb{C}^*$ is defined by $\text{det}(\chi)(x) = \text{det} T(x), x \in G$. Obviously, $\text{det} \chi \in \text{Lin}(G)$. 
- If $\chi \in \text{CF}(G)$, then $\overline{\chi} : G \to \mathbb{C}$ is defined by $\overline{\chi}(x) = \overline{\chi(x)}, x \in G$.
- If $X \subseteq \text{Irr}(G)$, then $X^* = X - 1_G$. In particular, $\text{Irr}^*(G)$ is the set of all nonprincipal characters of $G$. $\text{Irr}_1(G, p') = \{\chi \in \text{Irr}_1(G) \mid p \nmid \chi(1)\}$.
- $T_1(G, p') = \sum_{\chi \in \text{Irr}_1(G, p')} \chi(1)$.
- If $P \in \text{Syl}_p(G)$, then $T_1(G, P, p') = \sum_{\chi \in \text{Irr}_1(G, p')} \chi_{P} \chi(1)$.
- $\text{Kern } G = \{\ker \chi \mid \chi \in \text{Irr}_1(G)\}$.
- $v(x) = ||\chi(x) \mid x \in G||$.
- A character $\chi$ of $G$ is monolithic if $\chi \in \text{Irr}(G)$ and $G/\ker \chi$ is a monolith. $\text{Irr}_m(G)$ is the set of all monolithic characters of $G$, $\text{Irr}_{1,m}(G) = \text{Irr}_m(G) \cap \text{Irr}_1(G)$. 