3 Positive-Energy Representations of Noncompact Quantum Algebras

Summary
We construct positive-energy representations of noncompact quantum algebras at roots of unity. We give the general setting, and then we consider in detail the examples of the $q$-deformed anti de Sitter algebra $\mathcal{A}_q = U_q(\mathfrak{so}(3, 2))$ and $q$-deformed conformal algebra $\mathcal{C}_q = U_q(\mathfrak{su}(2, 2))$. For $\mathcal{A}_q$ we discuss in detail the singleton representations, while for $\mathcal{C}_q$ we discuss in detail the massless representations. When the deformation parameter $q$ is $N$-th root of unity, all irreducible representations are finite-dimensional. We give the dimensions of these representations and their character formulae. Generically, these dimensions are not classical, except in some special cases, including the deformations of the fundamental irreps of $\mathfrak{so}(3, 2)$ and $\mathfrak{su}(2, 2)$. We follow the papers [165, 212, 225, 231].

3.1 Preliminaries

Let $G$ be a simple connected noncompact Lie group with unitary highest-weight representations [264], and let $\mathfrak{g}_0$ be its Lie algebra. Thus, $\mathfrak{g}_0$ is one of the following Lie algebras: $\mathfrak{su}(m, n)$, $\mathfrak{so}(n, 2)$, $\mathfrak{sp}(2n, R)$, $\mathfrak{so}^*(2n)$, $\mathfrak{E}_6(-14)$, $\mathfrak{E}_7(-25)$. We consider $q$-deformations $U_q(\mathfrak{g}_0)$ constructed by the procedure proposed in [204] and reviewed in Section 1.5. The positive-energy irreps of $U_q(\mathfrak{g}_0)$ are realized as lowest-weight module $M$ of $U_q(\mathfrak{g}_0)$, where $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$, together with a hermiticity condition necessary for the construction of a scalar product in $M$. We take lowest instead of the more often used highest-weight modules since we want the energy to be bounded from below. We use the standard deformation $U_q(\mathfrak{g}_0)$ [251, 360] given in terms of the Chevalley generators $X_i^+$ and $H_i$, $i = 1, \ldots, r = \text{rank } \mathfrak{g}$ by the relations (1.19).

A lowest-weight module $M^\Lambda$ is given by the lowest-weight $\Lambda \in \mathcal{H}^*$ (where $\mathcal{H}^*$ is the dual of $\mathcal{H}$) and a lowest-weight vector $v_0$ so that $Xv_0 = 0$ if $X \in \mathcal{H}^-$ and $Hv_0 = \Lambda(H)v_0$ if $H \in \mathcal{H}$. In particular, we use the Verma modules $V^\Lambda$ which are the lowest-weight modules such that $V^\Lambda = U_q(\mathcal{H})v_0$. Thus the Poincaré–Birkhof–Witt theorem (cf., e. g., Section 2.5.1) tells us that the basis of $V^\Lambda$ consists of monomial vectors

$$\Psi_{\{k\}} = (Y_{i_1}^+)^{k_{i_1}} \cdots (Y_{i_n}^+)^{k_{i_n}}v_0 = \mathcal{P}_{\{k\}}v_0, \quad k_i \in \mathbb{Z}_+, \quad (3.1)$$

where $Y_i^+ \in \mathcal{H}^+$, $i_1 < i_2 < \ldots < i_n$, in some fixed ordering of the basis. A $U_q(\mathcal{H}_0)$-invariant scalar product in $V^\Lambda$ is given by:

$$\left(\Psi_{\{k'\}}, \Psi_{\{k\}}\right) = \left(\mathcal{P}_{\{k'\}}v_0, \mathcal{P}_{\{k\}}v_0\right) = \left(v_0, \omega(\mathcal{P}_{\{k'\}}\mathcal{P}_{\{k\}}v_0)\right), \quad (3.2)$$

with $(v_0, v_0) = 1$ and $\omega$ is the conjugation which singles out $\mathcal{H}_0$, which has the property that $\omega(X^+) \in \mathcal{H}^-$ if $X^+ \in \mathcal{H}^+$. 
We use the information on Verma modules as given in Chapter 2. Specifically, we recall that when the deformation parameter $q$ is a root of unity, the picture of the representations changes drastically. In this case all Verma modules $V^\Lambda$ are reducible [198], and all irreducible representations are finite-dimensional [175]. Let $q$ be a primitive $N$-th root of unity; that is, $q = e^{2\pi i/N}$, where $N \in \mathbb{N}$ and $N \geq 1 + n(G)$, where $n(G)$ is the ratio $(a_L, a_L)/[a_S, a_S]$, where $a_L$ is a long root, and $a_S$ a short root. The maximal dimension of any irreducible representation is equal to $d_N$ for $N$ odd [175]. There are singular vectors for all positive roots [198]. Condition (2.2) also has more content now because if $(\Lambda - \rho)(H_a) = -m \in \mathbb{Z}$, then (2.2) will be fulfilled for all $m + kN_a$, $k \in \mathbb{Z}$, $N_a = N/n(G)$ if $N \in n(G)N$ and $a$ is a long root and $N_a = N$ in all other cases. In particular, there is an infinite series of positive integers $m$ such that (2.2) is true [198]. For identical reasons, there is an infinite number of lowest weights $\Lambda$ such that (2.2) is satisfied for the same set of positive integers $m = m_a$. The structure of the corresponding finite-dimensional irreps is the same since it is fixed by these positive integers.

Some of the finite-dimensional irreducible representations can be unitary as we show in the examples in the next sections.

We also give an interpretation of the spectrum via character formulae.

### 3.2 Quantum Anti de Sitter Algebra

#### 3.2.1 Representations

Here we follow mostly [212, 231]. The first example we consider is the quantum anti de Sitter algebra; that is, we take $G_0 = so(3, 2)$ and $G = so(5, \mathbb{C})$. In this case $r = 2$ and the nonzero products between the simple roots are $(\alpha_1, \alpha_1) = 2$, $(\alpha_2, \alpha_2) = 4$, and $(\alpha_1, \alpha_2) = -2$; thus $a_{12} = -2$, $a_{21} = -1$. The non-simple positive roots are $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$. The Cartan–Weyl basis for the nonsimple roots is given by [198, 576]:

$$X_3^\pm = \pm q^{1/4}(q^{1/2}X_1^+X_2^+ - q^{-1/2}X_2^+X_1^+), \quad X_4^\pm = \pm (X_1^+X_3^+ - X_3^+X_1^+) / [2]_q. \quad (3.3)$$

All commutation relations now follow from the above relations. We mention, in particular:

$$[X_3^+, X_3^-] = [H_3]_q, \quad H_3 = H_1 + 2H_2, \quad [X_4^+, X_4^-] = [H_4]_q^2, \quad H_4 = H_1 + H_2, \quad (3.4)$$

where the Cartan generators $H_3, H_4$ corresponding to the nonsimple roots $\alpha_3, \alpha_4$ are chosen as in [242].

We choose the generators of $U_q(so(3, 2))$ as a real form of $U_q(so(5, \mathbb{C}))$ as follows [242]:
\[ M_{21} = H_1/2, \quad M_{31} = (X_1^+ + X_1^-)/2, \]
\[ M_{32} = i(X_1^+ - X_1^-)/2, \]
\[ M_{04} = (H_1 + H_2)/2, \quad M_{30} = i(X_3^+ + X_3^-)/2, \]
\[ M_{34} = (X_3^- - X_3^+)/2, \quad (3.5a) \]
\[ M_{04} = (H_1 + H_2)/2, \quad M_{30} = i(X_3^+ + X_3^-)/2, \]
\[ M_{34} = (X_3^- - X_3^+)/2, \quad (3.5b) \]
\[ M_{10} = i(X_4^+ + X_4^- + X_2^+ + X_2^-)/2, \]
\[ M_{20} = (X_4^+ - X_4^- - X_2^+ - X_2^-)/2, \quad (3.5c) \]
\[ M_{41} = (X_2^+ - X_2^- + X_4^+ - X_4^-)/2, \]
\[ M_{42} = (X_2^+ - X_2^- - X_4^+ - X_4^-)/2. \quad (3.5d) \]

Clearly, for \( q = 1 \) the ten generators \( M_{AB} = -M_{BA}, A, B = 0, 1, 2, 3, 4, \) satisfy the \( so(3,2) \) commutation relations (with \( \eta_{AB} = \text{diag}(+ - - +) \)):

\[ [M_{AB}, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}), \quad q = 1. \]

The commutation relations for \( U_q(so(3,2)) \) follow from (3.5) and the commutation relations of \( U_q(so(5, \mathbb{C})) \). The Cartan subalgebras of \( U_q(so(3,2)) \) and \( U_q(so(5, \mathbb{C})) \) are generated by the same generators \( M_{21}, M_{04} \) or \( M_{10}, M_{20} \). Note that the generators in (3.5a) and (3.5b) are compact; the rest are noncompact. In particular, those in (3.5a) generate a \( U_q(su(2)) \) subalgebra, those in (3.5b) a \( U_q(su(1,1)) \) subalgebra.

For \( |q| = 1 \) the generators in (3.5) are preserved by the following antilinear antinvolution \( \omega \) of \( U_q(so(5, \mathbb{C})) \) [231]:

\[ \omega(H_j) = H_j, \quad j = 1, 2, \quad \omega(X_1^+) = X_1^-, \quad \omega(X_2^+) = -X_2^-, \quad \omega(X_3^+) = X_3-, \quad k = 2, 3, 4. \quad (3.6) \]

The restriction \( |q| = 1 \) follows from requiring consistency between (3.3) and (3.6), which is necessary since the generators \( X_3^+, X_4^+ \) are given in terms of \( X_1^+, X_2^+ \). Thus in what follows we work with \( |q| = 1 \).

For the four positive roots of the root system of \( so(5, \mathbb{C}) \), one has from (2.2) (cf. [242]):

\[ m_1 = -\Lambda(H_1) + 1 = 2s_0 + 1, \quad (3.7a) \]
\[ m_2 = -\Lambda(H_2) + 1 = 1 - E_0 - s_0, \quad (3.7b) \]
\[ m_3 = -\Lambda(H_3) + 3 = m_1 + 2m_2 = 3 - 2E_0, \quad (3.7c) \]
\[ m_4 = -\Lambda(H_4) + 2 = m_1 + m_2 = 2 - E_0 + s_0. \quad (3.7d) \]

where the representations are labelled (as those of \( so(3,2) \)) by the lowest value of the energy \( E_0 \) and by the spin \( s_0 \in \mathbb{Z}_+ \) of the state with this energy.

Let us recall the list of the positive-energy representations of \( so(3,2) \) (cf. [191, 267, 289, 302]):
3.2 Quantum Anti de Sitter Algebra

Rac: \( D(E_0, s_0) = D(1/2, 0) \), Di: \( D(E_0, s_0) = D(1, 1/2) \),
\( D(E_0 > 1/2, s_0 = 0) \), \( D(E_0 > 1, s_0 = 1/2) \),
\( D(E_0 \geq s_0 + 1, s_0 \geq 1) \). (3.8)

The first two are the \textit{singleton representations}, which were first discovered by Dirac in [191], and the last ones for \( E_0 = s_0 + 1 \) correspond to the spin-\( s_0 \) \textit{massless representations} of \( \mathfrak{so}(3,2) \).

Let us consider (3.7) for this list. We note that in all cases \( m_1 \in \mathbb{N} \) (because \( s_0 \in \mathbb{Z}_+ \)) and \( m_2 \notin \mathbb{N} \) (because \( m_2 \leq 1/2 \)). Next, we note that \( m_3 \) is a positive integer only for \( E_0 = 1/2, 1 \), in which case \( m_3 = 2, 1 \), respectively. Similarly, \( m_4 \) is a positive integer only for \( E_0 - s_0 = 1 \), and that integer is \( m_4 = 1 \). Accordingly, we find the following singular vectors of the Verma module over \( U_q(\mathfrak{so}(3,2)) \) [231]:

\[
\begin{align}
\nu_1^a &= (X_1^+)^{2s_0+1}v_0, \quad s_0 \in \mathbb{Z}_+/2, \\
\nu_{31}^a &= ([2s_0]_q X_3^+ - (1 + q)X_2^+ X_1^+)v_0, \quad m_3 = 1, \\
\nu_{32}^a &= ((X_3^+)^2 - q^{1/2}[2]_q X_2^+ X_4^+)v_0, \quad m_3 = 2, \\
\nu_4^a &= ([2s_0]_q [2s_0 - 1]_q X_4^+ + q^{s_0} [1 - 2s_0]_q X_3^+ X_1^+ + X_2^+ (X_1^+)^2) v_0, \quad m_4 = 1.
\end{align}
\] (3.9a, 3.9b, 3.9c, 3.9d)

Note that (3.9b) for \( s_0 = 0 \) and (3.9d) for \( s_0 = 0, 1/2 \) are composite singular vectors being \textit{descendants} of (3.9a). We take the basis of the Verma module (3.1) in terms of the Cartan–Weyl generators as:

\[
\Psi_{\{k\}} = (X_4^+)^{k_4}(X_3^+)^{k_3}(X_2^+)^{k_2}(X_1^+)^{k_1}v_0, \quad k_j \in \mathbb{Z}_+.
\] (3.10)

Further, we concentrate on the \textit{singleton} representations. To obtain the irreducible factor-representations \( L_D \) with ground states denoted by \( |E_0, s_0 \rangle \), we have to impose the following null-state vanishing conditions (following from (3.9)):

Rac: \( \langle X_1^+ |1/2, 0 \rangle = 0 \), \( \langle (X_3^+)^2 - q^{1/2}[2]_q X_2^+ X_4^+ |1/2, 0 \rangle = 0 \); (3.11)

Di: \( \langle X_1^+ |1, 1/2 \rangle = 0 \), \( \langle X_3^+ - (1 + q)X_2^+ X_1^+ |1, 1/2 \rangle = 0 \). (3.12)

(For \( q = 1 \) formulae (3.9), (3.11), and (3.12) were obtained in [242].)

Now we give explicitly the basis of \( L_A \). We consider the monomials as in (3.10), but on the vacuum \( |E_0, s_0 \rangle \). Condition (3.11) means that in (3.10) we have \( k_1 = 0 \) and \( k_3 \leq 1 \), since we replace \( (X_3^+)^2 \) by \( X_2^+ X_4^+ \) (one may replace also \( X_2^+ X_4^+ \) by \( (X_3^+)^2 \) as in [242]). Similarly, (3.12) means that in (3.10) we have \( k_1 \leq 1 \) and \( k_3 = 0 \), since we replace \( X_3^+ \) by \( X_1^+ X_4^+ \). Thus, we see that the basis of \( L_A \) consists, as in the classical case [242], of the following monomials [231]:
Rac: \((X_n^+)^j(X_2^+)^k\{1/2, 0\}, \quad j, k = 0, 1, \ldots, \quad \epsilon = 0, 1, \quad (3.13)\)

Di: \((X_q^+)^j(X_2^+)^k\{1, 1/2\}, \quad j, k = 0, 1, \ldots, \quad \epsilon = 0, 1. \quad (3.14)\)

Note that each weight has multiplicity one, which was the reason these representations were called singletons [289].

Now we shall calculate the norms of these states. First we calculate some norms valid for any \(D\):

\[
\| (X_2^+)^j(X_1^+)^k | \Lambda \rangle \|^2 = [j]_q! [k]_q! \left( \prod_{\ell=1}^{k} [\Lambda(H_2) - k - 1 + \ell]_q \right) \times \\
\times \prod_{s=1}^{k} [1 - \Lambda(H_1) - s]_q, \quad (3.15a)
\]

\[
\| (X_3^+)^j(X_1^+)^k | \Lambda \rangle \|^2 = [j]_q! [k]_q! \left( \prod_{\ell=1}^{k} [\Lambda(H_3) - 1 + \ell]_q \right) \times \\
\times \prod_{s=1}^{k} [1 - \Lambda(H_1) - s]_q, \quad (3.15b)
\]

\[
\| (X_3^+)^j(X_2^+)^k \{|1/2, 0\} \rangle \|^2 = [j]_q! [k]_q! \left( \prod_{\ell=1}^{k} [\Lambda(H_3) + 2k - 1 + \ell]_q \right) \times \\
\times \prod_{s=1}^{k} [\Lambda(H_2) - 1 + s]_q, \quad (3.15c)
\]

\[
\| (X_q^+)^j(X_2^+)^k(X_1^+)^l | \Lambda \rangle \|^2 = [j]_q! [k]_q! [-\Lambda(H_1)]_q^e \times \\
\times \left( \prod_{\ell=1}^{j} [\Lambda(H_4) - 1 + \varepsilon + \ell]_q \right) \prod_{s=1}^{k} [\Lambda(H_2) - 1 - \varepsilon + s]_q, \quad (3.15d)
\]

\[
\| (X_q^+)^j(X_3^+)^e(X_2^+)^k \{|1/2, 0\} \rangle \|^2 = [j]_q! [k]_q! [\Lambda(H_3) + 2k]_q^e \times \\
\times \left( \prod_{\ell=1}^{j} [\Lambda(H_4) - 1 + \varepsilon + \ell]_q \right) \prod_{s=1}^{k} [\Lambda(H_2) - 1 + s]_q. \quad (3.15e)
\]

In all cases we consider we have \(\Lambda(H_1) = -2s_0\). Thus we get from (3.15a) with \(j = 0\)

\[
\| (X_1^+)^k | E_0, s_0 \rangle \|^2 = [k]_q! \prod_{\ell=1}^{k} [2s_0 + 1 - \ell]_q, \quad (3.16)
\]

which vanishes if \(k \geq 2s_0 + 1 = m_1\); the latter statement is clear also from the null-state condition. In the same way we see that (3.15a,b) vanish for \(k \geq 2s_0 + 1\) and any \(j\). To calculate the other norms we also use \(\Lambda(H_2) = E_0 + s_0\) (then \(\Lambda(H_3) = 2E_0\), \(\Lambda(H_4) = E_0 - s_0\)).
Finally, the norms of the basis states (3.13) and (3.14) are:

\[
\| (X^+_j)^k (X^+_j)^ℓ (X^+_j)^k|1/2, 0\rangle \|^2 = [2]^k q^j ![k] q^2! \left( \prod_{ℓ=1}^{j} [ℓ - 1/2 + \varepsilon] q^2 \right) \times \\
\times \prod_{s=1}^{k} [s - 1/2 + \varepsilon] q^2, \quad (3.17)
\]

\[
\| (X^+_j)^k (X^+_j)^ℓ (X^+_j)^k|1/2, 0\rangle \|^2 = [j] q^2 ![k] q^2! \left( \prod_{ℓ=1}^{j} [ℓ - 1/2 + \varepsilon] q^2 \right) \times \\
\times \prod_{s=1}^{k} [s + 1/2 - \varepsilon] q^2, \quad (3.18)
\]

### 3.2.2 Roots of Unity Case

In this subsection we consider the case where the deformation parameter is a root of unity, namely, \( q = e^{2\pi i/N} \), \( N = 3, 4, \ldots \)

Let us denote

\[
\tilde{N} = \begin{cases} 
N & \text{for } N \text{ odd} \\
N/2 & \text{for } N \text{ even}
\end{cases}
\quad N_j = \begin{cases} 
N & \text{for } j = 1, 3 \\
\tilde{N} & \text{for } j = 2, 4.
\end{cases} \quad (3.19)
\]

In this situation independently of the weight \( \Lambda \) there are singular vectors for all positive roots \( \alpha_j \), which are given by: \( (X^+_j)^{Nj} v_0, j = 1, 3 \), and \( (X^+_j)^{Nj} v_0, j = 2, 4, k = 1, 2, \ldots \) [198]. Thus we have to impose the following vanishing of null states in our representation spaces:

\[
(X^+_j)^N|E_0, m_0\rangle = 0, \quad j = 1, 3, \quad (X^+_j)^{Nj}|E_0, m_0\rangle = 0, \quad j = 2, 4. \quad (3.20)
\]

Taking into account condition (2.2) we see that if \( m_j = (\rho - \Lambda)(H_j) \in \mathbb{Z} \), \( j = 1, 2, 3, 4 \), there would be singular vectors of weights \( (n'_j + kN)\alpha_j \), where \( n'_j = \{m_j\}_{N_j}, \{x\}_{p} \) being the smallest positive integer equal to \( x \) (mod \( p \)), and \( k = 0, 1, \ldots \). Analogously, if \( m_j \in 1/2 + \mathbb{Z}, j = 2, 4, \) and \( N \) is odd, there would be singular vectors of weights \( (n'_j + kN)\alpha_j \), \( n'_j = \{m_j + N/2\}_{N}, k = 0, 1, \ldots \). In particular, we have to impose:

\[
(X^+_j)^{n'_j}|E_0, m_0\rangle = 0, \quad j = 1, 2. \quad (3.21)
\]

Further our representations will be characterized by the following positive integers:
\[ n_1 = [2s_0 + 1]_N = [m_1]_N, \]  

\[ n_2 = \begin{cases} 
[1 - E_0 - s_0]_\tilde{N} = [m_2]_{\tilde{N}}, & \text{if } E_0 + s_0 \in \mathbb{Z}, \\
[1 - E_0 - s_0 + N/2]_N = [m_2 + N/2]_N, & \text{if } E_0 + s_0 \in 1/2 + \mathbb{Z}, \\
\tilde{N}, & \text{otherwise},
\end{cases} \]  

Note that \( n_k \leq N_k, k = 1, 2. \)

Let us recall that the finite-dimensional irreducible representations of \( so(5, \mathbb{C}) \) (or of other real form of \( so(5, \mathbb{C}) \) and of the corresponding quantum algebras when \( q \) is not a root of unity) are parametrized by two arbitrary positive integers, say, \( p_1, p_2 \), and the dimension of such a representation is given by:

\[ d_{p_1, p_2}^c = \frac{1}{6} p_1 p_2 p_3 p_4, \]  

where \( p_3 = p_1 + 2p_2, p_4 = p_1 + p_2. \)

Now for \( N \) odd we divide our representations in classes depending on the values of \( n_3 = n_1 + 2n_2, n_4 = n_1 + n_2 \) and \( n_1, n_2 \):

\[ a) n_3, n_4 \leq N, \]  

\[ b) n_4 < N < n_3 < 2N, \]  

\[ b') n_4 = N < n_3 \leq 2N, \quad \text{or} \quad n_4 < n_3/2 = N, \]  

\[ c) n_1 < N < n_3, n_4 < 2N, \]  

\[ c') n_1 = N < n_3, n_4 \leq 2N, \quad \text{or} \quad n_1 < N < n_4 < n_3 = 2N, \]  

\[ d) n_2 < N < n_4 < 2N < n_3 < 3N, \]  

\[ d') n_2 = N < n_4 \leq 2N < n_3 \leq 3N. \]

The same classification is valid for \( U_q(so(5, \mathbb{C})) \), where (3.24a) is the regular case. This is a refinement of the classification of [231], the primed cases being separated out since, together with the regular case, these have the classical dimensions of the finite-dimensional irreps of \( so(5, \mathbb{C}) \); that is, a representation characterized by \( n_1, n_2 \) has dimension \( d_{n_1, n_2}^c \). In particular, in case \( d' \) with \( n_1 = n_2 = N \), we achieve the maximal possible dimension \( N^6 \) of an irrep of \( U_q(so(5, \mathbb{C})) \) (cf. [175]). On the other hand, in the unprimed cases \( b) - d) \), the dimension of a representation characterized by \( n_1, n_2 \) is strictly smaller than \( d_{n_1, n_2}^c \). The representations \( U_q(so(3, 2)) \) inherit all the structure from their \( U_q(so(5, \mathbb{C})) \) counterparts. Thus, the classification of the positive-energy representations of \( U_q(so(3, 2)) \) proceeds as follows.
Let us decompose: \( 2s_0 = 2S_0 + r_0N, 2S_0, r_0 \in \mathbb{Z}_+, 2S_0 < N \). Then we have:

\[
n_1 = 2S_0 + 1. \tag{3.25}
\]

Now the formulae for \( n_2 \) depend on the combination \( E_0 + s_0 \).

Suppose first that \( E_0 + s_0 \notin \mathbb{Z}/2 \). Then we have:

\[
n_2 = N, \quad n_3 = 2N + 2S_0 + 1 > 2N, \quad n_4 = N + 2S_0 + 1 > N, \quad \text{odd } N, \tag{3.26}
\]

which is case (3.24d).

Next we consider the case \( E_0 + s_0 \in \mathbb{Z} \). Taking into account the conditions of positive energy (3.8), we see that we have \( E_0 \geq s_0 + 1 \). Thus we set \( E_0 = s_0 + 1 + p + kN \), where \( p = 0, 1, \ldots, N - 1, k \in \mathbb{Z}_+ \). Let us also set \( \kappa = 2S_0 + p \). Note that \( 0 \leq \kappa \leq 2N - 2 \). Then we have for \( N \) odd:

\[
n_2 = N - \kappa,
\]

\[
n_3 = 2N - \kappa - p + 1 \begin{cases} \leq N & \text{for } \kappa + p > N, \\ > N & \leq 2N & \text{for } \kappa + p \leq N, \kappa > 0, \\ > 2N & \text{for } \kappa = 0, \end{cases}
\]

\[
n_4 = N - p + 1 \begin{cases} \leq N & \text{for } p > 0, \\ > N & \text{for } p = 0; \end{cases}
\]

\[
\kappa < N, \tag{3.27a}
\]

\[
n_2 = 2N - \kappa,
\]

\[
n_3 = 4N - \kappa - p + 1 \begin{cases} > N & \leq 2N & \text{for } \kappa + p \geq 2N + 1, \\ > 2N & \text{for } \kappa + p \leq 2N, \end{cases}
\]

\[
n_4 = 2N - p + 1 > N, \quad \kappa \geq N. \tag{3.27b}
\]

Thus we have case (3.24a) in (3.27a) when \( \kappa + p \geq N + 1 \) & \( p > 0 \), case (3.24b) in (3.27a) when \( \kappa + p \leq N \) & \( p > 0 \) (\( \Rightarrow \kappa > 0 \)), case (3.24c) in (3.27a) when \( p = 0 \) & \( \kappa > 0 \) and in (3.27b) when \( \kappa + p \geq 2N + 1 \), case (3.24d) in (3.27a) when \( \kappa = 0 \) (\( \Rightarrow p = 0 \)), and in (3.27b) when \( \kappa + p \leq 2N \).

Then we consider the case \( E_0 + s_0 \in 1/2 + \mathbb{Z} \) for \( N \) odd. Taking into account the conditions of positive energy (3.8), we see that we have \( E_0 \geq s_0 + 1/2 \). Thus we set \( E_0 = s_0 + 1/2 + p + kN \), where \( p = 0, 1, \ldots, N - 1, k \in \mathbb{Z}_+ \). As above we set \( \kappa = 2S_0 + p \) \( (0 \leq \kappa \leq 2N - 2) \). We also denote \( \tilde{N} = (N + 1)/2 \in \mathbb{N} + 1 \). Then we have:
\[
\begin{align*}
    n_2 &= \hat{N} - \kappa, \\
    n_3 &= N - \kappa - p + 2 \begin{cases} 
        \leq N & \text{for } \kappa + p \geq 2, \\
        > N \& \leq 2N & \text{for } \kappa + p \leq 1,
    \end{cases} \\
    n_4 &= \hat{N} - p + 1 < N, \\
    &\quad \kappa < \hat{N}. 
\end{align*}
\]

\[(3.28a)\]

\[
\begin{align*}
    n_2 &= N + \hat{N} - \kappa, \\
    n_3 &= 3N - \kappa - p + 2 \begin{cases} 
        \leq N & \text{for } \kappa + p \geq 2N + 2, \\
        > N \& \leq 2N & \text{for } N + 2 \leq \kappa + p \leq 2N + 1, \\
        > 2N & \text{for } \kappa + p \leq \hat{N} + 1,
    \end{cases} \\
    n_4 &= N + \hat{N} - p + 1 \begin{cases} 
        \leq N & \text{for } p > \hat{N}, \\
        > N & \text{for } p \leq \hat{N},
    \end{cases} \\
    &\quad \hat{N} \leq \kappa \leq N + \hat{N} \quad (3.28b)
\end{align*}
\]

\[
\begin{align*}
    n_2 &= 2N + \hat{N} - \kappa, \\
    n_3 &= 5N - \kappa - p + 2 > 2N, \\
    n_4 &= 2N + \hat{N} - p + 1 > N, \\
    &\quad \kappa \geq N + \hat{N} \quad (3.28c)
\end{align*}
\]

Thus we have case (3.24a) in (3.28a) when \(\kappa + p \geq 2\) and in (3.28b) when \(\kappa + p \geq 2N + 2\) \(\Rightarrow p > \hat{N}\), case (3.24b) in (3.28a) when \(\kappa + p \leq 1\) and in (3.28b) when \(p > \hat{N} \& \kappa + p \leq 2N + 1\) \(\Rightarrow \kappa + p \geq N + 2\), case (3.24c) in (3.28b) when \(p \leq \hat{N} \& \kappa + p \geq N + 2\) \(\Rightarrow \kappa + p \leq 2N + 1\), and case (3.24d) in (3.28b) when \(\kappa + p \leq N + 1\) \(\Rightarrow p \leq \hat{N}\) and in (3.28c).

After the above analysis it remains to mention that the singleton irreps, \((E_0, s_0) = (1/2, 0), (1, 1/2)\), belong to case (3.24b) (cf. (3.28a) with \(\kappa = 0, 1, p = 0\)), while the massless irreps, \(E_0 = s_0 + 1\), belong to case (3.24c).

This completes the classification of the positive-energy representations of \(U_q(so(3, 2))\) at odd roots of 1.

Further we treat in detail the singleton cases. In the case of the \(Rac\) besides (3.11) a new vanishing condition is:

\[
(X_2^+)^{n_2} |1/2, 0\rangle = 0, \quad n_2 = [(N + 1)/2]_{\text{int}},
\]

where \([x]_{\text{int}}\) is the biggest integer smaller or equal to \(x\); note that this condition is (3.20) for \(N\) even and (3.21) for \(N\) odd. Further using (1.21) we find that the following states from (3.13) have positive norms [231]:

\[
\| (X_2^-)^j (X_3^-)^k (X_2^+)^{n_2} |1/2, 0\rangle \|^2 > 0, \quad \text{iff} \quad
\begin{cases} 
    j, k \leq (N - 1 - 2e)/2 & \text{for } N \text{ odd} \\
    j, k \leq (N - 2)/2 & \text{for } N \text{ even}
\end{cases}
\]

(3.30)
Due to factors in (3.17): $[j - 1/2 + \epsilon]_q^2$, $[k - 1/2 + \epsilon]_q^2$ for $N$ odd, and $[j]_q^2$, $[k]_q^2$ for $N$ even; all other states from (3.13) have zero norm and decouple from the irrep. Thus we calculate the dimension of the Rac irrep by counting the states in (3.30), which are $(N + 1 - 2\epsilon)^2/4$ for $\epsilon = 0, 1$ and $N$ odd, and $N^2/4$ for $\epsilon = 0, 1$, and $N$ even. Thus we get [231]:

$$
\dim \text{Rac} = \begin{cases} 
\frac{N^2 + 1}{2}, & \text{for } N \text{ odd} \\
\frac{N^2}{2}, & \text{for } N \text{ even}
\end{cases}
\quad (3.31)
$$

In the case of the $Di$ besides (3.12) the new vanishing condition is:

$$(X_2^+)^{n_2}|1, 1/2\rangle = 0, \quad n_2 = \lfloor N/2\rfloor_{\text{int}},
\quad (3.32)
$$

again this is (3.20) for $N$ even and (3.21) for $N$ odd. Then we find from (3.18) that the following states have positive norms [231]:

$$
\| (X_1^+)^j (X_2^+)^k |1, 1/2\rangle \|^2 > 0, \quad \text{iff} \quad \begin{cases} 
j \leq (N - 1 - 2\epsilon)/2 & \text{and} \\
k \leq (N - 3 + 2\epsilon)/2 & \text{for } N \text{ odd} \\
j, k \leq (N - 2)/2 & \text{for } N \text{ even}
\end{cases}
\quad (3.33)
$$

and the counting of states gives [231]:

$$
\dim \text{Di} = \begin{cases} 
\frac{N^2 - 1}{2}, & \text{for } N \text{ odd} \\
\frac{N^2}{2}, & \text{for } N \text{ even}
\end{cases}
\quad (3.34)
$$

Thus the dimension of a singleton irrep for fixed $N$ is strictly smaller than the minimal dimension of a (semi-) periodic irrep of $U_q(\mathfrak{so}(5, \mathbb{C}))$, which is $N^2$ [177]. The interesting thing is that the sum of the dimensions of the two singletons is exactly $N^2$. Thus we are led to the conjecture that passing from a minimal (semi-) periodic irrep of $U_q(\mathfrak{so}(5, \mathbb{C}))$ to a lowest-weight module of $U_q(\mathfrak{so}(5, \mathbb{C}))$ (by setting the corresponding Casimir values to zero), we obtain a reducible representation which is the direct sum of two irreps. The latter irreps when restricted to $U_q(\mathfrak{so}(3, 2))$ are the two singleton representations.

### 3.2.3 Character Formulae

When $q$ is not a nontrivial root of 1, the spectrum of the singletons can be represented by the following character formulae (containing the same information as (3.13) and (3.14)):

$$
\text{ch} L_{\text{Rac}} = e(\Lambda)(1 + t_3) \sum_{j=0}^{\infty} t_j^2 \sum_{k=0}^{\infty} t_k^2,
\quad (3.35)
$$
\[ ch \ L_{Di} = e(\Lambda)(1 + t_4) \sum_{j=0}^{\infty} t_4^j \sum_{k=0}^{\infty} t_2^k, \]  
(3.36)  

where \( t_3 = e(\alpha_1 + \alpha_2) = t_1 t_2, \) \( t_4 = e(2\alpha_1 + \alpha_2) = t_1^2 t_2. \) (For \( q = 1 \) these formulae were given in a slightly different, but equivalent, form in \([242]\).) Now we note that the character formula for the Verma module with the same lowest weight here is:

\[ ch \ V^\Lambda = e(\Lambda)/(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4). \]  
(3.37)

Then we can rewrite the character formulae (3.35) and (3.36) as follows \([242]\):

\[ ch \ L_{Rac} = ch \ V^\Lambda(1 - t_1 - t_2 + t_3), \]  
(3.38)  
\[ ch \ L_{Di} = ch \ V^\Lambda(1 - t_1^2 - t_1 t_2 + t_2^2). \]  
(3.39)

These formulae represent alternating sign summations over part of the Weyl group of \( so(5, \mathbb{C}) \), which was called reduced Weyl group in \([196]\]).

Next we note that the spectrum given in (3.30) and (3.33) can be represented by the following character formulae for \( N \) odd:

\[ ch \ L_{Rac} = e(\Lambda) \left( \sum_{j=0}^{(N-1)/2} t_4^j \sum_{k=0}^{(N-1)/2} t_2^k + t_3 \sum_{j=0}^{(N-3)/2} t_4^j \sum_{k=0}^{(N-3)/2} t_2^k \right), \]  
(3.40)  
\[ ch \ L_{Di} = e(\Lambda) \left( \sum_{j=0}^{(N-1)/2} t_4^j \sum_{k=0}^{(N-3)/2} t_2^k + t_1 \sum_{j=0}^{(N-3)/2} t_4^j \sum_{k=0}^{(N-1)/2} t_2^k \right). \]  
(3.41)

Let us denote by \( L_{n_1,n_2}^c \) the finite-dimensional irreps of \( so(5, \mathbb{C}) \). The corresponding character formula, which is the classical Weyl character formula, is:

\[ ch \ L_{n_1,n_2}^c = ch \ V^\Lambda(1 - t_1^{n_1} - t_2^{n_2} - t_3^{n_3} - t_4^{n_4} + t_1^{n_1} t_2^{n_2} + t_3^{n_3} t_4^{n_4} + t_1^{n_1} t_2^{n_2} t_3^{n_3} + t_1^{n_1} t_2^{n_2}), \]  
(3.42)

where the eight terms represent (alternating sign) summation over the (eight element) Weyl group of \( so(5, \mathbb{C}) \).

As we mentioned, the dimension of a unitary irrep of \( U_q(so(3, 2)) \) characterized by \( n_1, n_2 \) is generically smaller than \( d_{n_1,n_2}^c \). In particular, for the Rac when \( N \) is odd we have \( (n_1, n_2) = (1, (N + 1)/2) \). We have that \( d_{n_1,(N+1)/2}^c = (N + 1)(N + 2)(N + 3)/24 \geq dim_{Rac} = (N^2 + 1)/2 \). It is easy to notice that \( dim_{Rac} \) may be represented as the difference of two dimensions:

\[ dim_{Rac} = d_{1,(N+1)/2}^c - d_{1,(N-3)/2}^c \]  
(3.43)

where the subtracted term corresponds to the weight \( \Lambda' = \Lambda + 2\alpha_3 \) with characterizing integers given by: \( n_j' = (\rho - \Lambda')(H_j) = n_j - 2\alpha_3(H_j) \); that is, \( (n_1', n_2') = (n_1 - 2, n_2 - 2) \). Correspondingly, the character formula for odd \( N \) is given by (cf. (3.40)):
\[ ch L_{\text{Rac}} = ch L_{1,(N+1)/2}^c - ch L_{1,(N-3)/2}^c = ch V^\Lambda (P_{1,(N+1)/2} - t_3^2 P_{1,(N-3)/2}), \]  
\[ (3.44) \]

where we have introduced the notation: \( ch L_{n_1,n_2}^c = ch V_D P_{n_1,n_2}^r. \)

Note that the subtraction term vanishes only for \( N = 3 \), which is the only case when the quantum Rac dimension coincides with a classical dimension, here of one of the fundamental irreps of \( so(3, 2) \) with \( d^c = 5 \).

Analogously, for the \( D_i \) when \( N \) is odd we have \( (n_1, n_2) = (2, (N - 1)/2) \). Here we have that \( d^c_{2,(N-1)/2} = (N^2 - 1)(N + 3)/12 \geq dim D_i = (N^2 - 1)/2, \) and equality is possible only for \( N = 3 \); then the dimension is of the other fundamental irrep, \( d^c = 4 \). Here we have to subtract the character \( ch L_{A'}^c \) with \( A' = \Lambda + \alpha_3 \), and \( (n_1', n_2') = (n_1, n_2 - 1) \). We have for odd \( N \):

\[ ch L_{D_i} = ch L_{2,(N-1)/2}^c - ch L_{2,(N-3)/2}^c = ch V^\Lambda (P_{2,(N-1)/2} - t_3 P_{2,(N-3)/2}), \]
\[ (3.45) \]

\[ \text{dim}_{D_i} = d^c_{2,(N-1)/2} - d^c_{2,(N-3)/2}. \]  
\[ (3.46) \]

### 3.3 Conformal Quantum Algebra

#### 3.3.1 Generic Case

The other example that we consider is the conformal algebra; that is, we take \( \mathcal{G}_0 = su(2, 2) \) and \( \mathcal{G} = sl(4, \mathbb{C}) \). In this case \( r = 3 \), and the nonzero products between the simple roots are \((\alpha_1, \alpha_j) = 2, j = 1, 2, 3 \) and \((\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1 \). The nonsimple positive roots are \( \alpha_{12} = \alpha_1 + \alpha_2, \alpha_{23} = \alpha_2 + \alpha_3, \alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3 \). The Cartan–Weyl basis for the nonsimple roots is given by [202, 360]:

\[ X_{jk}^\pm = \pm q^{\mp 1/4} (q^{1/4} X_k^+ X_j^+ - q^{-1/4} X_k^+ X_j^-), (jk) = (12), (23), \]
\[ X_{13}^\pm = \pm q^{\mp 1/4} (q^{1/4} X_1^+ X_{23}^+ - q^{-1/4} X_{23}^+ X_1^-) = \pm q^{\mp 1/4} (q^{1/4} X_{12}^+ X_3^- - q^{-1/4} X_3^+ X_{12}^-). \]  
\[ (3.47) \]

To single out \( U_q(sl(4, \mathbb{C})) \) we use the following antilinear anti-involution [165]:

\[ \omega(H) = H, \forall H \in \mathcal{H}, \quad \omega(X_{jk}^\pm) = \begin{cases} X_{jk}^\mp, & (jk) = (11), (33), \\ -X_{jk}^\pm, & \text{otherwise}. \end{cases} \]  
\[ (3.48) \]

For the six positive roots of the root system of \( sl(4, \mathbb{C}) \) one has from (2.2) that the Verma module \( V^\Lambda \) is reducible when:
where we use the classical labelling of the $su(2,2)$ representations: $2j_1, 2j_2$ are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, and $d > 0$ is the energy (or conformal dimension). First we note that $m_1$ and $m_3$ are positive, since $2j_1$ and $2j_2$ are non-negative integers. The corresponding singular vectors are:

$$v_1 = \left(X_1^+\right)^{2j_1+1}v_0, \quad v_3 = \left(X_3^+\right)^{2j_2+1}v_0,$$

and these are present for all representations we discuss. Next, it is clear that depending on the value of $d$ there may be other singular vectors. Since we are interested in the positive-energy irreps, we recall the list of these representations for $su(2,2)$ [449]:

1) $d > j_1 + j_2 + 2, \quad j_1j_2 \neq 0,$
2) $d = j_1 + j_2 + 2, \quad j_1j_2 \neq 0,$
3) $d > j_1 + j_2 + 1, \quad j_1j_2 = 0,$
4) $d = j_1 + j_2 + 1, \quad j_1j_2 = 0,$

(omitting the one-dimensional representation with $d = j_1 = j_2 = 0$). In case 1) there are no additional singular vectors. If $d = j_1 + j_2 + 2$, which is case 2) and is also possible in case 3), then $m_{13} = 1$, and there is an additional singular vector:

$$v_{13}^{(1)} = \left(2j_1\right)\left(2j_2\right)X_1^+X_2^+X_3^+ - \left(2j_1\right)\left(2j_2 + 1\right)X_1^+X_2^+X_3^+ - 2j_2X_3^+X_2^+X_1^+ - \left(2j_1 + 1\right)\left(2j_2 + 1\right)X_1^+X_2^+X_3^+ \right) v_0.$$

Further, we concentrate on case 4), that is, to the massless representations of $so(4,2)$ [165, 225, 449] for which $d = j_1 + j_2 + 1 \geq 1, j_1j_2 = 0$. For definiteness we choose first $j_2 = 0$. Then we see that in the case $j_1 \neq 0$, we have a singular vector corresponding to $m_{12} = 1$ [165, 225]:

$$v_{12} = \left(2j_1\right)X_{12}^+ - q^{ij}X_i^+X_j^+ v_0, \quad d = j_1 + 1, j_2 = 0, \quad m_{12} = 1,$$

and another one which corresponds to $m_{13} = 2$ [165, 225], which, however, is a composite one and is not relevant. When $j_1 = 0$ there is still another composite singular vector
corresponding to \( m_{23} = 1 \) [165, 225]. Furthermore, for \( j_1 = 0 \) the vector \( v_{12} = X_2^+X_1^+v_0 \) is also composite. Next we factor all invariant submodules built on these singular vectors. However, this factor representation is still reducible since it has an additional singular vector [225]:

\[
v_f = \left( X_{13}^+X_2^+ - q^{-1/2}X_{12}^+X_{23}^+ \right) \left| \bar{0} \right>,
\]

where \( \left| \bar{0} \right> \) denotes the ground-state vector of this factor representation. [This is actually a subsingular vector of the Verma module \( V^\Lambda \) (cf. [215]).] Factoring out the submodule built on \( v_f \), we obtain the irreducible lowest-weight representation \( L_\Lambda \) whose vacuum vector \( \left| \bar{0} \right> \) obeys [225]:

\[
\begin{align*}
\left( X_1^+ \right)^{2j_1+1} \left| \bar{0} \right> &= 0, \\
X_1^+ \left| \bar{0} \right> &= 0,
\end{align*}
\]

\[
\begin{align*}
\left( [2j_1]X_{12}^+ - q^jX_2^+X_1^+ \right) \left| \bar{0} \right> &= 0, \\
\left( X_{13}^+X_2^+ - q^{-1/2}X_{12}^+X_{23}^+ \right) \left| \bar{0} \right> &= 0.
\end{align*}
\]

Now we can give explicitly the basis of \( L_\Lambda \). We consider the monomials as in (3.1), but on the vacuum \( \left| \bar{0} \right> \). Taking into account all vanishing conditions we see that the basis of \( L_\Lambda \) consists of the following monomials [225]:

\[
\begin{align*}
\Phi_{k,\ell,n}^1 &= (X_{13}^+)^k(X_{12}^+)\ell(X_2^+)^n \left| \bar{0} \right>, & k, \ell, n &\in \mathbb{Z}_+, \\
\Phi_{k,\ell,n}^2 &= (X_{13}^+)^k(X_{23}^+)\ell(X_2^+)^n \left| \bar{0} \right>, & k, n &\in \mathbb{Z}_+, \ell \in \mathbb{N}, \\
\Phi_{k,\ell,n}^3 &= (X_{13}^+)^k(X_{12}^+)\ell(X_1^+)^n \left| \bar{0} \right>, & k, \ell, n &\in \mathbb{Z}_+, 1 \leq n \leq 2j_1
\end{align*}
\]

the third case being absent for \( j_1 = 0 \). We note that the different vectors in (3.56) have different weights. Thus each weight has multiplicity one and is represented by a single vector just as the singletons of \( so(3, 2) \) (cf. the previous section).

The norms squared of the basis vectors \( \| \Phi_{k,\ell,n}^a \|^2 = \langle \Phi_{k,\ell,n}^a | \Phi_{k,\ell,n}^a \rangle \) are explicitly given by [225]:

\[
\begin{align*}
\| \Phi_{k,\ell,n}^1 \|^2 &= [k]_q!\langle k + \ell \rangle_q! \langle \ell + n \rangle_q!\langle n + 2j_1 \rangle_q!/[2j_1]_q!
\end{align*}
\]

\[
\begin{align*}
\| \Phi_{k,\ell,n}^2 \|^2 &= [k]_q!\langle k + \ell \rangle_q! \langle \ell + n + 2j_1 \rangle_q!\langle n \rangle_q!/[2j_1]_q!\langle 2j_1 \rangle_q!
\end{align*}
\]

\[
\begin{align*}
\| \Phi_{k,\ell,n}^3 \|^2 &= [k]_q!\langle k + \ell + n \rangle_q! \langle \ell \rangle_q!\langle 2j_1 \rangle_q!/\langle 2j_1 - n \rangle_q!.
\end{align*}
\]

When \( q \) is not a root of unity these norms can have both signs. They are positive only for \( q = 1 \), which is the well-known classical case of \( su(2, 2) \) [449]. Note, however, that such a basis is new also for the algebra \( su(2, 2) \). Unitarity can be achieved also when \( q \) is a nontrivial root of unity, which case we consider in the next subsection.
3.3.2 Roots of 1 Case

Let us now turn to the case of the deformation parameter \( q \) being a nontrivial root of unity, namely, \( q = e^{2\pi i/N} \), \( N = 2, 3, \ldots \).

Independently of the weight \( \Lambda \) there are singular vectors for all positive roots \( \alpha \), which are given by: \((X^{\alpha}_+)^{kN}v_0, k = 1, 2, \ldots [202]\). Thus we have to impose the following vanishing of null states in our representation spaces:

\[
(X^{\alpha}_+)^{kN}|\rangle = 0. \quad (3.58)
\]

Taking into account condition (2.2) we see that if \( m_\alpha = (\rho - \Lambda)(H_\alpha) \in \mathbb{Z} \), there would be singular vectors of weights \((|m_\alpha|_N + kN)\alpha\), where \(|x|_p\) is the smallest positive integer equal to \( x \) (mod \( p \)), and \( k = 0, 1, \ldots \). In particular, we have to impose:

\[
(X^{\alpha}_+)^{(mj/N)}|\rangle = 0, \quad j = 1, 2, 3. \quad (3.59)
\]

Further our representations will be characterized by the following positive integers:

\[
\begin{align*}
n_1 &= |2j_1 + 1|_N = |m_1|_N \\
n_2 &= \begin{cases} 
-\frac{d}{N} - j_1 - j_2 + 1|_N = |m_2|_N, & \text{if } d + j_1 + j_2 \in \mathbb{Z}, \\
N, & \text{if } d + j_1 + j_2 \notin \mathbb{Z}, 
\end{cases} \\
n_3 &= |2j_2 + 1|_N = |m_3|_N.
\end{align*}
\]

(3.60)

Note that \( n_k \leq N, k = 1, 2, 3 \).

Let us recall that the finite-dimensional irreducible representations of \( sl(4, \mathbb{C}) \) (or of \( su(2, 2) \), of \( su(4) \), or of any other real form of \( sl(4, \mathbb{C}) \) and of the corresponding quantum algebras when \( q \) is not a root of unity) are parametrized by three arbitrary positive integers \( p_1, p_2, p_3 \), and the dimension of such a representation is given by:

\[
d^p_{p_1,p_2,p_3} = \frac{1}{12}p_1p_2p_3p_{12}p_{23}p_{13},
\]

(3.61)

where \( p_{12} = p_1 + p_2, p_{23} = p_2 + p_3, p_{13} = p_1 + p_2 + p_3 \).

Now the representations are divided into classes [165] depending on the values of \( n_{12} = n_1 + n_2, n_{23} = n_2 + n_3, n_{13} = n_1 + n_2 + n_3 \) and \( n_k \):

a) \( n_k \leq N \), \hspace{1cm} (3.62a)

b) \( n_{12}, n_{23} < N < n_{13} \leq 2N \), \hspace{1cm} (3.62b)

\( b' \) \( n_{12} < n_{23} = N < n_{13} \leq 2N \), or \( n_{12} \rightarrow n_{23} \), \hspace{1cm} (3.62b')

c) \( n_{12} \leq N < n_{23}, n_{13} \leq 2N \), \hspace{1cm} (3.62c)

\( c' \) \( n_{12} \leq N < n_{23}, n_{13} \leq 2N \), \hspace{1cm} (3.62c')
\(d) \ n_{23} \leq N < n_{12}, \ n_{13} \leq 2N, \quad n_1 < N, \quad (3.62d)\)

\(d') \ n_{23} \leq N < n_{12}, \ n_{13} \leq 2N, \quad n_1 = N, \quad (3.62d')\)

\(e) \ n < n_{12}, \ n_{23}, \ n_{13} \leq 2N, \quad n_2 + n_{13} < 3N, \quad (3.62e)\)

\(e') \ N < n_{12}, \ n_{23} < 2N, \quad n_2 = n_{13}/2 = N, \quad (3.62e')\)

\(f) \ n < n_{12}, \ n_{23} < 2N < n_{13} < 3N, \quad (3.62f)\)

\(f') \ n_1 = n_2 = N, \quad \text{or} \quad n_1 = n_3 = N, \quad \text{or} \quad n_2 = n_3 = N. \quad (3.62f')\)

The same classification is valid for \(U_q(sl(4, \mathbb{C}))\), where case (3.62a) is the so-called regular case. This is a refinement of the classification of [165], the primed cases being separated out since together with the regular case these have the classical dimensions of the finite-dimensional irreps of \(sl(4, \mathbb{C})\); that is, a representation characterized by \(n_1, n_2, n_3\) has dimension \(d^{(c)}_{n_1,n_2,n_3}\). In particular, in case \(f')\) with \(n_1 = n_2 = n_3 = N\) we achieve the maximal possible dimension \(N^6\) of an irrep of \(U_q(sl(4, \mathbb{C}))\) (cf. (2.113) and [175]). On the other hand, in the unprimed cases \(b) - f)\), the dimension of a representation characterized by \(n_1, n_2, n_3\) is strictly smaller than \(d^{(c)}_{n_1,n_2,n_3}\).

The representations \(U_q(sl(2, \mathbb{C}))\) inherit all the structure from their \(U_q(sl(4, \mathbb{C}))\) counterparts. Thus, the classification of the positive-energy representations of \(U_q(sl(2, \mathbb{C}))\) proceeds as follows.

Let us decompose: \(2j_k = 2f_k + r_k N, \quad 2j_k, r_k \in \mathbb{Z}_+, \quad 2j_k < N, \quad k = 1, 2\). Then we have:

\[n_1 = 2f_1 + 1, \quad n_3 = 2f_2 + 1. \quad (3.62)\]

Let us consider now the conditions of positive energy (3.51). We see that in cases 1) and 3) we have to distinguish whether \(d + j_1 + j_2\) is integer or not. If \(d + j_1 + j_2 \notin \mathbb{N}\) then \(n_2 = N, \ n_{12} = N + 2f_1 + 1 > N, \ n_{23} = N + 2f_2 + 1 > N, \ n_{13} = N + 2f_1 + 2f_2 + 2 > N.\) Thus, depending on \(n_{13}\), the possible cases are (3.62e,f).

Consider now the cases 1) and 3) of (3.51) with \(d = j_1 + j_2 + 3\) and we set \(d = p + j_1 + j_2 + 3 + kN, \quad \text{where} \quad p = 0, 1, \ldots, N - 1, \quad k \in \mathbb{Z}_+.\) Let us also set \(\kappa = 2f_1 + 2f_2 + 2 + p.\) Note that \(2 \leq \kappa \leq 3N - 1.\) Then we have:

\[n_2 = N - \kappa, \ n_{12} = N - 2f_2 - 1 - p < N, \quad (3.63a)\]

\[n_{23} = N - 2f_1 - 1 - p < N, \ n_{13} = N - p \leq N, \quad (3.63a)\]

\[\kappa < N \quad (3.63a)\]

\[n_2 = 2N - \kappa, \ n_{12} = 2N - 2f_2 - 1 - p, \quad (3.63b)\]

\[n_{23} = 2N - 2f_1 - 1 - p, \ n_{13} = 2N - p \leq 2N, \quad (3.63b)\]

\[N \leq \kappa < 2N \quad (3.63b)\]

\[n_2 = 3N - \kappa, \ n_{12} = 3N - 2f_2 - 1 - p > N, \quad (3.63c)\]

\[n_{23} = 3N - 2f_1 - 1 - p > N, \ n_{13} = 3N - p > 2N, \quad (3.63c)\]

\[2N \leq \kappa < 3N \quad (3.63c)\]
Thus, all cases of (3.62) are possible: we have case (3.62a) in (3.63a) and (3.62f,f') in (3.63c), while (3.63b) contains all cases (3.62b,b',e,e'), since both $n_{12}, n_{23}$ can be bigger or smaller than $N$.

We pass now to case 2) of (3.51), \( d = j_1 + j_2 + 2, j_1 j_2 \neq 0 \), setting $\kappa' = 2J_1 + 2J_2 + 1$. Note that $1 \leq \kappa' \leq 2N - 1$. Then we have:

\[
\begin{align*}
  n_2 &= N - \kappa', n_{12} = N - 2J_2 \leq N, \\
  n_{23} &= N - 2J_1 \leq N, n_{13} = N + 1 > N, \\
  \kappa' &< N \\
  n_2 &= 2N - \kappa', n_{12} = 2N - 2J_2 > N, \\
  n_{23} &= 2N - 2J_1 > N, n_{13} = 2N + 1 > 2N, \\
  N &\leq \kappa' < 2N
\end{align*}
\] (3.64a)

Thus, we have cases (3.62b,b') in (3.64a) and (3.62f,f') in (3.64b).

Finally we consider the massless case 4) of (3.51) \( d = j_1 + j_2 + 1, j_1 j_2 = 0 = J_1 J_2 \). We have:

\[
\begin{align*}
  n_2 &= N - 2J_1 - 2J_2, \\
  n_{12} &= N + 1 - 2J_2 \begin{cases} \leq N & \text{for } J_2 \neq 0, (J_1 = 0) \\ > N & \text{for } J_2 = 0 \end{cases} \\
  n_{23} &= N + 1 - 2J_1 \begin{cases} \leq N & \text{for } J_1 \neq 0, (J_2 = 0) \\ > N & \text{for } J_1 = 0 \end{cases} \\
  N < n_{13} &= N + 2 \leq 2N.
\end{align*}
\] (3.65)

Thus, we have case (3.62c) if $0 < J_2 < (N - 1)/2$, case (3.62c') if $J_2 = (N - 1)/2$, case (3.62d) if $0 < J_1 < (N - 1)/2$, case (3.62d') if $J_1 = (N - 1)/2$, case (3.62e) if $J_1 = J_2 = 0$ and $N > 2$. case (3.62e') if $J_1 = J_2 = 0$ and $N = 2$.

This completes the classification of the positive-energy representations of $U_q(su(2,2))$ at roots of 1.

### 3.3.3 Massless Case

Further we treat in detail the massless case at roots of 1. Since $j_1 j_2 = 0$, let us choose for definiteness $J_2 = 0$. The additional vanishing conditions (3.59) besides (3.55) and (3.58) are:

\[
\begin{align*}
  (X_1^+)^{n_1} |0\rangle &= 0, & \text{if } & n_1 < 2j_1 + 1, N, \\
  (X_2^+)^{N-2j_1} |1\rangle &= 0, & \text{if } & J_1 > 0.
\end{align*}
\] (3.66a) (3.66b)
To obtain the dimension $d(N, J_1)$ of these representations we first note that the norms given in (3.57) can be positive only in the following range of $j_1$ [165], [225]:

$$2rN \leq 2j_1 \leq (2r + 1)N - 1, \quad \forall r \in \mathbb{Z}^+;$$

that is, in terms of the decomposition $2j_1 = 2J_1 + r_1N$ we consider only $r_1 = 2r \in 2\mathbb{Z}^+$.

For fixed $j_1$ in the above range, the basis of the massless unitary irreducible representation is given by [225]:

$$\Phi^1_{k,\ell,n}, \quad k,\ell,n \in \mathbb{Z}^+, \quad k+\ell+n \leq N-1,$n \leq N-2J_1-1,$$

$$\Phi^2_{k,\ell,n}, \quad k,n \in \mathbb{Z}^+, \quad \ell \in \mathbb{N}, \quad k+\ell \leq N-1,$$n \leq N-2J_1-1,$$

$$\Phi^3_{k,\ell,n}, \quad k,\ell,n \in \mathbb{Z}^+, \quad k+\ell+n \leq N-1,$1 \leq n \leq 2J_1.$$ (3.68)

The norms of these vectors are given by (3.57) with $j_1$ replaced by $J_1$ and are strictly positive. Now we can find that the number of states in (3.68a), (3.68b) and (3.68c), respectively, is [225]:

$$\frac{1}{6}(N-2J_1)(2N^2 + N(4J_1 + 3) + 1 - 4J_1^2),$$
$$\frac{1}{6}(N-2J_1)(N-2J_1-1)(2N + 2J_1 - 1),$$
$$\frac{1}{3}J_1(3N^2 - 6NJ_1 - 1 + 4J_1^2).$$

The sum of these three numbers gives the dimension of the massless irreps (cf. [165],[225]):

$$d(N, J_1) = \frac{1}{3} \left[ 2N^3 - N(12J_1^2 - 1) + 3J_1(4J_1^2 - 1) \right].$$

(3.70)

We recall that in the classical case the massless unitary representations are infinite-dimensional. However, we may compare our representations with the undeformed non-unitary finite-dimensional representations which have the same quantum numbers $(n_1, n_2, n_3) = (2J_1 + 1, N - 2J_1, 1)$. We note that the dimension of the former is generically smaller than the dimension of the latter, which is given by:

$$d_{2J_1+1,N-2J_1,1}^c = \frac{1}{12} (2J_1 + 1)(N - 2J_1)(N + 1)(N - 2J_1)(N + 2),$$

(3.71)
except when \( N = 2, J_1 = 0 \), and then \( d(2, 0) = d^e = 6 \), or \( N = 2J_1 + 1, J_1 > 0 \), and then:

\[
d_0 \equiv d(2J_1 + 1, J_1) = d^e = \frac{1}{3} (J_1 + 1)(2J_1 + 1)(2J_1 + 3) = \frac{1}{6} N(N + 1)(N + 2).
\] (3.72)

The irreps for \( N = 2 \) with \( J_1 = 0, \frac{1}{2} \) are deformations of two of the three fundamental representations of \( su(2, 2) \) with dimensions six and four, respectively, [165].

Finally, we note that one considers the remaining massless representations with \( j_1 = 0 \) and \( j_2 \neq 0 \) in the same way. Thus, in the dimension formulae one has to exchange all subscripts \( 1 \rightarrow 3 \). Also one may introduce the helicity \( h = j_1 - j_2 \), then all the formulae above may be written in terms of \( |h| \). Thus, for the exceptional case \( N = 2|h| + 1, h \neq 0 \), we have (cf. (3.72)) [165]:

\[
d_0 = \frac{1}{3} (|h| + 1)(2|h| + 1)(2|h| + 3) = \frac{1}{6} N(N + 1)(N + 2).
\] (3.73)

In particular, for \( N = 2, J_2 = 1/2 \) one obtains a deformation of the third fundamental representation of \( su(2, 2) \) with dimension four [165].

Thus the maximal possible dimension of a massless irrep for fixed \( N \) is \( d_0 \) for \( N > 2 \) and six for \( N = 2 \). Note that this maximal dimension is strictly smaller than the minimal dimension of a (semi-) periodic irrep of \( U_q(sl(4, \mathbb{C})) \), which is \( N^3 \) [177].

### 3.3.4 Character Formulae

It is easy to see that the spectrum given in (3.56) can be represented by the following character formula [225]:

\[
ch L = e(\Lambda) \left( \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} t_{13}^k t_{12}^\ell t_2^n + \right.
\]

\[
+ \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} t_{13}^k t_{23}^\ell t_2^n + \right.
\]

\[
+ \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} t_{13}^k t_{12}^\ell t_1^n \right)
\] (3.74)

where \( t_{12} = e(a_{12}) = t_1 t_2 \), \( t_{23} = e(a_{23}) = t_2 t_3 \), \( t_{13} = e(a_{13}) = t_1 t_3 \). Next we note that the character formula for the Verma module with the same lowest weight here is:

\[
ch V^\Lambda = e(\Lambda)/(1-t_1)(1-t_2)(1-t_3)(1-t_{12})(1-t_{23})(1-t_{13}).
\] (3.75)

Now we can rewrite the character formula (3.74) as follows [225]:

\[
ch L_\Lambda = ch V^\Lambda Q(t_1, t_2, t_3) =
\]

\[
= ch V^\Lambda (1-t_1^n + t_3^n t_2^n - t_3 -
\] (3.76)
Finally, we can show that (3.78) may be represented as follows:

\[-t_1t_2 + t_1^{n_1}t_2 - t_1^{n_1}t_2t_3^2 + t_1t_2t_3^2 - t_1^{n_1}t_2^2t_3 + t_1^{n_1}t_2t_3^2 - t_1^{n_1}t_2^2t_3 - t_1^{n_1}t_2^2t_3^2 + t_1^{n_1}t_2t_3^2,\]

\[n_1 = 2j_1 + 1 \geq 1, \quad d = j_1 + 1, \quad j_2 = 0.\]

This formula is valid for all \(j_1 \in (1/2)\mathbb{Z}_+, j_2 = 0\). Note, however, that for \(j_1 = 1/2\) the terms in the fourth row cancel each other, while for \(j_1 = 0\) the terms in the third row cancel each other. To show that (3.76) coincides with (3.74) amounts to the explicit straightforward division of the polynomials:

\[
\frac{Q(t_1, t_2, t_3)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_1)(1 - t_2)(1 - t_3)}. \tag{3.77}
\]

The formula (3.76) represents an alternating sign summation over part of the Weyl group of \(\text{sl}(4, \mathbb{C})\) (called reduced Weyl group in [209]) and may be obtained using [381, 382]. Note, however, that the ultimate formula is (3.74), which is obtained in a straightforward manner.

Analogously, the spectrum given in (3.68) can be represented by the following character formula:

\[
\text{ch } L_A = e(\Lambda) \left( \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1-k} \sum_{n=0}^{\min(N-1-\ell, N-1-2j_1)} t_{13}^k t_{12}^n + \right. \\
+ \sum_{k=0}^{N-1} \sum_{\ell=1}^{N-1-k} \sum_{n=0}^{N-1-2j_1} t_{13}^k t_{23}^n + \\
\left. \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1-k} \sum_{n=1}^{\min(N-1-\ell, 2j_1)} t_{13}^k t_{12}^n \right) \tag{3.78}
\]

Finally, we can show that (3.78) may be represented as follows:

\[
\text{ch } L_A = \text{ch } L_{2j_1+1,N-2j_1,1}^{\mathbb{C}} - \text{ch } L_{2j_1-1,N-1-2j_1,2}^{\mathbb{C}} + \text{ch } L_{2j_1-1,N-1-2j_1,1}^{\mathbb{C}},
\]

\[J_1 \neq 0, \tag{3.79a}\]

\[= \text{ch } L_{1,N,1}^{\mathbb{C}} - \text{ch } L_{1,N-2,1}^{\mathbb{C}}, \quad J_1 = 0, \tag{3.79b}\]

where \(L_{n_1,n_2,n_3}^{\mathbb{C}}\), \(n_1, n_2, n_3 \in \mathbb{N}\), denote the finite-dimensional irreducible (non-unitary) representation of \(\text{su}(2, 2)\) with character formula (cf. [195]):

\[
\text{ch } L_{n_1,n_2,n_3} = \text{ch } V^{A} \left( 1 - t_1^{n_1} - t_2^{n_2} - t_3^{n_3} + t_1^{n_1}t_3^{n_3} + t_1^{n_1}t_2^{n_2} + +t_3^{n_3}t_2^{n_2} + t_1^{n_1}t_2^{n_2} + t_3^{n_3}t_2^{n_2} - t_1^{n_1}t_2^{n_2}t_3^{n_3} - \\
- t_1^{n_1}t_2^{n_2}t_3^{n_3} - t_1^{n_1}t_2^{n_2}t_3^{n_3} - t_1^{n_1}t_2^{n_2}t_3^{n_3} - \\
- (t_1t_2)^{n_1} - (t_2t_3)^{n_2} + t_1^{n_1}t_2^{n_2} + t_3^{n_3}t_2^{n_2} + \right)
\]
\[ + t_1^{n_1} (t_2 t_3)^{n_{13}} + (t_1 t_2)^{n_{12}} t_3^{n_{13}} + t_1^{n_{13}} (t_2 t_3)^{n_{23}} + \\
( t_1 t_2)^{n_{13}} t_3^{n_{12}} - t_1^{n_{12}} t_2^{n_{13}} t_3^{n_{13}} - t_1^{n_{13}} t_2^{n_{23} + n_{13}} t_3^{n_{23}} - \\
- (t_1 t_2 t_3)^{n_{13}} + (t_1 t_2 t_3)^{n_{13}} t_2^{n_{13}} \right) \tag{3.80} \]

and dimension \( d_{n_1, n_2, n_3}^c \) (cf. (3.61a)) and in (3.79) we use the convention \( chL_{n_1, n_2, n_3} = 0 \) if any \( n_k = 0 \), which happens for \( J_1 = 1/2 \) or for \( N = 2J_1 + 1 \). A simple consequence of (3.79) is:

\[
 d(N, J_1) = \begin{cases} 
 d_{2J_1+1, N-2J_1, 1}^c - d_{2J_1, N-1-2J_1, 2}^c + d_{2J_1-1, N-1-2J_1, 1}, & J_1 \neq 0, \\
 d_{1, N, 1}^c - d_{1, N-2, 1}, & J_1 = 0.
\end{cases} \tag{3.81} \]

As we noted the dimensions of the massless representations are generically smaller than the corresponding classical dimensions (the first terms on the RHS of (3.81)).