Preface

How do you get your arms around an infinite abelian group? Such groups are all around us—integers, rationals, reals, and complex numbers. Yet these examples do not even hint at the richness and complexities of abelian groups, as we will see. To help in getting through the complexity we want to focus in on some set of characteristics that completely determines the group. Specifically, we seek some invariants of a group such that if two groups share the same invariants, then they are of necessity isomorphic.

As it turns out, this problem cannot be solved in general. There is a trade-off between the size of the class of groups classified and the complexity of the invariants. This means that a key part of the classification task is finding classes of groups that admit to an insightful classification. Thus, the classification problem has three parts:

(1) Identify a class of groups to be classified. The broader the class the better. Much of the work in this area involves expanding the class. For example, Ulm was able to classify countable torsion groups, and later work expanded the classification to other torsion groups, and then to certain mixed groups (that is, those that are neither torsion nor torsion-free) with additional invariants. The ideal class will be closed under key operations to aid studying it. In particular, the class should be closed under direct sums and summands, and ideally G should be in the class if and only if both $p^{\alpha}G$ and $G/p^{\alpha}G$ are in the class.

(2) Identify an invariant, usually a cardinal, that can be determined for each group. It is somewhat misleading to say "an invariant" since it may actually consist of infinitely many values. Prove that it is indeed an invariant by proving that it is unchanged under isomorphism.

(3) Prove that the invariant classifies the groups in the class. Specifically, prove that if two groups in the class have the same invariants, then they are isomorphic. In some cases, we can expand the class by demanding something less than full isomorphism.

This exploration leads to the discovery of interesting classes of groups, and hence to the structure problem. The goal is to describe the structure or other defining characteristics of a class of groups. For example, we will see that groups in certain classes may be written and thought of as direct sums of certain cyclic groups. Often this structure suggests invariants to use for classification. Structure and classification are often combined into uniqueness theorems that present the class as the unique one that contains certain groups, is closed under certain operations and classified by certain invariants.

It can be argued that Gauss gave us the first classification result in 1801, the finite version of the primary decomposition theorem. This was even before groups and abelian groups were defined by Kronecker in 1870. The first classification result of infinite groups was due to Ulm in 1933. He developed cardinal invariants that com-

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pletely classified countable torsion groups. Soon after, Baer (1937) developed invariants called types that classify torsion-free groups of rank 1. This was the first work that systematically addressed groups that were not necessarily countable. Interest in abelian groups reignited in the mid-20th century with the publication of books on the topic [78, 24] by Kaplansky (1954) and Fuchs (1958), the latter containing 86 open problems. Hill extended Ulm's classification to a larger class of torsion groups, the totally projective groups. These torsion and torsion-free results seemed to be unrelated until Warfield developed a classification theory of a class of groups that includes both, but also some groups that do not split into a direct sum of torsion and torsion-free subgroups. An apparent way to look at these mixed groups is as extensions of their torsion subgroup by a torsion-free group. Warfield flipped this around and viewed these mixed groups in terms of a torsion-free basis, characterized by the Warfield invariants, with the quotient over this basis torsion and totally projective, and thus characterized by the Ulm invariants.

We extend this work in two ways. One is to weaken the requirement for isomorphism. Barwise and Eklof extended Ulm's theorem to all torsion groups up to partial isomorphism. Since partial isomorphism implies isomorphism on countable groups, this is a true generalization. In Chapter 5, we define a class of groups that includes the Warfield groups and classify it up to partial isomorphism. Partial isomorphism has an interesting model-theoretic counterpart that we explore.

The other extension looks at topological groups. In Chapters 6 and 7, we use Pontrjagin duality to determine the dual counterparts of various constructs in topological groups. Every group result has a corresponding result in compact abelian groups. In particular, this gives us classifications for various classes of compact abelian groups.

The classes of groups all have a range of characterizations. We explore each class in terms of presentations, homology, increasing sequences of subgroups, decomposition bases, and many others. Many of the chapters culminate in a list of varied characterizations that are proved equivalent.

We assume the reader is familiar with the definition of group, and key constructs, such as subgroups, factor groups, homomorphisms, and isomorphisms. When we use the word "group", we will mean additive abelian group. At times, we will consider groups as modules over the integers or over $\mathbb{Z}_p$, a ring that will be useful for looking at groups one prime at a time. Necessary background in the key characteristics of infinite abelian groups is provided in Chapter 1.