6 Invariant $q$-Difference Operators Related to $GL_q(n)$

Summary
This chapter is devoted to the detailed consideration of the $q$-difference operators related to $GL_q(n)$. We consider in detail several special cases, in particular, the case of $U_q(sl(3))$ and the polynomial solutions of $q$-difference equations. The relation of these solutions with the Gelfand–(Weyl)–Zetlin basis is studied in detail, also in the case of roots of unity, where new features are discovered. The case of $U_q(sl(4))$ is developed also in preparation for the subsequent chapter. This chapter is based mainly on [211, 220, 230, 244–246].

6.1 Representations Related to $GL_q(n)$

In this section we follow mainly [211, 220]. We consider again the matrix quantum group $A_q = GL_q(n)$, $q \in \mathbb{C}$, introduced in Section 4.1 though replacing $q^{1/2}$ by $q$. Thus, we set instead of (4.4) ($\lambda = q - q^{-1}$):

\begin{align*}
M_{i\ell}M_{ij} &= qM_{ij}M_{i\ell}, \quad \text{for } \ell > j, \\
M_{kj}M_{ij} &= qM_{ij}M_{kj}, \quad \text{for } k > i, \\
M_{kj}M_{i\ell} &= M_{i\ell}M_{kj}, \quad \text{for } k > i, \ell > j, \\
M_{ij}M_{k\ell} &= M_{k\ell}M_{ij} - \lambda M_{i\ell}M_{kj}, \quad \text{for } k > i, \ell > j.
\end{align*}

This algebra has determinant $\mathcal{D}$ given by (4.6) but with

$$
e(w) = \prod_{j<k} (-q^{-1}) = (-q^{-1})^{\epsilon(w)}.
$$

Next one defines the left and right quantum cofactor matrix $A_{ij}$ [462]:

$$
A_{ij} = \sum_{w(i)=j} \frac{\epsilon(w-\sigma_i)}{\epsilon(\sigma_i)} M_{1,w(1)} \cdots \tilde{M}_{ij} \cdots M_{n,w(n)} = \sum_{w(j)=i} \frac{\epsilon(w-\sigma'_j)}{\epsilon(\sigma'_j)} M_{w(1),1} \cdots \tilde{M}_{ij} \cdots M_{w(n),n},
$$

where $\sigma_i$ and $\sigma'_j$ denote the cyclic permutations:

$$
\sigma_i = \{i, \ldots, 1\}, \quad \sigma'_j = \{j, \ldots, n\}.
$$
and the notation \( \hat{x} \) indicates that \( x \) is to be omitted. Now one can show that [462]:

\[
\sum_j M_{ij} A_{\ell j} = \sum_j A_{ji} M_{j \ell} = \delta_{\ell \ell} \mathcal{D}, \tag{6.5}
\]

and obtain the left and right inverse [462]:

\[
M^{-1} = \mathcal{D}^{-1} A = A \mathcal{D}^{-1}. \tag{6.6}
\]

Thus, the antipode in \( GL_q(n) \) is [462] (cf. also (4.10)):

\[
\gamma_{\mathcal{D}}(M_{ij}) = D_{ji}^{-1} = A_{ji} D^{-1}. \tag{6.7}
\]

Next we introduce a basis of \( GL_q(n) \) which consists of monomials

\[
f = (M_{21})^{p_{21}} \ldots (M_{n,n-1})^{p_{n,n-1}} (M_{11})^{\ell_1} \ldots (M_{nn})^{\ell_n} \times
\]

\[
(M_{n-1,n})^{n_{n-1,n}} \ldots (M_{12})^{n_{12}} = \bar{f}_{\ell, p, n}, \tag{6.8}
\]

where \( \ell, p, n \) denote the sets \( \{ \ell_i \}, \{ p_{ij} \}, \{ n_{ij} \} \), respectively, \( \ell_i, p_{ij}, n_{ij} \in \mathbb{Z}_+ \) and we have used the so-called normal ordering of all elements \( M_{ij} (1 \leq i, j \leq n) \). Namely, we first put the elements \( M_{ij} \) with \( i > j \) in lexicographic order; that is, if \( i < k \) then \( M_{ij} (i > j) \) is before \( M_{k\ell} (k > \ell) \) and \( M_{ti} (t > i) \) is before \( M_{tk} (t > k) \); then we put the elements \( M_{ij} \); finally we put the elements \( M_{ij} \) with \( i < j \) in antilexicographic order; that is, if \( i > k \) then \( M_{ij} (i < j) \) is before \( M_{k\ell} (k < \ell) \) and \( M_{ti} (t < i) \) is before \( M_{tk} (t < k) \). Note that the basis (6.8) includes also the unit element \( 1_{\mathcal{D}} \) of \( \mathcal{D} \) when all \( \{ \ell_i \}, \{ p_{ij} \}, \{ n_{ij} \} \) are equal to zero; that is:

\[
f_{0,0,0} = 1_{\mathcal{D}}. \tag{6.9}
\]

We need the algebra in duality with \( GL_q(n) \). This is the algebra \( \mathcal{U}_g = U_q(sl(n)) \otimes U_q(\mathcal{D}) \), where \( U_q(\mathcal{D}) \) is central in \( \mathcal{U}_g \) [209, 233]. Let us denote the Chevalley generators of \( sl(n) \) by \( H_i, X_i^\pm, i = 1, \ldots, n - 1 \). Then we take (as in (1.52)) for the rational “Chevalley” generators of \( \mathcal{U} = U_q(sl(n)) \) : \( k_i = q^{\ell_i/2}, k_i^{-1} = q^{-\ell_i/2}, X_i^\pm, i = 1, \ldots, n - 1, \) with the following algebra relations:

\[
k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1_{\mathcal{U}_g}, \quad k_i X_i^\pm = q^{\pm a_{ij}} X_i^\pm k_i \tag{6.10a}
\]

\[
[X_i^+, X_j^-] = \delta_{ij} (k_i^2 - k_i^{-2})/\lambda, \tag{6.10b}
\]

\[
(X_i^\pm)^2 - [2]_{q} X_i^\pm X_j^\pm X_i^\pm + X_i^\pm (X_i^\pm)^2 = 0, |i - j| = 1 \tag{6.10c}
\]

\[
[X_i^+, X_j^-] = 0, |i - j| \neq 1 \tag{6.10d}
\]
where \(a_{ij}\) is the Cartan matrix of \(\mathfrak{sl}(n)\), and coalgebra relations:

\[
\begin{align*}
\delta_i (k^+_i) &= k^+_i \otimes k^+_i, \\
\delta_i (X^+_i) &= k^+_i \otimes k_i + k_i^{-1} \otimes X^+_i, \\
\epsilon_i (k^+_i) &= 1, \quad \epsilon_i (X^+_i) = 0, \\
\gamma_i (k_i) &= k_i^{-1}, \quad \gamma_i (X^+_i) = -q^{-1}X^+_i,
\end{align*}
\]  

(6.11a–d)

where \(k^+_i = k_i\), \(k_i^- = k_i^{-1}\). Further, we denote the generator of \(\mathcal{X}\) by \(H\) and the generators of \(U_q(\mathcal{X})\) by \(k = q^{\frac{H}{2}}, k^{-1} = q^{-\frac{H}{2}}, kk^{-1} = k^1 = 1_{\mathcal{U}_g}\). The generators \(k, k^{-1}\) commute with the generators of \(\mathcal{Y}\), and their coalgebra relations are as those of any \(k_i\). From now on we shall give most formulae only for the generators \(k_i, X^+_i, k\), since the analogous formulae for \(k_i^{-1}, k^{-1}\) follow trivially from those for \(k_i, k\), respectively.

The bilinear form giving the duality pairing between \(\mathcal{U}_g\) and \(\mathcal{A}_g\) is given by [233]:

\[
\begin{align*}
\langle k_i, M_{j\ell} \rangle &= \delta_{j\ell} q^{(\delta_{ij} - \delta_{i+1,j})/2}, \\
\langle X^+_i, M_{j\ell} \rangle &= \delta_{j+1,\ell} \delta_{ij}, \\
\langle X^-_i, M_{j\ell} \rangle &= \delta_{j-1,\ell} \delta_{ij}, \\
\langle k, M_{j\ell} \rangle &= \delta_{j\ell} q^{1/2}.
\end{align*}
\]  

(6.12a–d)

The pairing between arbitrary elements of \(\mathcal{U}_g\) and \(f\) follows then from the properties of the duality pairing. The pairing (6.12) is standardly supplemented with

\[
\langle y, 1_{\mathcal{A}_g} \rangle = \epsilon_{\mathcal{U}_g} (y).
\]  

(6.13)

It is well known that the pairing provides the fundamental representation of \(\mathcal{U}_g\):

\[
F(y)_{j\ell} = \langle y, M_{j\ell} \rangle, \quad y = k_i, X^+_i, k.
\]  

(6.14)

Of course, \(F(k) = q^{1/2}I_n\), where \(I_n\) is the unit \(n \times n\) matrix.

### 6.1.1 Actions of \(U_q(\mathfrak{gl}(n))\) and \(U_q(\mathfrak{sl}(n))\)

We begin by defining two actions of the quantum algebra in duality \(\mathcal{U}_g\) on the basis \(6.8\) of \(\mathcal{A}_g\).

First we introduce the left regular representation of \(\mathcal{U}_g\) which in the \(q = 1\) case is the infinitesimal version of:

\[
\pi(Y) M = Y^{-1} M, \quad Y, M \in \mathbb{GL}(n).
\]  

(6.15)
Explicitly, we define the action of $U_g$ as follows:

$$\pi(y)M_{ij} \doteq \left(F\left(\gamma^0_{U_g}(y)\right)M\right)_{ij} = \sum_j \langle \gamma^0_{U_g}(y), M_{ij} \rangle M_{ij} = \sum_j \gamma^0_{U_g}(y) M_{ij} M_{ij}$$

where $y$ denotes the generators of $U_g$ and $\gamma^0_{U_g}$ is the antipode $\gamma_{U_g}$ for $q = 1$, the possible pairs being given explicitly by:

$$(y, \gamma^0_{U_g}(y)) = (k_1, k^{-1}_1), (X^+_i, X^-_i), (k, k^{-1}).$$

From (6.16) we find the explicit action of the generators of $U_g$:

$$\pi(k_i)M_{ij} = q^{(\delta_{i+1} - \delta_{ij})/2} M_{ij},$$
$$\pi(X^+_i)M_{ij} = -\delta_{ij} M_{j+1}M_{ij},$$
$$\pi(X^-_i)M_{ij} = -\delta_{i+1} M_{j-1}M_{ij},$$
$$\pi(k)M_{ij} = q^{1/2} M_{ij}.$$ (6.18)

The above is supplemented with the following action on the unit element of $U_q$:

$$\pi(k_i)1_{U_q} = 1_{U_q}, \quad \pi(X^+_i)1_{U_q} = 0, \quad \pi(k)1_{U_q} = 1_{U_q}.$$ (6.19)

In order to derive the action of $\pi(y)$ on arbitrary elements of the basis (6.8), we use the twisted derivation rule consistent with the coproduct and the representation structure, namely, we take: $\pi(y)\varphi\psi = \pi(\delta'_{U_q}(y))(\varphi \otimes \psi)$, where $\delta'_{U_q} = \sigma \circ \delta_{U_q}$ is the opposite coproduct ($\sigma$ is the permutation operator). Thus, we have:

$$\pi(k_i)\varphi\psi = \pi(k_i)\varphi \cdot \pi(k_i)\psi,$$ (6.20a)
$$\pi(X^+_i)\varphi\psi = \pi(X^+_i)\varphi \cdot \pi(k^{-1}_i)\psi + \pi(k_i)\varphi \cdot \pi(X^+_i)\psi,$$ (6.20b)
$$\pi(k)\varphi\psi = \pi(k)\varphi \cdot \pi(k)\psi.$$ (6.20c)

From now on we suppose that $q$ is not a nontrivial root of unity. Applying the above rules one obtains:

$$\pi(k_i)(M_{ij})^n = q^n(\delta_{i+1} - \delta_{ij})/2 (M_{ij})^n,$$ (6.21a)
$$\pi(X^+_i)(M_{ij})^n = -\delta_{ij} c_n (M_{ij})^{n-1} M_{j+1}M_{ij},$$ (6.21b)
$$\pi(X^-_i)(M_{ij})^n = -\delta_{i+1} c_n M_{j-1}M_{ij} (M_{ij})^{n-1},$$ (6.21c)
$$\pi(k)(M_{ij})^n = q^{n/2} (M_{ij})^n.$$ (6.21d)
where
\[ c_n = q^{(n-1)/2}[n]_q, \quad [n]_q = (q^n - q^{-n})/\lambda. \quad (6.22) \]

Note that (6.19) and (6.18) are partial cases of (6.21) for \( n = 0 \) and \( n = 1 \) respectively (cf. (6.9)).

Analogously, we introduce the right action (see also [465]) which in the classical case is the infinitesimal counterpart of:
\[ 0^R(Y)M = MY, \quad Y, M \in GL(n). \quad (6.23) \]

Thus, we define the right action of \( U_g \) as follows:
\[ 0^R(y_{i\ell})M_{j\ell}^n = (MF(y))_{i\ell}^n = \sum_j M_{ij}^n(y, M_{j\ell}), \quad (6.24) \]

where \( y \) denotes the generators of \( U_g \).

From (6.24) we find the explicit right action of the generators of \( U_g \):
\[ 0^R(k_i)M_{j\ell}^n = q^{(\delta_{i\ell} - \delta_{i+1,\ell})/2} M_{j\ell}^n, \quad (6.25a) \]
\[ 0^R(X_{i+}^+)M_{j\ell}^n = \delta_{i+1,\ell} M_{j,\ell-1}, \quad (6.25b) \]
\[ 0^R(X_{i-}^-)M_{j\ell}^n = \delta_{i\ell} M_{j,\ell+1}, \quad (6.25c) \]
\[ 0^R(k)M_{j\ell}^n = q^{1/2} M_{j\ell}, \quad (6.25d) \]

supplemented by the right action on the unit element:
\[ 0^R(k_i)1_{U_g} = 1_{U_g}, \quad 0^R(X_{i+}^+)1_{U_g} = 0, \quad 0^R(k)1_{U_g} = 1_{U_g}. \quad (6.26) \]

The twisted derivation rule is now given by \( 0^R(y)\varphi \psi = \varphi 0^R(y)(\varphi \otimes \psi) \); that is,
\[ 0^R(k_i)\varphi \psi = \varphi 0^R(k_i)\psi, \quad (6.27a) \]
\[ 0^R(X_{i+}^+)\varphi \psi = \varphi 0^R(X_{i+}^+)\psi + \sum R(k_i)^{-1}\varphi \cdot 0^R(X_{i+}^+)\psi, \quad (6.27b) \]
\[ 0^R(k)\varphi \psi = \varphi 0^R(k)\psi. \quad (6.27c) \]

Using this, we find:
\[ 0^R(k_i)(M_{j\ell})^n = q^{n(\delta_{i\ell} - \delta_{i+1,\ell})/2}(M_{j\ell})^n, \quad (6.28a) \]
\[ 0^R(X_{i+}^+)M_{j\ell}^n = \delta_{i+1,\ell} c_n M_{j,\ell-1}(M_{j\ell})^{n-1}, \quad (6.28b) \]
\[ 0^R(X_{i-}^-)(M_{j\ell})^n = \delta_{i\ell} c_n (M_{j\ell})^{n-1} M_{j,\ell+1}, \quad (6.28c) \]
\[ 0^R(k)(M_{j\ell})^n = q^{n/2} (M_{j\ell})^n. \quad (6.28d) \]
6.1.2 Representation Spaces

Let us now introduce the elements $\varphi$ as formal power series of the basis (6.8):

$$\varphi = \sum_{\ell, m, \ell \in \mathbb{Z}_+} \mu_{\ell, m, n}(M_{21})^{m1} \cdots (M_{n,n-1})^{m_{n-1}}(M_{11})^{\ell 1} \cdots (M_{mn})^{\ell n} \times \times (M_{n-1,n})^{n_{n-1}} \cdots (M_{12})^{n_{12}}. \quad (6.29)$$

By (6.21) and (6.28) we have defined left and right action of $U$ on $\varphi$. As in the classical case the left and right actions commute, and as in [197] we shall use the right covariance to reduce the left regular representation. In particular, we would like the right action to mimic some properties of a highest-weight module, that is, annihilation by the raising generators $X_i^+$ and scalar action by the (exponents of the) Cartan operators $k_i, k$. However, first we have to make a change of basis using the $q$-analogue of the classical Gauss decomposition. For this we have to suppose that the principal minor determinants of $M$:

$$D_m = \sum_{w \in S_m} e(w)M_{1,w(1)} \cdots M_{m,w(m)} =$$

$$= \sum_{w \in S_m} e(w)M_{w(1),1} \cdots M_{w(m),m}, \quad m \leq n, \quad (6.30)$$

are invertible; note that $D_n = D, D_{n-1} = A_{nn}$.

Further, for the ordered sets $I = \{i_1 < \cdots < i_\ell\}$ and $J = \{j_1 < \cdots < j_r\}$, let $\xi_{ij}^r$ be the $r$-minor determinant with respect to rows $I$ and columns $J$ such that

$$\xi_{ij}^r = \sum_{w \in S_r} e(w)M_{i_{w(1)j_1}} \cdots M_{i_{w(r)j_r}}. \quad (6.31)$$

Note that $\xi_{1 \cdots i}^1 = D_i$. Then one has [63] $(i, j, \ell, i, 1, \ldots, n)$:

$$M_{i\ell} = \sum_j B_{ij}Z_{i\ell}, \quad B_{i\ell} = \xi_{1 \cdots i \ell}^{-1} D_{i-1}^{-1}, \quad Z_{i\ell} = D_{i-1}^{-1} \xi_{1 \cdots i-1 \ell}^{-1}, \quad (6.32)$$

$B_{i\ell} = 0$ for $i < \ell, Z_{i\ell} = 0$ for $i > \ell$, (which follows from the obvious extension of (6.31) to the case when $I, J$, resp. $I$, is not ordered). Then $Z_{ij}, i < j$, may be regarded as a $q$-analogue of local coordinates of the coset $B_iGL(n)$.

For our purposes we need a refinement of this decomposition:

$$B_{i\ell} = \check{Y}_{i\ell} D_{i\ell}, \quad \check{Y}_{i\ell} = \xi_{1 \cdots i \ell}^{-1} D_{i-1}^{-1}, \quad D_{i\ell} = D_{i-1}^{-1}, \quad (D_0 = 1_{cg}), \quad (6.33)$$

where $\check{Y}_{j\ell}, j > \ell$, may be regarded as a $q$-analogue of local coordinates of the coset $GL(n)/DZ$. 


Clearly, we can replace the basis (6.8) of $A_{\ell}$ with a basis in terms of $\tilde{Y}_{i\ell}^{}$, $i > \ell$, $\mathcal{R}_{\ell}$, $Z_{i\ell}^{}$, $i < \ell$. (Note that we set $\tilde{Y}_{i\ell}^{} = Z_{i\ell}^{} = 1_{A_{\ell}}$. ) Thus, we consider formal power series:

$$
\varphi = \sum_{\tilde{\ell} \in \mathbb{Z}, \tilde{m} \in \mathbb{Z}_,} \mu_{\tilde{\ell}, \tilde{m}, \tilde{n}} (\tilde{Y}_{21}^{})^{m_{21}} \cdots (\tilde{Y}_{n,n-1}^{})^{m_{n,n-1}} (\mathcal{R}_{1}^{\ell_1} \cdots (\mathcal{R}_{n}^{\ell_n})^n \times
\times (Z_{n-1,n}^{})^{n-1,n} \cdots (Z_{12}^{})^{n,12}.
$$

(6.34)

Now, let us impose right covariance (cf. [197]) with respect to $X_i^+$, that is, we require:

$$
\pi_R^{}(X_i^+) \varphi = 0.
$$

(6.35)

First we notice by a direct calculation that:

$$
\pi_R^{}(X_i^+) \xi_j^l = 0, \quad \text{for } J = \{1, \ldots, j\}, \forall I,
$$

(6.36)

from which follow:

$$
\pi_R^{}(X_i^+) \mathcal{R}_j^{} = 0, \quad \pi_R^{}(X_i^+) \tilde{Y}_{j\ell}^{} = 0.
$$

(6.37)

On the other hand $\pi_R^{}(X_i^+)$ acts nontrivially on $Z_{j\ell}^{}$:

$$
\pi_R^{}(X_i^+) Z_{j\ell}^{} = \delta_{i+1,\ell} q_{ij/2}^{} Z_{j\ell-1}^{}.
$$

(6.38)

Thus, (6.35) simply means that our functions $\varphi$ do not depend on $Z_{j\ell}^{}$. Thus, the functions obeying (6.35) are:

$$
\varphi = \sum_{\tilde{\ell} \in \mathbb{Z}, \tilde{m} \in \mathbb{Z}_,} \mu_{\tilde{\ell}, \tilde{m}, \tilde{n}} (\tilde{Y}_{21}^{})^{m_{21}} \cdots (\tilde{Y}_{n,n-1}^{})^{m_{n,n-1}} (\mathcal{R}_{1}^{\ell_1} \cdots (\mathcal{R}_{n}^{\ell_n})^n \times
\times (Z_{n-1,n}^{})^{n-1,n} \cdots (Z_{12}^{})^{n,12}.
$$

(6.39)

Next, we impose right covariance with respect to $k_i^+, k_i$:

$$
\pi_R^{}(k_i^+) \varphi = q_{i/2}^{\ell_j} \varphi, \\
\pi_R^{}(k_i) \varphi = q_{i/2}^{\ell_j} \varphi,
$$

(6.40a)

(6.40b)

where $r_i$, $\tilde{r}$ are parameters to be specified below. On the other hand using (6.27a,c), ((6.28)a,c) we have:

$$
\pi_R^{}(k_i) \xi_j^l = q_{ij/2}^{\ell_j} \xi_j^l, \quad \pi_R^{}(k) \xi_j^l = q_{j/2}^{\ell_j} \xi_j^l, \quad \text{for } J = \{1, \ldots, j\}, \forall I,
$$

(6.41)

from which follows:

$$
\pi_R^{}(k_i) \mathcal{R}_j^{} = q_{ij/2}^{\ell_j} \mathcal{R}_j^{}, \quad \pi_R^{}(k) \mathcal{R}_j^{} = q_{j/2}^{\ell_j} \mathcal{R}_j^{},
$$

(6.42a)

$$
\pi_R^{}(k_i) \tilde{Y}_{j\ell}^{} = \tilde{Y}_{j\ell}^{}, \quad \pi_R^{}(k) \tilde{Y}_{j\ell}^{} = \tilde{Y}_{j\ell}^{},
$$

(6.42b)
and thus we have:

\[
\pi_R(k_i) \varphi = q^{\ell_i/2} \varphi, \quad (6.43a)
\]
\[
\pi_R(k) \varphi = q^{\sum_{j=1}^{n} \ell_j/2} \varphi. \quad (6.43b)
\]

**Remark 6.1.** For \( q = 1 \) the elementary representations (in particular, the right covariance conditions) for a complex semisimple Lie group \( G_c \) are given by (cf. Volume 1):

\[
C_{\Lambda, \Lambda'} = \{ \mathcal{F} \in C^\infty(G_c) | \mathcal{F}(g x n) = e^{\Lambda(X) + \Lambda'(\overline{X})} \cdot \mathcal{F}(g) \}, \quad \Lambda(X) - \Lambda'(X) \in \mathbb{Z}, \quad (6.44)
\]

where \( x = \exp(X), X \in \mathcal{H}_c, n \in G^+_c = \exp(\mathcal{H}_c) \) of the Lie algebra \( \mathcal{H}_c \) of \( G_c \), and the last condition in (6.44) is necessary to ensure uniqueness on the Cartan subgroup \( H_c = \exp(\mathcal{H}_c) \) of \( G_c \). In the quantum group setting above, for simplicity, we are using infinitesimal holomorphic representations for which \( \mathcal{D} = 0 \). For \( U_q(\mathfrak{sl}(2)) \) with \( \mathcal{D} \neq 0 \) we refer to Section 5.3 where this construction was carried out for a \( q \)-deformed Lorentz algebra.

Comparing right covariance conditions (6.40) with the direct calculations (6.43) we obtain:

\[
\ell_i = r_i, \quad \text{for} \quad i < n, \quad \sum_{j=1}^{n} \ell_j = \tilde{r}. \quad \text{This means that} \quad r_i, \tilde{r} \in \mathbb{Z} \quad \text{and that there is no summation in} \quad \ell_i, \quad \text{also} \quad \ell_n = (\tilde{r} - \sum_{i=1}^{n-1} r_i) / n.
\]

Thus, the reduced functions obeying (6.35) and (6.40) are:

\[
\varphi = \sum_{m \in \mathbb{Z}^+} \mu_m (\tilde{Y}_{21})^{m_{21}} \ldots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} (\mathcal{D}_1)^{\ell_1} \ldots (\mathcal{D}_{n-1})^{\ell_{n-1}} (\mathcal{D}_n)^{\ell_n}, \quad (6.45)
\]

where \( \tilde{\ell} = (\tilde{r} - \sum_{i=1}^{n-1} r_i) / n. \)

Next we would like to derive the \( \mathcal{U} \) action \( \pi \) on \( \varphi \). First, we notice that \( \mathcal{U} \) acts trivially on \( \mathcal{D}_n = \mathcal{D} \):

\[
\pi(X_i^\pm) \mathcal{D} = 0, \quad \pi(k_i) \mathcal{D} = \mathcal{D}. \quad (6.46)
\]

Then we note:

\[
\pi(k) \mathcal{D}_j = q^{-\ell_j/2} \mathcal{D}_j, \quad \pi(k) \tilde{Y}_{j\ell} = \tilde{Y}_{j\ell}, \quad (6.47)
\]

from which follows:

\[
\pi(k) \varphi = q^{-\tilde{r}/2} \varphi. \quad (6.48)
\]

Thus, the action of \( \mathcal{U} \) involves only the parameters \( r_i, i < n \), while the action of \( U_q(\mathcal{D}) \) involves only the parameter \( \tilde{r} \). Thus we can consistently also from the representation theory point of view restrict to the matrix quantum group \( SL_q(n) \); that is, we set:
Then the quantum algebra in duality is $\mathcal{U} = U_q(sl(n))$. This is justified as in the $q = 1$ case [197] since for our considerations only the semisimple part of the algebra is important.

Thus, the reduced functions for the $\mathcal{U}$ action are:

$$\tilde{\Phi}(\tilde{Y}, \mathcal{D}) = \sum_{m \in \mathbb{Z}_+} \mu_m(\tilde{Y}_2)_{m+1} \ldots (\tilde{Y}_{n-1})_{m+n-1} \times$$

$$\times (\mathcal{D}_1)^{r_1} \ldots (\mathcal{D}_{n-1})^{r_{n-1}} = \tilde{\Phi}(\tilde{Y})(\mathcal{D}_1)^{r_1} \ldots (\mathcal{D}_{n-1})^{r_{n-1}}$$  \hspace{2cm} (6.50a)

$$= \tilde{\Phi}(\tilde{Y}_l)_{\ell} \mathcal{D}_i \ldots (\mathcal{D}_{n-1})^{r_{n-1}}$$  \hspace{2cm} (6.50b)

where $\tilde{Y}, \mathcal{D}$ denote the variables $\tilde{Y}_{ij}, i > \ell, \mathcal{D}_i, i < n$.

Further we note the commutation relations of the $\tilde{Y}_{ij}$ and $\mathcal{D}_i$ variables:

$$\tilde{Y}_{ij} \tilde{Y}_{ij} = q \tilde{Y}_{ij} \tilde{Y}_{ij}, i > \ell > j,$$  \hspace{2cm} (6.51a)

$$\tilde{Y}_{ij} \mathcal{D}_i = \mathcal{D}_i \tilde{Y}_{ij}, i > \ell > j,$$  \hspace{2cm} (6.51b)

$$\tilde{Y}_{ij} \tilde{Y}_{ij} = \tilde{Y}_{ij} \tilde{Y}_{ij}, k > i > j,$$  \hspace{2cm} (6.51c)

$$\tilde{Y}_{ij} \mathcal{D}_i = \mathcal{D}_i \tilde{Y}_{ij}, k > i > \ell > j,$$  \hspace{2cm} (6.51d)

$$\tilde{Y}_{ij} \tilde{Y}_{ij} = \tilde{Y}_{ij} \tilde{Y}_{ij} + \lambda \tilde{Y}_{ij} \tilde{Y}_{ij}, k > i > \ell > j,$$  \hspace{2cm} (6.51e)

$$\tilde{Y}_{ij} \tilde{Y}_{ij} = q^{-1} \tilde{Y}_{ij} \tilde{Y}_{ij} + q^{-1} \lambda \tilde{Y}_{ij} \tilde{Y}_{ij}, k > i > j,$$  \hspace{2cm} (6.51f)

$$Y_{j\ell} \mathcal{D}_i = \mathcal{D}_i Y_{j\ell}, j > \ell > i,$$  \hspace{2cm} (6.51g)

$$Y_{j\ell} \tilde{Y}_{ij} = q \mathcal{D}_i Y_{j\ell}, j > i > \ell,$$  \hspace{2cm} (6.51h)

$$Y_{j\ell} \mathcal{D}_i = \mathcal{D}_i Y_{j\ell}, i > \ell > j,$$  \hspace{2cm} (6.51i)

where in (6.51d) we use $\tilde{Y}_{ij} = 0$ when $i < \ell$. Note that (6.51a-d) may be obtained by replacing $M_{ij}$ with $\tilde{Y}_{ij}$ in (6.1a-d). Note that the structure of the $q$-coset for general $n$ is exhibited already for $n = 4$, while for $n = 3$ relations (6.51c,d) are not present. The commutation relations between the $Z$ and $\mathcal{D}$ variables are obtained from (6.51) by just replacing $Y_{st}$ by $Z_{ts}$ in all formulae.

Note that for real $q$ the $q$-coset is invariant under the antilinear anti-involution $\tilde{\omega}$ acting as:

$$\tilde{\omega}(\tilde{Y}_{j\ell}) = \tilde{Y}_{n+1-\ell,n+1-j}.$$  \hspace{2cm} (6.52)

Thus it can be considered as a $q$-coset of the quantum group $SU_q^{([\lfloor n+1/2 \rfloor_{\text{int}}, \lfloor n/2 \rfloor_{\text{int}}])}$, where $[x/2]_{\text{int}}$ is the biggest integer number not greater than $x$. The same invariance holds for the $Z$ coordinate $q$-coset.
Next we calculate:

\[ \pi(k_i) \mathcal{D}_j = q^{-\delta_{ij}/2} \mathcal{D}_j, \quad (6.53) \]
\[ \pi(X_i^+) \mathcal{D}_j = -\delta_{ij} \tilde{Y}_{j+1,\ell} \mathcal{D}_j, \quad (6.54) \]
\[ \pi(X_i^-) \mathcal{D}_j = 0, \]
\[ \pi(k_i) \tilde{Y}_{j\ell} = q^{1/2} (\delta_{i+1,\ell} - \delta_{i+1,\ell-1}) \tilde{Y}_{j\ell}, \]
\[ \pi(X_i^+) \tilde{Y}_{j\ell} = -\delta_{ij} \tilde{Y}_{j+1,\ell} + \delta_{\ell-1} q^{-1} \tilde{Y}_{j-1,\ell} \tilde{Y}_{j\ell} + \delta_{\ell+1,\ell} (q^{-1} \tilde{Y}_{j,\ell-1} - \tilde{Y}_{j+1,\ell-1} \tilde{Y}_{j\ell}), \]
\[ \pi(X_i^-) \tilde{Y}_{j\ell} = -\delta_{i+1,\ell} q^{-1/2} \tilde{Y}_{j-1,\ell}. \]

These results have the important consequence that the degrees of the variables \( \mathcal{D}_j \) are not changed by the action of \( \mathcal{V} \); that is, the parameters \( r_i \) characterize this action. Furthermore it is easy to check that \( \pi(y) \) satisfy (6.10). Thus, we have obtained representations of \( \mathcal{V} \). These are analogues of the elementary representations of the classical case \( q = 1 \).

To obtain these representations more explicitly one just applies (6.53), (6.54) to the basis in (6.50) using (6.20). In particular, we have:

\[ \pi(k_i) \mathcal{D}_j^n = q^{-n\delta_{ij}/2} (\mathcal{D}_j)^n, \quad n \in \mathbb{Z}, \quad (6.55) \]
\[ \pi(X_i^+) \mathcal{D}_j^n = -\delta_{ij} \tilde{c}_n \tilde{Y}_{j+1,\ell} (\mathcal{D}_j)^n, \quad n \in \mathbb{Z}, \]
\[ \pi(X_i^-) \mathcal{D}_j^n = 0, \quad n \in \mathbb{Z}, \]
\[ \pi(k_i) \tilde{Y}_{j\ell}^n = q^{\frac{n}{2}} (\delta_{i+1,\ell} - \delta_{i+1,\ell-1}) (\tilde{Y}_{j\ell})^n, \quad n \in \mathbb{Z}, \quad (6.56) \]
\[ \pi(X_i^+) \tilde{Y}_{j\ell}^n = -\delta_{ij} \tilde{c}_n (\tilde{Y}_{j\ell})^{n-1} \tilde{Y}_{j+1,\ell} + \delta_{\ell+1,\ell} \tilde{c}_n (q^{-1} \tilde{Y}_{j,\ell-1} (\tilde{Y}_{j\ell})^{n-1} - \tilde{Y}_{j+1,\ell-1} (\tilde{Y}_{j\ell})^n) + \delta_{\ell-1} q^{-1} \tilde{c}_n (\tilde{Y}_{j-1,\ell} (\tilde{Y}_{j\ell})^{n-1} - \tilde{Y}_{j+1,\ell-1} (\tilde{Y}_{j\ell})^n), \quad n \in \mathbb{Z}, \]
\[ \pi(X_i^-) \tilde{Y}_{j\ell}^n = -\delta_{i+1,\ell} q^{-\delta_{i+1,\ell}/2} \tilde{c}_n \tilde{Y}_{j-1,\ell} (\tilde{Y}_{j\ell})^{n-1}, \quad n \in \mathbb{Z}, \]
\[ \tilde{c}_n = q^{(1-n)/2}[n]_q. \quad (6.57) \]

We shall denote by \( \mathcal{C}_r \) the representation space of functions in (6.50) which have covariance properties (6.35), (6.40a). The representation acting in \( \mathcal{C}_r \) we denote by \( \tilde{\pi}_r \) doing also a renormalization to simplify things later, namely, we set:

\[ \tilde{\pi}_r(k_i) = \pi(k_i), \quad \tilde{\pi}_r(X_i^+) = q^{(r_i-1)/2} \pi(X_i^+). \quad (6.58) \]

Then \( \tilde{\pi}_r \) also satisfy (6.10).
Further, since the action of $\mathcal{U}$ is not affecting the degrees of $D_i$, we introduce (as in [197]) the restricted functions $\tilde{\phi}(\tilde{Y})$ by the formula which is prompted in (6.50b):

$$\tilde{\phi}(\tilde{Y}) = (\tilde{\mathcal{A}} \tilde{\phi})(\tilde{Y}) \equiv \tilde{\phi}(\tilde{Y}, \mathcal{D}_1 = \cdots = \mathcal{D}_{n-1} = 1_{\mathcal{A}_g}),$$  

(6.59a)

$$\tilde{\phi}(\tilde{Y}) = \sum_{m \in \mathbb{Z}_+} \mu_m (\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}}.$$

(6.59b)

We denote the representation space of $\tilde{\phi}(\tilde{Y})$ by $\tilde{\mathcal{C}}_{\tilde{Y}}$ and the representation acting in $\tilde{\mathcal{C}}_{\tilde{Y}}$ by $\tilde{\mathcal{R}}$. Thus, the operator $\tilde{\mathcal{A}}$ acts from $\tilde{\mathcal{C}}_{\tilde{Y}}$ to $\tilde{\mathcal{C}}_{\tilde{Y}}$. The properties of $\tilde{\mathcal{C}}_{\tilde{Y}}$ follow from the intertwining requirement for $\tilde{\mathcal{A}}$ [197]:

$$\tilde{\mathcal{R}} = \tilde{\mathcal{A}} = \tilde{\mathcal{A}} \circ \tilde{\mathcal{R}}.$$  

(6.60)

For the more compact exposition of the representation formulae we shall need below also the following operators (corresponding to each of the variables $\tilde{Y}_{j\ell}$):

$$\tilde{M}_{j\ell}(\tilde{Y}) = \sum_{m \in \mathbb{Z}_+} \mu_m \tilde{M}_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}},$$

(6.61)

$$\tilde{M}_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} = (\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{j\ell})^{m_{j\ell}+1} \cdots \times \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}},$$

(6.62)

$$\tilde{T}_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} = m_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}}$$

Next we introduce also the homogeneity (number) operators $N_{j\ell}$ for the variable $\tilde{Y}_{j\ell}$:

$$N_{j\ell}(\tilde{Y}) = \sum_{m \in \mathbb{Z}_+} \mu_m N_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}}$$

(6.63)

$$N_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} = m_{j\ell}(\tilde{Y}_{21})^{m_{21}} \cdots (\tilde{Y}_{n,n-1})^{m_{n,n-1}}$$

Clearly, we have the relation:

$$T_{j\ell} = q^{N_{j\ell}}.$$  

(6.64)

Using the above we define the $q$-difference operators which admit a general definition on a larger domain than polynomials, but on polynomials are well defined as follows:

$$\tilde{\mathcal{D}}_{j\ell}(\tilde{Y}) = \frac{1}{\tilde{M}_{j\ell}} (T_{j\ell} - T_{j\ell}^{-1}) \tilde{\phi}(\tilde{Y}) =$$

$$= \tilde{M}_{j\ell}^{-1}[N_{j\ell} \tilde{\phi}(\tilde{Y})]$$

(6.65)
from which follows:

\[
\hat{\mathcal{D}}_{x^0}(\hat{Y}_n^{m_{21}} \ldots (\hat{Y}_{n,n-1})^{m_{n,n-1}} = [m_{\hat{x},q}]_q (\hat{Y}_n^{m_{21}} \ldots (\hat{Y}_{n})^{m_{n-1}} \ldots (\hat{Y}_{n,n-1})^{m_{n,n-1}}
\]

(6.66)

Note that although \( \hat{M}_{x^0}^{-1} \) is not defined on \((\hat{Y}_n)^{m_{21}} \ldots (\hat{Y}_{n,n-1})^{m_{n,n-1}} \) for \( m_{\hat{x},q} = 0 \), the operator \( \hat{\mathcal{D}}_{x^0} \) is well defined on such terms, and the result is zero (given by the action of \((T_{x^0} - T_{x^0}^{-1})\)). Of course, for \( q \to 1 \) we have \( \hat{\mathcal{D}}_{x^0} \to \partial_{x^0} \equiv \partial / \partial Y_{x^0} \). Note that the above operators for different variables commute; that is, with these we have actually passed to commuting variables.

For the intertwining operators between partially equivalent representations we need the action of \( \pi_R(X_i) \) on \( \hat{Y}_{x^0} \) and \( \mathcal{D}_{x^0} \). Using (6.28) and (6.27) we obtain:

\[
\pi_R(X_i)(\mathcal{D}_{x^0})^n = \delta_{i,n} c_n(\mathcal{D}_{x^0})^n Z_{x^0,\ell+1},
\]

(6.67a)

\[
\pi_R(X_i)(\hat{Y}_{x^0})^n = \delta_{i,n} q^{n-3/2} n_q(\hat{Y}_{x^0})^{n-1} \hat{Y}_{x^0,\ell+1} \mathcal{D}_{x^0} \mathcal{D}_{x^0}^{-2} \mathcal{D}_{x^0,\ell-1}
\]

(6.67b)

where, as usual, we use \( \hat{Y}_{ij} = 1_{x^0} = \mathcal{D}_0 \). We shall use also the repeated action of \( \pi_R(X_i) \) so in addition we need:

\[
\pi_R(X_i)Z_{x^0} = \delta_{i,1} Z_{x^0,\ell+1} - \delta_{i,0} q^{-1/2} n_q Z_{x^0,\ell+1} Z_{x^0} + \delta_{i,0} \mathcal{D}_{x^0}^{-1} \mathcal{D}_{x^0,\ell-1} \mathcal{D}_{x^0,\ell-1} \mathcal{D}_{x^0,\ell-1} \cdot
\]

(6.68)

\[
\pi_R(k_i)Z_{x^0} = q^{(\delta_{i,1} - \delta_{i,0})/2} Z_{x^0}.
\]

(6.69)

### 6.1.3 Reducibility and Partial Equivalence

We have defined the representations \( \tilde{\pi}_r \) for \( r_i \in \mathbb{Z} \). However, notice that we can consider the restricted functions \( \tilde{\phi}(\hat{Y}) \) for arbitrary complex \( r_i \). We shall make this extension from now on, since this gives the same set of (holomorphic) representations for \( U_q(sl(n)) \) as in the case \( q = 1 \).

Now we make some statements which are true in the classical case [197], and will be illustrated below. For any \( i, j \), such that \( 1 \leq i \leq j \leq n - 1 \), define:

\[
m_{ij} \equiv r_i + \cdots + r_j + j - i + 1,
\]

(6.70)

note \( m_i = m_{ii} = r_i + 1, m_{ij} = m_i + \cdots + m_j \). Note that the possible choices of \( i, j \) are in one-to-one correspondence with the positive roots \( \alpha = a_{ij} = a_i + \cdots + a_j \) of the root system of \( sl(n) \), the cases \( i = j = 1, \ldots, n - 1 \) enumerating the simple roots \( a_i = a_{ii} \). In general, \( m_{ij} \in \mathbb{C} \) for the representations \( \tilde{\pi}_r \), while \( m_{ij} \in \mathbb{Z} \) for the representations \( \pi_r \). If \( m_{ij} \notin \mathbb{N} \), for all possible \( i, j \) the representations \( \tilde{\pi}_r \), \( \pi_r \) are irreducible. If \( m_{ij} \in \mathbb{N} \), for some \( i, j \) the representations \( \tilde{\pi}_r \), \( \pi_r \) are reducible. The corresponding irreducible subrepresentations are still infinite-dimensional unless \( m_i \in \mathbb{N} \) for all \( i = 1, \ldots, n - 1 \).
The representation spaces of the irreducible subrepresentations are invariant irreducible subspaces of our representation spaces. These invariant subspaces are spanned by functions depending on all variables $Y_{j\ell}$, except when for some $s \in \mathbb{N}$, $1 \leq s \leq n-1$, we have $m_s = m_{s+1} = \cdots = m_{n-1} = 1$. In the latter case these functions depend only on the $(s-1)(2n-s)/2$ variables $Y_{j\ell}$ with $\ell < s$, (the unrestricted subrepresentation functions depend still on $\mathcal{D}_\ell$ with $\ell < s$). In particular, for $s = 2$ the restricted subrepresentation functions depend only on the $(n-1)$ variables $Y_{1\ell}$. The latter situation is relatively simple also in the $q$ case since these variables are $q$-commuting: $Y_{j\ell}Y_{k\ell} = q^{\ell-k}Y_{k\ell}Y_{j\ell}$, $j > k$. (For $s = 1$ the irreducible subrepresentation is one-dimensional, hence no dependence on any variables.)

Furthermore, for $m_{ij} \in \mathbb{N}$ the representation $\hat{\pi}_r$, $\pi_r$ respectively is partially equivalent to the representation $\hat{\pi}_{r'}$, $\pi_{r'}$, respectively with $m_{ij}' = r_{ij}' + 1$ being explicitly given as follows [197]:

$$m_{ij}' = \begin{cases} m_{ij} & , \text{for } \ell \neq i-1, i, j, j+1 , \\ m_{ij} & , \text{for } \ell = i-1 , \\ -m_{ij} & , \text{for } \ell = i \leq j , \\ -m_{ij+1} & , \text{for } \ell = j > i , \\ -m_{ij} & , \text{for } \ell = j = i , \\ m_{ij} & , \text{for } \ell = j+1 . \end{cases}$$ (6.71)

These partial equivalences are realized by intertwining operators:

$$\mathcal{I}_{ij} : \mathcal{C}_r \longrightarrow \mathcal{C}_{r'} , \quad m_{ij} \in \mathbb{N} ,$$ (6.72a)

$$\hat{\mathcal{I}}_{ij} : \hat{\mathcal{C}}_r \longrightarrow \hat{\mathcal{C}}_{r'} , \quad m_{ij} \in \mathbb{N} ,$$ (6.72b)

that is, one has:

$$\mathcal{I}_{ij} \circ \pi_r = \pi_{r'} \circ \mathcal{I}_{ij} , \quad m_{ij} \in \mathbb{N} ,$$ (6.73a)

$$\hat{\mathcal{I}}_{ij} \circ \hat{\pi}_r = \hat{\pi}_{r'} \circ \hat{\mathcal{I}}_{ij} , \quad m_{ij} \in \mathbb{N} .$$ (6.73b)

The invariant irreducible subspace of $\hat{\pi}_r$ (respectively, $\pi_r$) discussed above is the intersection of the kernels of all intertwining operators acting from $\hat{\pi}_r$ (respectively, $\pi_r$). When all $m_i \in \mathbb{N}$ the invariant subspace is finite-dimensional with dimension $\prod_{1 \leq i \leq j \leq n-1} m_{ij}/\prod_{i=1}^{n-1} t_i$, and all finite-dimensional (holomorphic) irreps of $U_q(sl(n))$ can be obtained in this way.

We restate now the canonical procedure for the derivation of these intertwining operators (cf. [197, 211]) in the current setting. By the procedure one should take as intertwiners (up to nonzero multiplicative constants):

$$\mathcal{I}_{ij}^m = \mathcal{P}_{ij}^m (\pi_R(X_{i1}), \ldots, \pi_R(X_{ij})) , \quad m = m_{ij} \in \mathbb{N} ,$$ (6.74a)

$$\hat{\mathcal{I}}_{ij}^m = \mathcal{P}_{ij}^m (\hat{\pi}_R(X_{i1}), \ldots, \hat{\pi}_R(X_{ij})) , \quad m = m_{ij} \in \mathbb{N} ,$$ (6.74b)
where $\mathcal{P}_{ij}^m$ is a homogeneous polynomial in each of its $(j - i + 1)$ variables of degree $m$, while the operators $l_{ij}^m$ are defined through $\mathcal{I}_{ij}^m$ and the operator $\mathcal{A}$. The polynomial $\mathcal{P}_{ij}^m$ gives a singular vector $v_{ij}$ in the Verma module $V^{\Lambda(\tilde{r})}$ with highest weight $\Lambda(\tilde{r})$, that is,

$$v_{ij} = \mathcal{P}_{ij}^m(X_i, \ldots, X_j)v_0,$$

(6.75)

where $v_0$ is the highest-weight vector of $V^{\Lambda(\tilde{r})}$. The explicit expression for $v_{ij}$ with $j = i + p - 1$ is given in (2.95). In particular, in the case of the simple roots, that is, when $m_i = m_{ii} = r_i + 1 \in \mathbb{N}$, we have:

$$I_{m_i}^{m_{ii}} = (\pi_R(X_i^-))^m_{m_{ii}}, m_i \in \mathbb{N}.\quad (6.76)$$

Implementing the above one should be careful since $\pi_R(X_i^-)$ is not preserving the reduced spaces $\mathcal{E}_{\tilde{r}}, \hat{\mathcal{E}}_{\tilde{r}}$, which is of course a prerequisite for (6.73), (6.74), (6.76).

### 6.2 The Case of $U_q(sl(3))$

In this section we also follow [211]. In this section we consider in more detail the case $n = 3$. (The case $n = 2$ was discussed in Section 5.1.3. It can also be obtained by restricting the construction for the complexification of the Lorentz quantum algebra to one of its $U_q(sl(2))$ subalgebras, see Section 5.3.)

Let us now for $n = 3$ denote the coordinates on the $q$-flag manifold by: $\xi = Y_{21}$, $\eta = Y_{32}$, $\zeta = Y_{31}$. We note for future use the commutation relations between these coordinates:

$$\xi\eta = q\eta\xi - \lambda\zeta, \quad \eta\zeta = q\zeta\eta, \quad \zeta\zeta = q\zeta\zeta.\quad (6.77)$$

The reduced functions for the $U_q(sl(3))$ action are (cf. (6.50)):

$$\tilde{\varphi}(\tilde{Y}, \tilde{\mathcal{D}}) = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \xi^j \eta^n \zeta^\ell \mathcal{D}_1^{(1)}(\mathcal{D}_2)^{(2)} = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \varphi_{j,n,\ell},\quad (6.78a)$$

$$\tilde{\varphi}_{j,n,\ell} = \xi^j \eta^n \zeta^\ell \mathcal{D}_1^{(1)}(\mathcal{D}_2)^{(2)}.\quad (6.78b)$$

Now the action of $U_q(sl(3))$ on (6.78) is given explicitly by:

$$\pi(k_1)\varphi_{j,n,\ell} = q^{j+(n-\ell-r_1)/2} \varphi_{j,n,\ell},\quad (6.79a)$$

$$\pi(k_2)\varphi_{j,n,\ell} = q^{\ell+(n-j-r_2)/2} \varphi_{j,n,\ell}.\quad (6.79b)$$
As a consequence of the intertwining property (5.40) we obtain that
\[
\pi(X_1^+)\varphi_{j\ell n} = q^{(1+n-\ell r_1)/2}[n + j - \ell - r_1]q\hat{\varphi}_{j+1,n\ell} +
+q^{j-(n-3r_1-1)/2}[\ell]q\hat{\varphi}_{j,n+1,\ell-1},
\]
(6.79c)
\[
\pi(X_2^+)\varphi_{j\ell n} = q^{(1+n-j-r_2)/2}[\ell - r_2]q\hat{\varphi}_{j,n+1,\ell-1} -
- q^{j-(n+j+r_2-1)/2}[\ell]q\hat{\varphi}_{j-1,n+1,\ell},
\]
(6.79d)
\[
\pi(X_1^-)\varphi_{j\ell n} = q^{(\ell-n+r_1-1)/2}[\ell]q\hat{\varphi}_{j-1,n\ell},
\]
(6.79e)
\[
\pi(X_2^-)\varphi_{j\ell n} = -q^{(n-j+r_2-1)/2}[\ell]q\hat{\varphi}_{j,n-1,\ell} -
- q^{j-(n+j-r_2-1)/2}[\ell]q\hat{\varphi}_{j+1,n-1,\ell}.
\]
(6.79f)

It is easy to check that \(\pi(k_1), \pi(X_1^+)\) satisfy (6.10). It is also clear that we can remove the inessential phases by setting:
\[
\tilde{\pi}_{r_1,r_2}(k_1) = \pi(k_1), \quad \tilde{\pi}_{r_1,r_2}(X_1^+) = q^{\ell(r_1-1)/2}\pi(X_1^+).
\]
(6.80)

Then \(\tilde{\pi}_{r_1,r_2}\) also satisfy (6.10).

Then we consider the restricted functions (cf. (6.59)):
\[
\tilde{\varphi}(\tilde{Y}) = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \xi^j \zeta^n \eta^\ell = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \varphi_{j\ell n},
\]
(6.81a)
\[
\tilde{\varphi}_{j\ell n} = \xi^j \zeta^n \eta^\ell.
\]
(6.81b)

As a consequence of the intertwining property (5.40) we obtain that \(\varphi_{j\ell n}\) obey the same transformation rules (6.79) as \(\varphi_{j\ell n}\), that is, we have:
\[
\tilde{\pi}_{r_1,r_2}(k_1)\varphi_{j\ell n} = q^{j+(n-\ell-r_1)/2}\varphi_{j\ell n},
\]
(6.82a)
\[
\tilde{\pi}_{r_1,r_2}(k_2)\varphi_{j\ell n} = q^{\ell+(n-j-r_2)/2}\varphi_{j\ell n},
\]
(6.82b)
\[
\tilde{\pi}_{r_1,r_2}(X_1^+)\varphi_{j\ell n} = q^{(n-\ell)/2}[n + j - \ell - r_1]q\hat{\varphi}_{j+1,n\ell} +
+ q^{j-(r_1-1)(n-\ell)/2}[\ell]q\hat{\varphi}_{j,n+1,\ell-1},
\]
(6.82c)
\[
\tilde{\pi}_{r_1,r_2}(X_2^+)\varphi_{j\ell n} = q^{(n-j)/2}[\ell - r_2]q\hat{\varphi}_{j,n+1,\ell-1} -
- q^{j-(r_2-1)(n-j)/2}[\ell]q\hat{\varphi}_{j-1,n+1,\ell},
\]
(6.82d)
\[
\tilde{\pi}_{r_1,r_2}(X_1^-)\varphi_{j\ell n} = q^{(\ell-n)/2}[\ell]q\hat{\varphi}_{j-1,n\ell},
\]
(6.82e)
\[
\tilde{\pi}_{r_1,r_2}(X_2^-)\varphi_{j\ell n} = -q^{(n-j)/2}[\ell]q\hat{\varphi}_{j,n-1,\ell} -
- q^{j-(n-j)/2}[\ell]q\hat{\varphi}_{j+1,n-1,\ell}.
\]
(6.82f)
Let us introduce the following operators acting on our functions:

\[
\hat{M}^\pm \hat{\phi}(\hat{Y}) = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \hat{M}^\pm \hat{\phi}_{jn\ell},
\]

\[
T_k \hat{\phi}(\hat{Y}) = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} T_k \hat{\phi}_{jn\ell},
\]

where \( \kappa = \xi, \eta, \zeta \), and the explicit action on \( \hat{\phi}_{jn\ell} \) is defined by:

\[
\hat{M}^\pm \hat{\phi}_{jn\ell} = \hat{\phi}_{j\pm 1,n\ell}, \quad (6.84a)
\]

\[
T^\pm \hat{\phi}_{jn\ell} = q^\pm \hat{\phi}_{jn\ell}, \quad (6.84b)
\]

\[
T \hat{\phi}_{jn\ell} = q^\ell \hat{\phi}_{jn\ell}, \quad (6.84c)
\]

\[
\hat{\phi}_{jn\ell} \rightarrow q^{-1} \hat{\phi}_{jn\ell}. \quad (6.84d)
\]

Now we define the \( q \)-difference operators by:

\[
\hat{D} \hat{\phi}(\hat{Y}) = 1 + \hat{M}^\mp \left( T_k - T_k^{-1} \right) \hat{\phi}(\hat{Y}), \quad (6.85)
\]

Thus, we have:

\[
\hat{D}_\xi \hat{\phi}_{jn\ell} = [\ell] \hat{\phi}_{j,n\ell+1}, \quad (6.86a)
\]

\[
\hat{D}_\eta \hat{\phi}_{jn\ell} = [\ell] \hat{\phi}_{j,n\ell-1}, \quad (6.86b)
\]

\[
\hat{D}_\zeta \hat{\phi}_{jn\ell} = [n] \hat{\phi}_{j,n-1,\ell}, \quad (6.86c)
\]

Of course, for \( q \rightarrow 1 \) we have \( \hat{D}_\kappa \rightarrow \partial / \partial \kappa \).

In terms of the above operators the transformation rules (6.82) are written as follows:

\[
\hat{\pi}_{1,2}^{(1)}(k_1) \hat{\phi}(\hat{Y}) = q^{-r_1/2} T_\xi T_\zeta^{-1/2} T_\eta^{-1/2} \hat{\phi}(\hat{Y}), \quad (6.87a)
\]

\[
\hat{\pi}_{1,2}^{(2)}(k_2) \hat{\phi}(\hat{Y}) = q^{-r_2/2} T_\eta T_\zeta^{-1/2} T_\xi^{-1/2} \hat{\phi}(\hat{Y}), \quad (6.87b)
\]

\[
\hat{\pi}_{1,2}^{(3)}(X^1_1) \hat{\phi}(\hat{Y}) = (1/\lambda) \hat{M}_\xi T_\zeta^{1/2} T_\eta^{-1/2} \times \hat{M}_\zeta \hat{\phi}(\hat{Y}) + q^{-r_1} \hat{D}_\eta T_\xi T_\zeta^{-1} T_\eta^{-1} \hat{\phi}(\hat{Y}), \quad (6.87c)
\]

\[
\hat{\pi}_{1,2}^{(4)}(X^2_2) \hat{\phi}(\hat{Y}) = (1/\lambda) \hat{M}_\eta T_\xi^{1/2} T_\zeta^{-1/2} \left( q^{-r_2} T_\eta - q^{-2} T_\eta^{-1} \right) \hat{\phi}(\hat{Y}) - q^{-r_2} \hat{D}_\xi T_\zeta^{-1/2} T_\eta^{-1} \hat{\phi}(\hat{Y}), \quad (6.87d)
\]
\[ \hat{\pi}_{r_1,r_2}(X^+_r) \hat{\phi}(\tilde{Y}) = \mathcal{P}_\xi T^{1/2}_\xi - T^{-1/2}_\eta \hat{\phi}(\tilde{Y}), \]  
(6.87e)

\[ \hat{\pi}_{r_1,r_2}(X^-_r) \hat{\phi}(\tilde{Y}) = -\mathcal{P}_\eta T^{1/2}_\eta - T^{-1/2}_\xi \hat{\phi}(\tilde{Y}) - \hat{M}_\xi \mathcal{P}_\eta T^{1/2}_\eta - T^{-1/2}_\xi \hat{\phi}(\tilde{Y}), \]  
(6.87f)

where \( \hat{M}_\xi = \hat{M}_\xi^* \).

Notice that it is possible to obtain a realization of the representation \( \hat{\pi}_{r_1,r_2} \) on monomials in three commuting variables \( x, y, z \). Indeed, one can relate the noncommuting algebra \( \mathbb{C}[\xi, \eta, \zeta] \) with the commuting one \( \mathbb{C}[x, y, z] \) by fixing an ordering prescription. However, such realization in commuting variables may be obtained much more directly as is done by other methods (cf. Section 6.3.1 below). Here we are interested in the noncommutative case and we continue to work with the noncommuting variables \( \xi, \eta, \zeta \).

Now we can illustrate some of the general statements of the previous section. Let \( m_2 = r_2 + 1 \in \mathbb{N} \). Then it is clear that functions \( \hat{\phi} \) from (6.81) with \( \mu_{j,n,\ell} = 0 \) if \( \ell \geq m_2 \) form an invariant subspace since:

\[ \hat{\pi}_{r_1,r_2}(X^+_r) \hat{\phi}_{m2} = -q^{-1+(j-n)/2}[j]_q \hat{\phi}_{j-1,n+1,r_2}, \]  
(6.88)

and all other operators in (6.82) either preserve or lower the index \( \ell \). The same is true for the functions \( \hat{\phi} \). In particular, for \( m_2 = 1 \) the functions in the invariant subspace do not depend on the variable \( \eta \). In this case we have functions of two \( q \)-commuting variables \( \xi = q \zeta \) which are much easier to handle that the general noncommutative case (6.77).

The intertwining operator (6.76) for \( m_2 \in \mathbb{N} \) is given as follows. First we calculate:

\[ (\pi_R(X^+_r))^s \hat{\phi}_{jnt} = (\pi_R(X^-_r))^s \xi^j \zeta^n \eta^t \mathcal{P}_1^r_1 \mathcal{P}_2^r_2 = \]  
(6.89)

\[ = \xi^j \zeta^n \sum_{t=0}^s a_{st} \eta^t \mathcal{T}_1^{t+1} \mathcal{T}_2^{t+s-t} (\xi_2^{12})^{s-t}, \]

\[ a_{st} = q^{(t+2s)/2-(s+t)(s+t+1)/6} \frac{[s]_q [r_2 - t]_q! [\ell]_q!}{[r_2 - s]_q! [\ell - t]_q!}, \]

where \( \binom{s}{t}_q = [n]_q ![k]_q ![n - k]_q !, [m]_q ! = [m]_q [m - 1]_q \ldots [1]_q \). Thus, indeed \( \pi_R(X^+_r) \) is not preserving the reduced space \( \mathcal{C}_{r_1,r_2} \), and furthermore there is the additional variable \( \xi_2^{12} \). Since we would like \( \pi_R(X^+_r) \) to some power to map to another reduced space this is only possible if the coefficients \( a_{st} \) vanish for \( s \neq t \). This happens iff \( s = r_2 + 1 = m_2 \). Thus we have (in terms of the representation parameters \( m_i = r_i + 1 \)):

\[ (\pi_R(X^+_r))^{m_2} \xi^j \zeta^n \eta^t \mathcal{P}_1^{m_1-1} \mathcal{P}_2^{m_2-1} = \]  
(6.90)

\[ = q^{m_2(\ell - m_2/2)} [\ell]_q ! \xi^j \zeta^n \eta^{m_2} \mathcal{P}_1^{m_1-2} \mathcal{P}_2^{m_2-1}. \]
Comparing the powers of \( D_i \) we recover at once (2.77) for our situation, namely, \( m'_1 = m_{12}, m'_2 = -m_2 \). Thus, we have shown (6.72a) and (6.73a). Then (6.72b) and (6.73b) follow using (5.40). This intertwining operator has a kernel which is just the invariant subspace discussed above – from the factor \( 1/[\ell - m_2]_q! \) in (6.90) it is obvious that all monomials with \( \ell < m_2 \) are mapped to zero.

For the restricted functions we have:

\[
(n_R(X_2))^{m_2} \hat{\phi}_{jn\ell} = q^{m_2(\ell - m_2/2)} \frac{[\ell]_q!}{[\ell - m_2]_q!} \hat{\phi}_{jn,\ell - m_2} = q^{-3m_2/2} (D_\eta T_\eta)^{m_2} \hat{\phi}_{jn\ell}. \tag{6.91}
\]

Thus, renormalizing (6.76b) by \( q^{-3m_2/2} \) we finally have:

\[
I^{m_2}_2 = (D_\eta T_\eta)^{m_2}. \tag{6.92}
\]

For \( q = 1 \) this operator reduces to the known result: \( I_2 = (\partial_{\xi} + \eta \partial_{\xi})^{m_2} \) [202].

Let now \( m_1 \in \mathbb{N} \). In a similar way, though the calculations are more complicated, we find:

\[
(n_R(X^-))^{m_1} \xi^{m_1} \eta^{m_1} \xi^{m_2} \eta^{m_2} = q^{m_1(j + n - \ell - m_1/2)} \sum_{t=0}^{m_1} q^{-t(t+3+2j)/2} \times
\]

\[
\times (m_1) \left( \frac{[j]_q! [n]_q!}{[j - m_1 + t]_q! [n - t]_q!} \hat{\phi}_{t+1, \ell - m_1, n - t, \ell + t} = q^{-m_1(3/2+m_1)} T_{\xi}^{m_1} \sum_{t=0}^{m_1} \hat{M}_{\eta}^{\ell} (q D_\xi T_\xi)^{m_1 - t} T_\eta^{m_1} \hat{\phi}_{t+1, \ell - m_1, n - t, \ell + t}. \tag{6.93}
\]

Comparing the powers of \( D_i \) we recover (2.77) for our situation, namely, \( m'_1 = -m_1, m'_2 = m_{12} \). Thus, we have shown (6.72) and (6.73).

For the restricted functions we have:

\[
(n_R(X^-))^{m_1} \hat{\phi}_{jn\ell} = q^{m_1(j + n - \ell - m_1/2)} \sum_{t=0}^{m_1} q^{t(t+3+2j)/2} \times
\]

\[
\times (m_1) \left( \frac{[j]_q! [n]_q!}{[j - m_1 + t]_q! [n - t]_q!} \hat{\phi}_{t+1, \ell - m_1, n - t, \ell + t} = q^{-m_1(3/2+m_1)} T_{\xi}^{m_1} \sum_{t=0}^{m_1} \hat{M}_{\eta}^{\ell} (q D_\xi T_\xi)^{m_1 - t} T_\eta^{m_1} \hat{\phi}_{t+1, \ell - m_1, n - t, \ell + t}. \tag{6.94}
\]

Then, renormalizing (5.43b) we finally have:

\[
I^{m_1}_1 = T_{\xi}^{m_1} \sum_{t=0}^{m_1} \hat{M}_{\eta}^{\ell} (q D_\xi T_\xi)^{m_1 - t} T_\eta^{m_1}. \tag{6.95}
\]

For \( q = 1 \) this operator reduces to the known result: \( I^{m_1}_1 = (\partial_{\xi} + \eta \partial_{\xi})^{m_1} \) [202].
Finally, let us consider the case \( m = m_1 + m_2 \in \mathbb{N} \), first with \( m_1, m_2 \notin \mathbb{Z}_+ \). In this case the intertwining operator is given by (6.74), (6.75) using singular vector from Section 2.4 and [198]:

\[
\mathcal{P}^m_{12}(X_1, X_2) = \sum_{s=0}^{m} a_s(X_1)^{m-s}(X_2)^m(X_1)^s, \tag{6.96}
\]

\[
a_s = (-1)^s a \frac{[m_1]_q}{[m_1 - s]_q} \binom{m}{s}, \text{ } s = 0, \ldots, m, a \neq 0. \tag{6.97}
\]

Let us illustrate the resulting intertwining operator in the case \( m = 1 \). Then, we have, setting in (6.96) \( a = [1 - m_1]_q \):

\[
\mathcal{I}^1_{12} = [1 - m_1]_q \pi_R(X_1) \pi_R(X_2) + [m_1]_q \pi_R(X_2) \pi_R(X_1). \tag{6.98}
\]

Then we can see at once the intertwining properties of \( \mathcal{I}^1_{12} \) by calculating:

\[
\mathcal{I}^1_{12} \xi^r \eta^s \mathcal{I}^{m_1 - 1} \mathcal{I}^{m_2 - 1} = \frac{q^{i+n-2-m_1} [j]_q [\ell]_q \xi^{i-1} \eta^{i-1} \mathcal{I}^{m_1 - 2} \mathcal{I}^{m_2 - 2} + q^{n-2} [n]_q [\ell + m_1]_q \xi^{n-1} \eta^{m_1 - 2} \mathcal{I}^{m_1 - 2} \mathcal{I}^{m_2 - 2}. \tag{6.99}
\]

Comparing the powers of \( \mathcal{I}_i \) we recover (2.77) for our situation, namely, \( m_1' = -m_2 = m_1 - 1, m_2' = -m_1 = m_2 - 1 \).

For the restricted functions we have:

\[
\left( [1 - m_1]_q \pi_R(X_1) \pi_R(X_2) + [m_1]_q \pi_R(X_2) \pi_R(X_1) \right) \hat{\phi}_{j \ell} = \frac{q^{n-2 j - m_1} [j]_q [\ell]_q \hat{\phi}_{j-1, n, \ell-1} + q^{n-2} [n]_q [\ell + m_1]_q \hat{\phi}_{j, n-1, \ell} = q^{-2} \left( q^{-m_1} \mathcal{D}_\xi T_\zeta \mathcal{D}_\eta + (1/\lambda) \mathcal{D}_\zeta (q^{m_1} T_\eta - q^{-m_1} T_\eta^{-1}) \right) T_\zeta \hat{\phi}_{j \ell}.
\]

Rescaling (6.74b) we finally have:

\[
I^1_{12} = \left( q^{-m_1} \mathcal{D}_\xi T_\zeta \mathcal{D}_\eta + (1/\lambda) \mathcal{D}_\zeta (q^{m_1} T_\eta - q^{-m_1} T_\eta^{-1}) \right) T_\zeta. \tag{6.100}
\]

For \( q = 1 \) this operator is: \( I_{12} = \partial_\xi \partial_\eta + (m_1 + \eta \partial_\eta) \partial_\xi \) [202].

Above we have supposed that \( m_1, m_2 \notin \mathbb{Z}_+ \). However, after the proper choice of \( a \) in (6.96), (e. g., as made above in (6.97) we can consider the singular vector (6.96) and the resulting intertwining operator also when \( m_1, m_2 \in \mathbb{Z}_+ \). In these cases the singular vector is reduced in four different ways (cf. (2.89)). Accordingly, the intertwining
The four expressions were used to prove commutativity of the hexagon diagram of $U_q(sl(3, \mathbb{C}))$ [198]. This diagram involves six representations which are denoted by $V_{00}, V_{00}^1, V_{00}^2, V_{00}^{12}, V_{00}^{21}, V_{00}^3$, in (29) of [198] and which in our notation are connected by the intertwiners in (6.101) as follows:

\[
\begin{align*}
\hat{C}_{m_1, m_2} &\quad \stackrel{I_{1}^{m_1}}{\longrightarrow} \quad \hat{C}_{-m_1, -m} \quad \stackrel{I_{2}^{m}}{\longrightarrow} \quad \hat{C}_{m_2, -m} \quad \stackrel{I_{1}^{m_2}}{\longrightarrow} \quad \hat{C}_{-m_2, -m_1} \quad \text{(6.102a)} \\
\hat{C}_{m_1, m_2} &\quad \stackrel{I_{2}^{m_2}}{\longrightarrow} \quad \hat{C}_{m_1, m_2} \quad \stackrel{I_{1}^{m_1}}{\longrightarrow} \quad \hat{C}_{-m_1, -m_1} \quad \stackrel{I_{2}^{m_1}}{\longrightarrow} \quad \hat{C}_{-m_2, -m_1} \quad \text{(6.102b)} \\
\hat{C}_{m_1, m_2} &\quad \stackrel{I_{1}^{m_1}}{\longrightarrow} \quad \hat{C}_{-m_1, m_1} \quad \stackrel{I_{12}^{m_1}}{\longrightarrow} \quad \hat{C}_{-m_1, -m_1} \quad \stackrel{I_{1}^{m_2}}{\longrightarrow} \quad \hat{C}_{-m_2, -m_1} \quad \text{(6.102c)} \\
\hat{C}_{m_1, m_2} &\quad \stackrel{I_{2}^{m_2}}{\longrightarrow} \quad \hat{C}_{m_1, m_2} \quad \stackrel{I_{12}^{m_1}}{\longrightarrow} \quad \hat{C}_{m_2, -m_1} \quad \stackrel{I_{1}^{m_2}}{\longrightarrow} \quad \hat{C}_{-m_2, -m_1} \quad \text{(6.102d)}
\end{align*}
\]

Of these six representations only $\hat{C}_{m_1, m_2}$ has a finite-dimensional irreducible subspace iff $m_1 m_2 > 0$, the dimension being $m_1 m_2 m_2/2$ [198]. If $m_1 = 0$ the intertwining operators with superscript $m_1$ become the identity (since in these cases the intertwined spaces coincide) and the compositions in (6.101) and (6.102) are shortened to two arrows in cases (a,b,d) and one arrow in case (c) (respectively, for $m_2 = 0$, two arrows in cases (a,b,c), one arrow in (d)). (Such considerations are part of the multiplet classification given in [198].)

### 6.3 Polynomial Solutions of $q$-Difference Equations in Commuting Variables

In this section we follow mainly [246]. A new approach to the theory of polynomial solutions of $q$-difference equations is proposed. The approach is based on the representation theory of simple Lie algebras $\mathfrak{g}$ and their $q$-deformations and is presented here for $U_q(sl(n))$. First a $q$-difference realization of $U_q(sl(n))$ in terms of $n(n - 1)/2$ commuting variables and depending on $n - 1$ complex representation parameters $r_i$ is constructed. From this realization lowest-weight modules (LWMs) are obtained which are studied in detail for the case $n = 3$ (the well-known $n = 2$ case is also recovered). All reducible LWM are found and the polynomial bases of their invariant irreducible
subrepresentations are explicitly given. This also gives a classification of the quasi-
exactly solvable operators in the present setting. The invariant subspaces are obtained
as solutions of certain invariant $q$-difference equations, that is, these are kernels of
invariant $q$-difference operators, which are also explicitly given. Such operators were
not used until now in the theory of polynomial solutions. Finally the states in all
subrepresentations are depicted graphically via the so called Newton diagrams.

### 6.3.1 Procedure for the Construction of the Representations

The procedure is iterative. In fact, we have to use also $U_q(gl(n))$. Let us introduce
first some notation. The basic $q$-number notation $[a] = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}$ will be used also for
diagonal operators $H$ replacing $a$. Following Biedenharn and Lohe [101] our represent-
ations will be given in terms of $n(n - 1)/2$ variables. For our purposes we denote these
variables by $z_i^k$, $2 \leq k \leq n$, $1 \leq i \leq k - 1$. Next we introduce the number operator $N_i^k$ for
the coordinate $z_i^k$, that is, $N_i^k z_j^m = \delta_{mk} \delta_{ij} z_j^m$ and the $q$-difference operators $D_i^k$, which
admit a general definition on a larger domain than polynomials, but on polynomials
are well defined as follows:

$$ D_i^k = \frac{1}{z_i^k} [N_i^k]. $$(6.103)

Further we note that the representations of $U_q(sl(n))$ will be characterized by $n - 1$ complex parameters $r_k \in \mathbb{C}$, $1 \leq k \leq n - 1$.

We rewrite formulae (5.3), (6.10), and (6.22) from [101] in the following way:

$$ \Gamma_n(E_{ij}) = \Gamma_{n-1}(E_{ij}) q^{1/4} (N_i^p - N_i^q) + q^{1/4} \Gamma_{n-1}(E_{ii} - E_{jj}) z_i^p D_j^n $$ (6.104a)

$$ i < j < n $$

$$ \Gamma_n(E_{ij}) = \Gamma_{n-1}(E_{ij}) q^{1/4} (N_j^p - N_j^q) + q^{1/4} \Gamma_{n-1}(E_{ii} - E_{jj}) z_i^p D_j^n $$ (6.104b)

$$ n > i > j $$

$$ \Gamma_n(E_{ii}) = \Gamma_{n-1}(E_{ii}) + N_i^n, i < n, $$ (6.104c)

$$ \Gamma_n(E_{mm}) = \Gamma_{n-1}(E_{mm}) - \sum_{i=1}^{n-1} N_i^n, $$ (6.104d)

$$ \Gamma_n(E_{ii}) = q^{1/2} \left( z_i^{i-1} \Gamma_{n-1}(E_{ii}) - \sum_{i=1}^{n-1} \Gamma_{n-1}(E_{ii}) D_i^n \right), i < n $$ (6.104e)

$$ \Gamma_n(E_{in}) = q^{n_i} z_i^n \left[ \Gamma_{n-1}(E_{mn}) - \Gamma_{n-1}(E_{ii}) - \sum_{k=1}^{n-1} N_k^n - \right. $$

$$ - \sum_{j=1}^{n-1} q^{n_j} z_j^n \Gamma_{n-1}(E_{ij}), i < n, $$ (6.104f)
where

\[ q^n_{ij} = q^{-\frac{1}{2}}(\sum_{k=0}^{i-1} r_i^{n-1} \Gamma_{n-1}(E_{kk}) - \sum_{k=j+1}^{n-1} \Gamma_{n-1}(E_{kk})) q^{\left(\frac{1}{2}(\Gamma_{n-1}(E_{mm}) - \sum_{k=j+1}^{n-1} N^{n-1}_k) + \frac{3}{2}\right)} \]

\[ \times q^{\left(\frac{1}{2}(N^{n-1}_i + N^{n-1}_j)\right)} \left(\sum_{j=0}^{i-1} N^{n-1}_j \right) q^{(r_0)}, \] (6.105)

\[ \left(\frac{r_0}{0}\right) = \begin{cases} + & \text{for } i < j, \\ - & \text{for } i > j, \\ 0 & \text{for } i = j, \end{cases} \]

\( \Gamma_{n-1}(E_{ij}) \) are defined at the previous step, except \( \Gamma_{n-1}(E_{nn}) \) which adds the representation parameter \( r_{n-1} \) and is given by:

\[ \Gamma_{n-1}(E_{nn}) = \sum_{k=0}^{n-1} r_k = r^{n-1} + r_0. \] (6.106)

The parameter \( r_0 \) represents the center of \( U_q(gl(n)) \) and is decoupled later. The additional input with respect to [101] is: 1) notation – we put indices on \( \Gamma \) corresponding to the case we consider – thus, this is made an iterative procedure; 2) we give values to the Cartan generators which values are consistent with previous knowledge from representation theory; 3) we introduce \( q \)-difference operators \( D^n \) to replace \( \overline{z}^n \); 4) we have done also something artificial by using \( \Gamma_{n-1}(E_{mm}) \) – this is given by the “total number” \( r^{n-1} \) (which for finite-dimensional representations is the number of boxes of the Young tableaux minus \( n - 1 \)) plus the number \( r_0 \) representing the center of \( U_q(gl(n)) \).

Denoting \( \mathcal{Z}^n = \sum_{i=1}^{n} E_{ii} \) we have:

\[ \Gamma_n(\mathcal{Z}^n) = \sum_{i=1}^{n} \Gamma_n(E_{ii}) = r^{n-1} + r_0 + \Gamma_{n-1}(\mathcal{Z}^{n-1}) = \]

\[ = \sum_{i=1}^{n-1} r^i + nr_0 = \sum_{i=0}^{n-1} (n - i)r_i. \] (6.107)

Thus, as expected, \( \Gamma_n(\mathcal{Z}^n) \) is central. Then the generators \( H^i_n = \Gamma_n(E_{ii} - E_{i+1,i+1}) \), \( 1 \leq i < n \), \( \Gamma_n(E_{ij}) \), \( i \neq j \), form a \( q \)-difference operator realization of \( U_q(sl(n)) \).

It is straightforward to obtain the explicit expressions for \( \Gamma_n(E_{ij}) \). In particular, we have

\[ \Gamma_n(E_{ij}) = \sum_{j=0}^{n-i-1} N^{n-j}_i - \sum_{j=1}^{i-1} N^i_j + \sum_{j=0}^{i-1} r_j, \] (6.108)

with the usual convention that a sum is zero if the upper limit is smaller than the lower limit. From this we obtain the expressions for the Cartan generators \( H^i_n \) (as defined above):
\[ H_i^n = 2N_i^{i+1} - r_i + \sum_{j=0}^{n-i-2} (N_i^{n-j} - N_i^{n-j+1}) + \sum_{j=1}^{i-1} (N_j^{i+1} - N_j^i) \quad i < n. \]  

(6.109)

Let us illustrate things for \( n = 2, 3 \).

For \( n = 2 \) we have (we use only (6.104e,f) and (6.109)):

\[
\begin{align*}
\Gamma_2(E_{12}) &= z_1^2 r_1 - N_1^2 = x[r - N_x], \\
\Gamma_2(E_{21}) &= D_1^2 = D_x, \\
H_1^2 &= 2N_x - r_1, \\
\end{align*}
\]

(6.110)

where we have denoted \( z_1^2 = x, N_1^2 = N_x \). This reproduces the known realization [309] of the \( U_q(sl(2)) \) representations with \( X^+ = \Gamma_2(E_{12}), X^- = \Gamma_2(E_{21}), H = H_1^2 \), depending on the representation parameter \( r_1 \) (\( r_0 \) being cancelled as expected). For \( q = 1 \) this coincides with the classical \( sl(2) \) vector-filed realization.

Next we take \( n = 3 \) setting \( z_1^3 = z, z_2^3 = y, N_1^3 = N_z, N_2^3 = N_y, D_1^3 = D_z, D_2^3 = D_y, r = r^2 = r_1 + r_2 \). Due to our recursive procedure we inherit form the case \( n = 2 \) the variable \( z_1^2 = x \), and the operators \( N_x, D_x \). Besides this we renormalize the generators \( \Gamma_3(E_{13}) \) and \( \Gamma_3(E_{31}) \) so that they obey the standard \( U_q(sl(3)) \) relations (these are different in [101], cf. (6.17) and (6.20)):

\[
\begin{align*}
\Gamma_3(E_{13}) &= \Gamma_3(E_{12})\Gamma_3(E_{23}) - q^{1/2}\Gamma_3(E_{23})\Gamma_3(E_{12}), \\
\Gamma_3(E_{31}) &= \Gamma_3(E_{32})\Gamma_3(E_{21}) - q^{-1/2}\Gamma_3(E_{21})\Gamma_3(E_{32}). \\
\end{align*}
\]

(6.111)

Thus, we have:

\[
\begin{align*}
H_1^3 &= 2N_1^2 - r_1 + N_1^3 - N_2^3 = 2N_x - r_1 + N_z - N_y, \\
H_2^3 &= 2N_2^2 - r_2 + N_1^3 - N_2^3 = 2N_y - r_2 + N_z - N_x, \\
\Gamma_3(E_{12}) &= \Gamma_2(E_{12})q^{1/2(N_2^3 - N_1^3)} + q^{1/2}\Gamma_2(E_{23})z_2^3D_2^3 = \\
&= x[r_1 - N_x]q^{1/2(N_z - N_y)} + zD_yq^{1/2(r_1 - 2N_x)}, \\
\Gamma_3(E_{21}) &= \Gamma_2(E_{21})q^{1/2(N_1^3 - N_2^3)} + q^{1/2}\Gamma_2(E_{31})z_2^3D_1^3 = \\
&= D_xq^{1/2(N_z - N_y)} + yD_yq^{1/2(r_1 - 2N_x)}, \\
\Gamma_3(E_{23}) &= q^{-1/2}\Gamma_2(E_{31})z_2^3[\Gamma_2(E_{33}) - \Gamma_2(E_{23}) - \sum_{k=1}^{2} N_k^3] - \\
&= -q^{2N_1^2}z_2^3\Gamma_2(E_{21}) = \\
&= y[r_2 + N_x - N_z - N_y]q^{-1/2(N_x + r_0)} - \\
&= -zD_xq^{-1/2(2r_2 + N_x - N_z - N_y + 1)}, \\
\Gamma_3(E_{32}) &= q^{1/2}\Gamma_2(E_{11})D_2^3 = D_yq^{1/2(r_0 + N_y)},
\end{align*}
\]

(6.112)
We now rescale the generators \( E_{3i}, E_{ij} \) (\( i = 1, 2 \)) so as to absorb the parameter \( r_0 \). (Such a rescaling should be done also for the general \( U_q(\mathfrak{gl}(n)) \) case.) Thus the realization of \( U_q(\mathfrak{sl}(3)) \) depends only on the parameters \( r_1, r_2 \), as in the classical case which may be obtained from (6.112) by setting \( q = 1 \).

### 6.3.2 Reducibility of the Representations and Invariant Subspaces

#### 6.3.2.1 Lowest-Weight Representations

Let us apply the realization (6.104) to the function 1. Using the fact that \( N_{n}^{i}1 = 0 = D_{n}^{i}1 \) we have:

\[
\Gamma_n(E_{ii}) 1 = \sum_{j=0}^{i-1} r_j = r^{i-1} , \quad H_n^{i} 1 = -r_i , \quad i \leq n , \tag{6.113a}
\]

\[
\Gamma_n(E_{mi}) 1 = 0 , \quad i < n , \tag{6.113b}
\]

\[
\Gamma_n(E_{ij}) 1 = \Gamma_{n-1}(E_{ij}) 1 = \cdots = \Gamma_{j}(E_{ij}) 1 = 0 , \quad j < i < n \tag{6.113c}
\]

\[
\Gamma_n(E_{in}) 1 = q^{\frac{1}{4}} \left( \sum_{k=1}^{n-1} k^{j-1} \sum_{k=1}^{n-1} k^{j-1} \right) z_{n}^{i} \left[ r_i + \cdots + r_{n-1} \right] - \sum_{s=i+1}^{n-1} q^{d_{i,s}} z_{s}^{n} \Gamma_{s}(E_{is}) 1 , \quad i < n , \tag{6.113e}
\]

where

\[
q^{d_{i,s}} = q^{\frac{1}{4}} \left( \sum_{k=s+1}^{n-1} r^{k-1} \sum_{k=1}^{n-1} k^{j-1} \right) q^{\frac{1}{2}} \left( \sum_{k=1}^{n} N_{k}^{i} \right) q^{\frac{1}{2}} \left( \sum_{k=1}^{n} N_{k}^{j} \right) N_{k}^{i} \times
\]

\[
q^{\frac{1}{2}} \left( N_{i}^{i} + N_{i}^{s} \right) q^{\frac{1}{2}} \left( \sum_{k=i+1}^{s} N_{k}^{i} \right) , \quad i < s. \tag{6.114}
\]

It is straightforward to obtain the explicit expressions for \( \Gamma_n(E_{ij}) 1, i < j \leq n \), applying recursively (6.113d,e). In particular, we have:

\[
\Gamma_n(E_{i,i+1}) 1 = \Gamma_{i+1}(E_{i,i+1}) 1 = q^{\frac{1}{4}} \sum_{k=1}^{i-1} r^{k-1} \left[ r_i \right] z_{i}^{i+1} . \tag{6.115}
\]

Thus, we have obtained an LWM with lowest-weight vector 1 (it is annihilated by the lowering generators \( \Gamma_n(E_{ij}), j < i \leq n \), and lowest-weight \( \Lambda \) such that \( \Lambda(H_i) = -r_i \) (cf.
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Generically this LWM is irreducible and then it is isomorphic to the Verma module with this lowest weight. The states in it correspond to the monomials of the Poincaré–Birkhoff–Witt basis of \( U_q(\mathcal{G}^+) \), where \( \mathcal{G}^+ \) is the subalgebra of the raising generators. This is isomorphic to the monomials in the variables \( z_i^k \). When the representation parameters \( r_i \) or certain combinations thereof are non-negative integers our representations are reducible. Below we consider in detail the cases \( n = 2 \) and \( n = 3 \).

6.3.2.2 Case \( U_q(sl(2)) \)

We start with \( n = 2 \) (though this example is well known). Using (6.110) we apply \( H, X^+, X^- \) to the function 1. We use the fact that \( N_x 1 = 0 = D_x 1 \). Thus:

\[
H 1 = -r, \quad X^+ 1 = x[r], \quad X^- 1 = 0.
\]  

(6.116)

Thus, we obtain an LWM with lowest-weight vector 1 and lowest-weight \( \Lambda \) such that \( \Lambda(H) = -r \). All states are given by powers of \( x \), that is, the basis is \( x^k \) with \( k \in \mathbb{Z}_+ \) and the representation is infinite dimensional. The action of \( U_q(sl(2)) \) is given by:

\[
X^+ x^k = [r-k]x^{k+1}, \quad X^- x^k = [k]x^{k-1}, \quad H x^k = (2k-r)x^k.
\]

(6.117)

Clearly, if \( r \notin \mathbb{Z}_+ \) this representation is irreducible. Furthermore all states may be obtained by the application of \( X^+ \) to the LWV; that is:

\[
(X^+)^k 1 = x^k[r][r-1] \ldots [r-k+1], \quad k \in \mathbb{Z}_+.
\]

(6.118)

Let \( r \in \mathbb{Z}_+ \), then \( (X^+)^{r+1} = X^+ x^r[r!] = 0 \). Thus, the states \( x^k \) with \( k = 0, 1, \ldots r \) form a finite-dimensional subrepresentation with \( \text{dim} = r + 1 \). Note that the complement of this subrepresentation; that is, the states \( x^k \) with \( k > r \), is not an invariant subspace.

Clearly, any polynomial in \( H, X^\pm \), will preserve this invariant subspace and thus would be a quasi-exactly solvable operator.

The invariant subspace may be obtained as the solution of either one of the following equations:

\[
(X^+)^{r+1} f(x) = 0, \quad (X^-)^{r+1} f(x) = 0,
\]

(6.119a, 6.119b)

in the space of formal power series \( f(x) = \sum_{k \in \mathbb{Z}_+} \mu_k x^k \). Note, however, that only (6.119b) (which is enough) was expected – this is an artefact of \( n = 2 \) simplifications. Indeed, only the operator in (6.119b) has the intertwining property (as in the classical case [202]):

\[
(X^-)^{r+1} \Gamma_2(X)_r = \Gamma_2(X)_r (X^-)^{r+1}, \quad r' = -r - 2,
\]

(6.120)

where \( X = H, X^\pm \), and \( \Gamma_2(X) \), is from (6.110) with explicit notation for the representation parameter of the two representations which are intertwined.
6.3.2.3 Case \( U_q(sl(3)) \)

Let us apply (6.112) to the function 1:

\[
H_1 1 = -r_1, \quad H_2 1 = -r_2, \\
\Gamma_3(E_{12}) 1 = x[r_1], \quad \Gamma_3(E_{21}) 1 = 0, \\
\Gamma_3(E_{23}) 1 = y[r_2], \quad \Gamma_3(E_{32}) 1 = 0, \\
\Gamma_3(E_{13}) 1 = q^{-\frac{1}{2}}r_1 z[r] - q^{\frac{1}{2}(2r_2 + 1)}yx[r_1], \quad \Gamma_3(E_{31}) 1 = 0.
\]

Thus, we obtain a lowest-weight module with lowest-weight vector 1 and lowest-weight \( \Lambda \) such that \( \Lambda(H_k) = -r_k \). All states are given by powers of \( x, y, z \); that is, the basis is generated by \( x^j z^k y^\ell \) with \( j, k, \ell \in \mathbb{Z}_+ \). The action of \( U_q(sl(3)) \) is given by:

\[
\begin{align*}
H_1 x^j z^k y^\ell &= (-r_1 + 2j - \ell + k)x^j z^k y^\ell, \\
H_2 x^j z^k y^\ell &= (-r_2 - j + 2\ell + k)x^j z^k y^\ell, \\
\Gamma_3(E_{12}) x^j z^k y^\ell &= [r_1 - j]q^{\frac{1}{2}(k-\ell)}x^{j+1} z^k y^\ell + \\
&\quad + [\ell]q^{\frac{1}{2}(r_1 - 2j)}x^j z^{k+1}y^{\ell - 1}, \\
\Gamma_3(E_{21}) x^j z^k y^\ell &= [j]q^{\frac{1}{2}(k-\ell)}x^{j+1} z^k y^\ell + \\
&\quad + [k]q^{\frac{1}{2}(r_1 - 2j)}x^j z^{k+1}y^{\ell - 1}, \\
\Gamma_3(E_{23}) x^j z^k y^\ell &= q^{-\frac{1}{4}}[r_2 + j - k - \ell]x^j z^k y^{\ell + 1} - \\
&\quad - [j]q^{-\frac{1}{4}(r_1 + 2r_2 + j - k - \ell + 1)}x^{j-1} z^{k+1}y^{\ell}, \\
\Gamma_3(E_{13}) x^j z^k y^\ell &= [\ell]q^{\frac{1}{4}(r_1 - 2j)}x^j z^k y^{\ell - 1}, \\
\Gamma_3(E_{32}) x^j z^k y^\ell &= [k]q^{\frac{1}{4}(r_1 - 2j)}x^j z^k y^{\ell - 1}, \\
\Gamma_3(E_{13}) x^j z^k y^\ell &= q^{\frac{1}{4}(j - r_1 - 2\ell)}[r - j - k - \ell]x^j z^{k+1} y^\ell - \\
&\quad - q^{\frac{1}{4}(2r_2 + j - k - 3\ell + 1)}[r_1 - j]x^{j+1} z^{k+1} y^{\ell + 1}, \\
\Gamma_3(E_{31}) x^j z^k y^\ell &= [k]q^{\frac{1}{4}(r_1 - 2j)}x^j z^{k-1} y^{\ell - 1}.
\end{align*}
\]

Further, in this section we show the following results which parallel the classical situation (cf. [202]):

1. If \( r_1, r_2, \text{ or } r + 1 \in \mathbb{Z}_+ \), this representation is reducible. It contains an irreducible subrepresentation which is infinite-dimensional, except when both \( r_1, r_2 \in \mathbb{Z}_+ \);
2. If \( r_1, r_2, r + 1 \notin \mathbb{Z}_+ \), this representation is irreducible and infinite-dimensional.

Clearly, if \( r_1 \in \mathbb{Z}_+ \), the representation (6.122) becomes reducible since the monomials \( x^j z^k y^\ell \) with \( j \leq r_1 \) form an invariant subspace since from (6.122c,g) we have:

\[
\begin{align*}
\Gamma_3(E_{12}) x^j z^k y^\ell &= [\ell]q^{-\frac{1}{4}r_1}x^j z^{k+1}y^{\ell - 1}, \\
\Gamma_3(E_{13}) x^j z^k y^\ell &= [r_2 - k - \ell]q^{-\frac{1}{4}\ell}x^j z^{k+1}y^{\ell}.
\end{align*}
\]
and all other operators are either lowering or preserving the powers of \( x \). This invariant subspace may be described as the solution of the following \( q \)-difference equation:

\[
(D_x)^{r_1+1} f(x, y, z) = 0.
\]  

(6.124)

Note that the operator in (6.124) has the intertwining property (as in the classical case [202]):

\[(D_x)^{r_1+1} \Gamma_3(X)_{r_1, r_2} = \Gamma_3(X)_{r_1', r_2'} (D_x)^{r_1+1}, \quad r_1' = -r_1 - 2, r_2' = r + 1, \]

(6.125)

where \( X = E_{ii} - E_{i+1,i+1}, E_{ij}, i \neq j, \Gamma_3(X)_{r_1,r_2} \) is taken from (6.112) with explicit dependence of the representation parameters of the two representations which are intertwined.

The subrepresentation obtained is infinite-dimensional if \( r_2 \not\in \mathbb{Z}_+ \) since the powers of \( y, z \) are still unrestricted by (6.122e,g).

If \( r_2 \in \mathbb{Z}_+ \) the representation in (6.122) becomes reducible. In the classical case \((q = 1)\) the equation which singles out the invariant subspace is [202]:

\[
(x \partial_x + \partial_y)^{r_2+1} f(x, y, z) = 0, \quad q = 1.
\]  

(6.126)

For the quantum case we have the following expression:

\[
q^{r_2+1} f(x, y, z) = 0, \quad q = \frac{1}{2}(r_1 + 1)(r_2 + 1)(r + 2).
\]  

(6.127)

\[
q^{r_2} = \sum_{s=0}^{k} \left( \begin{array}{c} k \\ s \end{array} \right) \frac{q^{s(N_x-r_1)+\frac{1}{2}(s-k)N_y-\frac{1}{4}kN_y}}{x^s D_2^k D_y^s q^{\frac{1}{4}s(N_x-r_1)+\frac{1}{2}(s-k)N_y-\frac{1}{4}kN_y}},
\]

which coincides with (6.126) for \( q = 1 \). The invariant subspace is infinite-dimensional if \( r_1 \not\in \mathbb{Z}_+ \).

As in the classical case [202] the explicit form (6.127) of this operator may be checked by the intertwining property:

\[
q^{r_2+1} \Gamma_3(X)_{r_1,r_2} = \Gamma_3(X)_{r_1',r_2'} q^{r_2+1}, \quad r_1' = r + 1, r_2' = -r_2 - 2,
\]

(6.128)

where \( X = E_{ij}, i \neq j, E_{ii} - E_{i+1,i+1}, \Gamma_3(X)_{r_1,r_2} \) is from (6.112) as in (6.125).

Further in this subsection we consider the case when both \( r_k \in \mathbb{Z}_+ \). Then there is a finite-dimensional irreducible subspace of dimension:

\[
d_{r_1,r_2} = \frac{1}{2}(r_1 + 1)(r_2 + 1)(r + 2).
\]  

(6.129)

Thus, we recover the complete list of the finite-dimensional irreps of \( U_q(sl(3)) \) and \( SL(3) \), and by default, also the complete list of the finite-dimensional unitary irreps of \( U_q(su(3)) \) and \( SU(3) \) (we have assumed that \( q \) is not a nontrivial root of 1).
Next we use the following general formula valid for arbitrary \( r_k \):

\[
v_{\ell kj} \equiv \Gamma_3(E_{23})^j \Gamma_3(E_{13})^k \Gamma_3(E_{12})^j 1 =
\]

\[
= k \sum_{s=0}^{\ell} \sum_{n=0}^{\ell} (-1)^{s-n} \binom{k}{s} q^{\frac{1}{4}((j-r_1)k-j+1)(r_1+2r_2-k-\ell+2)} \times
\]

\[
\times \frac{\Gamma_q(r_1+1) \Gamma_q(r-j-s+1)}{\Gamma_q(r_1-j-s+1) \Gamma_q(r-j-k+1)} \times \frac{\Gamma_q(r_2+j+s-k-n+1)}{\Gamma_q(r_2+j+s-k-\ell+1)} \frac{[j+s]!}{[j+s-n]!}
\]

\[
\times y^{\ell+s-n} z^{k-s+n} x^{j+s-n},
\]

\[
\ell + k + j \leq r, \quad 0 \leq j \leq r_1, \quad 0 \leq \ell \leq r_2.
\]

(6.130)

One of the main results of [246] is that the basis of the finite-dimensional irrep with dimension \( d_{r_1,r_2} \) for \( r_1, r_2 \in \mathbb{Z}_+ \) (cf. (6.129)) is given by \( v_{\ell kj} \) iff \( \ell + k + j \leq r, 0 \leq j \leq r_1, 0 \leq \ell \leq r_2 \). In the next section we relate this basis to the standard Gel’fand–Zetlin basis.

For later reference we note the special polynomial \( v_{0r0} \) which corresponds to the highest-weight vector (as we shall see later):

\[
v_{0r0} \equiv \Gamma_3(E_{13})^j 1 = q^{\frac{1}{4}r_1} [r] q^1 \sum_{s=0}^{r_1} (-1)^s \binom{r_1}{s} q^{\frac{1}{4}s(r-r_1+2)} x^s z^{r-s} y^s
\]

(6.131)

Note also that when \( r_1 = 0 \) there is no dependence on \( x \) in (6.130), all states being the monomials \( v_{\ell k0} \sim y^\ell z^k \).

Also for later reference we note the explicit value of \( v_{\ell kj} \) for \( z = 0 \) (given by the term \( s = k \) and \( n = 0 \)):

\[
v_{\ell kj}|_{z=0} = (-1)^k q^{\frac{1}{4}((j-r_1)k-j+1)(r_1+2r_2-k-\ell+2)} \times
\]

\[
\times \frac{\Gamma_q(r_1+1)}{\Gamma_q(r_1-j-k+1)} \frac{\Gamma_q(r_2+j+1)}{\Gamma_q(r_2+j-\ell+1)} y^{j+k} x^{\ell+k}
\]

(6.132)

Note that the RHS of (6.132) is equal to zero when \( j + k \geq r_1 + 1 \) (because of the \( \Gamma_q(r_1-j-k+1) \) in the denominator). In this case one applies \((D_z)^{j+k-r_1}\) to both sides of (6.130) and then sets \( z = 0 \).

Next, we discuss the case when \( r + 1 \in \mathbb{Z}_+ \), but \( r_k \notin \mathbb{Z}_+ \). Following our procedure for invariant differential operators, we use the \( U_q(sl(3)) \) singular vector in Chevalley basis (cf. f-la (27) from [198] or Section 2.4):
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$$v_{r}^{s+2} = \sum_{s=0}^{r+2} \frac{(-1)^s}{[r_1 +1-s]_q} \binom{r+2}{s}_q \left(\hat{X}_1^{r+2-s} \hat{X}_2^s\right)$$ (6.133)

In the latter we substitute the corresponding action of the simple root generators $\hat{X}_1^+$, $\hat{X}_2^+$ by the operators $q\hat{D}_1 = D_x$ and by $q\hat{D}_2$ from (6.127) to obtain the invariant $q$-difference operator:

$$q\hat{D}_3^{r+2} = \sum_{s=0}^{r+2} \frac{(-1)^s}{[r_1 +1-s]_q} \binom{r+2}{s}_q \left[D_x^{r+2-s}(\hat{D}_2)^s\right]$$ (6.134)

Thus, our subspace is singled out by the following explicit equation [246]:

$$q\hat{D}_3^{r+2}f(x,y,z) = 0,$$ (6.135)

$$q\hat{D}_3^{r+2} = \sum_{s=0}^{r+2} \sum_{t=0}^{s} \frac{q^{\frac{1}{4}(r+2-t-2s)}r_1^{\frac{1}{2}(r+2)t}r_2^{\frac{1}{2}(r+1)(s-t)}\Gamma_q(-1-r_1)}{[s]!t!\cdot [r+2-s-t]!\Gamma_q(r-r_1+2-s)} \times$$

$$\times D_z^{r+2-t}D_y^tD_x^t \prod_{u=1}^{r+2-s-t} [N_x - t + 1 - u] q^{\frac{(2r-r_2-2)}{4}N_x + \frac{r_1}{4}N_y + \frac{r(r+2)}{4}N_z}$$ (6.136)

As in the classical case [202] the explicit form of this operator may be checked by the intertwining property:

$$q\hat{D}_3^{r+2}\Gamma_3(X)_{r_1,r_2} = \Gamma_3(X)_{r_1',r_2'} q\hat{D}_3^{r+2}, \quad r_1' = -r_2 - 2, \quad r_2' = -r_1 - 2,$$ (6.136)

where $X = E_{ii} - E_{i+1,i+1}, E_{ij}, i \neq j, \Gamma_3(X)_{r_1,r_2}$ is from (6.112) as in (6.125) and (6.128).

The states in the subrepresentation are given by $\nu_{k\ell}j$, with $\ell, k, j \in \mathbb{Z}_+$, $k \leq r + 1$.

Let us illustrate the above by the simplest limiting case of $r = -1$. Then the states are:

$$\nu_{k\ell}j_{r=-1} = \sum_{n=0}^{\ell} \frac{(-1)^n}{n!} \binom{\ell}{n} q^{\frac{1}{4}(-n)(r_1+\ell)} \times$$

$$\times \frac{\Gamma_q(r_1 + 1)}{\Gamma_q(r_1 - 1 - j + \ell)} \frac{\Gamma_q(1 + r_1 - j + \ell)}{\Gamma_q(1 + r_1 - j + n)} \frac{[j]!}{[j-n]!} \times$$

$$\times x^{j-n}z^n y^{-n} =$$

$$= q^{\frac{1}{2}r_1} \frac{\Gamma_q(r_1 + 1)}{\Gamma_q(r_1 - 1 - j + \ell)} \frac{\Gamma_q(j - r_1)}{\Gamma_q(j - r_1 - \ell)} \times$$

$$\times x^j y^j F_1^q(-j, -\ell; r_1 - 1 + 1; q^{\frac{1}{2}(r_1+\ell)} \frac{z}{xy}).$$ (6.137)

Alternatively one may check that this is the general solution of (6.135) for $r = -1$. 
6.3.3 Newton Diagrams

In this section we give a visualization of the representation spaces. Each state is represented by a point on an integer lattice in \( n(n-1)/2 \) dimensions, that is, on \( \mathbb{Z}_{+}^{n(n-1)/2} \). For a finite-dimensional subrepresentation the number of these points is finite and the hull of these points is a convex polyhedron in \( \mathbb{R}^{n(n-1)/2} \). Such a polyhedron (not necessarily convex) was called a Newton diagram [43]. In the present context this notion was introduced in [580], where also some examples in the case of functions in one and two variables were given (for \( q = 1 \), when the figures are planar (polygons). Below, we give explicitly the Newton diagrams for \( n = 3 \). Moreover, we introduce also infinite Newton diagrams to depict the infinite-dimensional nontrivial subrepresentations.

6.3.3.1 Finite Newton Diagrams for \( n = 3 \)

Fix \( r_k \in \mathbb{Z}_{+} \). Then the Newton diagram is given by the points with integer coordinates \( j, \ell, k \) in \( \mathbb{Z}_{+}^{3} \) such that:

\[
\begin{align*}
0 \leq j + k + \ell & \leq r, \quad (6.138a) \\
0 \leq j & \leq r_1, \quad (6.138b) \\
0 \leq \ell & \leq r_2, \quad (6.138c)
\end{align*}
\]

(cf. below Figure 6.1 taken from [246]). The polyhedron formed by these points is planar only for \( r_1 = 0 \) or \( r_2 = 0 \) in which case it is a triangle (only (6.138a) is relevant since \( r = r_2 \) or \( r = r_1 \)). (The case \( r_1 = 0 \) was given in [580].)

Fix a point \( j, \ell, k \). This is represented by the state \( v_{\ell kj} \). Then, the number of states is:

\[
\begin{align*}
\sum_{j=0}^{r_1} \sum_{k=0}^{r_1-j} \sum_{\ell=0}^{\min(r-k-j,r_2)} 1 &= \sum_{j=0}^{r_1} \sum_{k=0}^{r_1-j} \sum_{\ell=0}^{r_2} 1 + \sum_{k=r_1-j}^{r_1} \sum_{\ell=0}^{r_2} 1 \\
&= \frac{(r_1 + 1)(r_1 + 2)(r_2 + 1)}{2} + \frac{(r_1 + 1)r_2(r_2 + 1)}{2} = d_{r_1,r_2}, \quad (6.139)
\end{align*}
\]

as expected (cf. (6.129)).

Note that such diagrams have an advantage over the usual weight diagrams for \( sl(3) \) and \( su(3) \) which are degenerate. For instance, consider the adjoint representation obtained for \( r_1 = r_2 = 1 \). The weight diagram consists of two orbits of the Weyl group, one with six points with multiplicity one, and the other with one point with multiplicity two. To the latter point in our diagram correspond the two linearly independent states:

\[
\begin{align*}
v_{101} &= q^{-\frac{1}{2}} ([2]_{q} xy - q^{-1} z), \quad (6.140a) \\
v_{010} &= q^{-\frac{1}{2}} (z - q xy), \quad (6.140b)
\end{align*}
\]
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\[ k = r_1 + r_2 \]

Figure 6.1: Newton diagram for the finite-dimensional representations of \( U_q(\mathfrak{sl}(3)) \)

### 6.3.3.2 Infinite Newton diagrams for \( n=3 \)

Here either \( r_1 \notin \mathbb{Z}_+ \) or \( r_2 \notin \mathbb{Z}_+ \) and the considerations run in parallel with considerations of the polynomial basis. Below \( j, \ell, k \in \mathbb{Z}_+ \).

1. For \( r_1 \in \mathbb{Z}_+ \) and \( r_2, r+1 \notin \mathbb{Z}_+ \) the Newton diagram is given by the points with coordinates:

\[
0 \leq k, \quad 0 \leq j \leq r_1, \quad 0 \leq \ell. \tag{6.141}
\]

2. For \( r_2 \in \mathbb{Z}_+ \) and \( r_1, r+1 \notin \mathbb{Z}_+ \) the Newton diagram is given by the points with coordinates:

\[
0 \leq k, \quad 0 \leq j, \quad 0 \leq \ell \leq r_2. \tag{6.142}
\]

3. For \( r+1 \in \mathbb{Z}_+ \) and \( r_1, r_2 \notin \mathbb{Z}_+ \) the Newton diagram is given by the points with coordinates:

\[
0 \leq k \leq r+1, \quad 0 \leq j, \quad 0 \leq \ell. \tag{6.143}
\]
4. For \( r_1, r + 1 \in \mathbb{Z}_+ \) and \( r_2 + 1 \in \mathbb{N} \) the Newton diagram is given by two sets of points with coordinates:

\[
0 \leq k, \quad -r_2 - 1 \leq j \leq r_1, \quad 0 \leq \ell.
\]

\[
0 \leq k \leq r + 1, \quad 0 \leq j \leq -r_2 - 2, \quad 0 \leq \ell.
\]

5. For \( r_1 = r + 1 \in \mathbb{Z}_+ \) and \( r_2 = -1 \) the Newton diagram is given by (6.141). It can be obtained formally from the previous case by setting \( r_2 = -1 \), then (6.144) coincides with (6.141), while (6.145) is empty.

6. For \( r_2, r + 1 \in \mathbb{Z}_+ \) and \( r_1 + 1 \in \mathbb{N} \) the Newton diagram is given by two sets of points with coordinates:

\[
0 \leq k, \quad 0 \leq j - r_1 - 1 \leq r_2,
\]

\[
0 \leq k \leq r + 1, \quad 0 \leq \ell, \quad 0 \leq j \leq -r_2 - 2.
\]

7. For \( r_2 = r + 1 \in \mathbb{Z}_+ \) and \( r_1 = -1 \) the Newton diagram is given by (6.142). It can be obtained formally from the previous case by setting \( r_1 = -1 \), then (6.146) coincides with (6.142), while (6.147) is empty.

### 6.4 Application of the Gelfand–(Weyl)–Zetlin Basis

#### 6.4.1 Correspondence with the GWZ Basis

In this section we follow [230, 244, 245]. We would like to establish the correspondence between our basis for the finite-dimensional irreducible representations given by the states \( v_{\ell k j} \) (cf. (6.130)) and the \( SU(3) \) Gel’fand–Weyl–Zetlin basis:

\[
\begin{pmatrix}
m_1 & m_2 & m_3 \\
m_2 & m_3 \\
m_3 & m_1
\end{pmatrix}
\]

(Note that in the literature this basis is most often called Gel’fand–Zetlin basis, here we keep the usage from [230, 244, 245].) In fact, the above is for \( U(3) \), and we shall set \( m_{33} = 0 \) to restrict to \( SU(3) \). Further we need the operators corresponding to isospin \( \hat{I}_2 \), third component of isospin \( \hat{I}_z \), and hypercharge \( \hat{Y} \):

\[
\hat{I}_z = \frac{1}{2} \hat{H}_1, \quad \hat{Y} = \frac{1}{2} (\hat{H}_1 + 2\hat{H}_2)
\]

\[
\hat{I} = E_{21}E_{12} + [\frac{1}{2}\hat{H}_1]_q[\frac{1}{2}\hat{H}_1 + 1]_q
\]
Note that $\hat{I}$ is the Casimir of the $U_q(\mathfrak{sl}(2))$ quantum subgroup generated by $E_{21}, E_{12}, H_1$. It is easy to see that, like the GWZ states, also the $v_{\ell kj}$ states are eigenvectors of $\hat{I}_z$ and $\hat{Y}$, but they are not eigenvectors of $\hat{I}$. In fact we have:

$$\Gamma_3(\frac{1}{2}H_1) v_{\ell kj} = (j + k - \frac{1}{2}(r_1 + \ell + k)) v_{\ell kj}$$

$$(6.150)$$

$$\Gamma_3\left(\frac{1}{3}(H_1 + 2H_2)\right) v_{\ell kj} = \left( r_1 + k + \ell - \frac{2}{3}(r + r_1) \right) v_{\ell kj}$$

$$\Gamma_3(E_{21})\Gamma_3(E_{12}) v_{\ell kj} = \left( (j + 1)(r_1 - j) + \ell(k + 1) \right) v_{\ell kj} +$$

$$+ kv_{\ell+1,k-1,j+1} + (r_1 - j + 1)\ell v_{\ell-1,k+1,j-1}$$

The last formula is given for $q = 1$ since it is only to illustrate our point. We shall diagonalize $\Gamma_3(E_{21})\Gamma_3(E_{12})$ in the next subsection and find explicit polynomial eigenvectors. Here we find alternatively an explicit correspondence between (m) and the appropriate linear combination of $v_{\ell kj}$’s. But first we place the labels $\ell, k, j$ in a GWZ pattern.

First, we fix the correspondence between the two representations, namely, between the labels $\{m_{13}, m_{23}\}$ and $\{r, r_1\}$, by considering the lowest-weight vector. This is the GWZ vector [65]:

$$\begin{pmatrix}
m_{13} & m_{23} & 0 \\
m_{23} & 0 \\
0 & & \\
\end{pmatrix}$$

(6.151)

which has $I = -I_z = m_{23}/2$ and $Y = -\frac{1}{3}(2m_{13} - m_{23})$. In our realization the lowest-weight vector is $v_{000} = 1$ and thus from (6.150) we get that $I_z = -r_1/2$ and $Y = -\frac{1}{3}(2r - r_1)$. Therefore we find $m_{13} = r$ and $m_{23} = r_1$.

For further use we record explicitly the patterns corresponding to the highest-weight state (h.w.s.) and to the lowest-weight state (l.w.s.):

(h.w.s.) $$= \begin{pmatrix}
r & r_1 & 0 \\
r_1 & 0 & \\
r & & \\
\end{pmatrix}$$

(l.w.s.) $$= \begin{pmatrix}
r & r_1 & 0 \\
r_1 & 0 & \\
0 & & \\
\end{pmatrix}$$

(6.152a)

(6.152b)

Remark 6.2. Notice that the well-known conjugation of representation $[m_{13}, m_{23}, 0] \rightarrow [m_{13}, m_{13} - m_{23}, 0]$ [65] corresponds to the exchange of $r_1$ with $r_2$ and that the dimension of the representation $[m_{13}, m_{23}, 0]$, namely, $\frac{1}{2}(m_{13} + 2)(m_{23} + 1)(m_{13} - m_{23} + 1)$, matches (6.129).
To place the \( v_{\ell,k,j} \) states in a GWZ pattern we split them as in [246] (cf. (66)) in two subsets depending whether \( j + k \leq r_1 \) or \( j + k > r_1 \). In the first case the correspondence is given by:

\[
v_{\ell,k,j} = \begin{pmatrix} r & \ell & r_1 & 0 \\ r_1 + \ell & k & 0 \\ j + k & & & \\
\end{pmatrix}, \quad j + k \leq r_1.
\]  

(6.153)

In the second case the correspondence is given by:

\[
v_{\ell,k,j} = \begin{pmatrix} r & \ell & r_1 & 0 \\ j + k + \ell & r_1 - j & 0 \\ j + k & & & \\
\end{pmatrix}, \quad j + k > r_1,
\]  

(6.154)

which is valid also for the boundary case \( j + k = r_1 \), when it coincides with (6.153). The betweenness constraint

\[
m_{ij} \geq m_{i,j-1} \geq m_{i+1,j},
\]  

(6.155)

typical of the GWZ pattern then gives the constraints \( 0 \leq j \leq r_1 \), \( 0 \leq \ell \leq r_2 \) and \( 0 \leq j + k + \ell \leq r \) found above for the finite-dimensional representations.

The actual correspondence is proved using well-known techniques of raising and lowering operators developed for classical groups and adapted to quantum groups (cf. [65, 100, 101, 516, 582]). Identifying the lowest-weight states one can find explicitly a polynomial \( p(m) \) in \( U_q(\mathfrak{g}^+) \) which corresponds to \( (m) \).

Let us denote by \( \hat{1} \) the lowest-weight state of any realization of the \( U_q(\mathfrak{sl}(3)) \) finite-dimensional representation with parameters \( r_1, r_2 \). Then we have (up to multiplicative normalization constant) [516, 582]:

\[
p_{(m)} \hat{1} = (E_{23})^{m_{12} - m_{11}} \tilde{c}^{r - m_{12}} (E_{32})^{1 - m_{22}} (E_{13})^{r} \hat{1} = \]

\[
= [m_{12} + m_{22} - r_1]q! \sum_{t=0}^{r-m_{22}} \sum_{u \in \mathbb{Z}_+} (-1)^t \binom{m_{12} - m_{11} + t}{u} q^t \times
\]

\[
\times \frac{[m_{22} - t - r_1 + u]_q! [m_{12} - m_{22} + r_1 + u]_q!}{[t]_q! [m_{22} - t]_q! [m_{12} - m_{22} + 1 + t]_q!} \times
\]

\[
\times \frac{[m_{12} - m_{22} + 1]_q! [t + r_1 - m_{22}]_q!}{[m_{11} - m_{12} - m_{22} + r_1 + u]_q!} \times
\]

\[
\times \frac{[m_{12} + m_{22} - m_{11} - u]_q!}{[m_{12} + m_{22} - r_1 - u]_q!} \times
\]

\[
\times (E_{23})^u (E_{32})^{m_{12} + m_{22} - r_1 - u} (E_{13})^{m_{11} - m_{12} - m_{22} + r_1 + u} \hat{1},
\]  

(6.156a)

(6.156b)
\[ \tilde{C} = E_{31} [H_1 + 1] + E_{21} E_{32} q^{H_1+1} = E_{32} E_{21} [H_1 + 1] - E_{21} E_{32} [H_1] \] (6.156c)

**Remark 6.3.** We would like to stress the peculiarity of (6.156a). One gets (in (6.156b)) a correspondence of the GWZ states with polynomials in \( U(\mathfrak{g}^+) \) but formula (6.156a) first gives us a one-to-one correspondence of the GWZ states with monomials in the \( q \)-deformed enveloping algebra \( U_q(\mathfrak{g}^-) \) of the lowering generators. Note that the latter monomials are not in the standard Poincaré–Birkhoff–Witt basis of \( U_q(\mathfrak{g}^-) \), namely, instead of the generator \( E_{31} \) one has the generator of the same weight \( \tilde{C} \) (cf. formula (6.3) of [516]). These monomials produce the polynomials of \( U(\mathfrak{g}^+) \) since they act on \((E^\ell)^{\ell} \tilde{1} \) which is in \( U(\mathfrak{g}^+) \) and \((E^\ell)^{\ell} \tilde{1} \) is the highest-weight vector. Finally, we note that there exists a similar description of this correspondence only in terms of raising generators, in particular, involving an analogue of \( \tilde{C} \) in \( U(\mathfrak{g}^+) \). However, the present description is simpler for our purposes here, while the other is used in Section 6.4.7, where it is more useful.

Finally, we get the correspondence we need using (6.156):

**Theorem 6.1.** A realization of the GWZ basis as polynomials in three variables (real or complex) is given by the formula:

\[
\phi_{(\mathbf{m})} = \Gamma_3(p_{(\mathbf{m})} 1 = [m_{12} + m_{22} - r_1] q! x \\
\times \sum_{t=0}^{r-m_{12}} \sum_{u \in \mathbb{Z}_+} \left( \frac{m_{12} - m_{11} + t}{u} \right) q \\
\times (-1)^{t} \frac{q^t (m_{22} - t - r_1)(m_{12} + m_{22} - r_1) + q^{u-2m_{22} - m_{12} + m_{11} + r_1 + t}}{[t] q! [m_{22} - t] q! [m_{12} - m_{22} + 1 + t] q!} \\
\times \left[ \frac{[m_{12} - m_{22} + 1]}{[m_{11} - m_{12} - m_{22} + r_1 + u]} q! \right] [t + r_1 - m_{22}] q! \left[ \frac{m_{12} + m_{22} - m_{11} - u}{m_{12} + m_{22} - r_1 - u} q! \right] \\
\times v_{u, m_{12} + m_{22} - r_1 - u, m_{11} - m_{12} - m_{22} + r_1 + u} \right] \] (6.157)

**Proof.** Straightforward using (6.156) and our formula for \( v_{\ell kj} \) (6.130).

For later reference we note the explicit value of \( \phi_{(\mathbf{m})} \) for \( z = 0 \) (using (6.132)):

\[
\phi_{(\mathbf{m})}|_{z=0} = \frac{A_{(\mathbf{m})}^+}{r_1 - m_{11} + 1} x^{m_{11}} y^{m_{12} + m_{22} - r_1}, \] (6.158a)
\[ \mathcal{N}^+_m = [r_1]_q! [m_{12} + m_{22} - r_1]_q! \sum_{t=0}^{r-m_{12}} \sum_{u \in \mathbb{Z}^+} (-1)^t u^{m_{12}+m_{22}+r_1} \times \]
\[ \times \binom{m_{12} - m_{11} + t}{u}_q \binom{r + m_{11} - m_{12} - m_{22} + 1}{u}_q \]

\[ \times q^{\frac{1}{2}((u-m_{12}-m_{22}+r_1)(m_{12}-r+1)+t(m_{12}+m_{22}-r_1)(m_{11}/2-r_1))} \times \]

\[ \times \frac{[t]_q! [m_{22} - t]_q! [m_{12} - m_{22} + 1 + t]_q!}{[m_{11} - m_{12} - m_{22} + r_1 + u]_q! [m_{12} + m_{22} - r_1 - u]_q!} \]

which is useful for \( r_1 - m_{11} + 1 > 0 \). Otherwise it is zero (due to the singled out factor \( \Gamma_q(r_1 - m_{11} + 1) \)), and to obtain a nonzero value one first has to differentiate \( m_{11} - r \) times \((6.157)\) w.r.t. \( z \).

We note also the expression for the lowest-weight state obtained from \((6.156)\) and \((6.157)\) for \( m_{12} = r_1, m_{11} = m_{22} = 0 \):

\[ p_{\text{lws}} \hat{1} = (E_{21})^{r_1} \hat{C}^{r-r_1} (E_{32})^{r_1} (E_{13})^{r_1} \hat{1} = \]

\[ = \mathcal{N}_{\text{lws}}^+ \hat{1} = ([r_1]_q)^3 \hat{1} \] \hspace{1cm} (6.159a)

\[ \phi_{\text{lws}} = \Gamma_q \langle p_{\text{lws}} \rangle 1 = ([r_1]_q)^3 \] \hspace{1cm} (6.159b)

which of course differ from \( \hat{1}, 1 \), respectively, by a constant – the corresponding value of \( \mathcal{N}^+_m \).

### 6.4.2 \( q \)-Hypergeometric Realization of the GWZ Basis

In the previous section we exhibited the relation of the GWZ basis and the polynomial basis \( v_{\ell k j} \). By formula \((6.157)\) this provides also a polynomial realization of the GWZ basis in the same variables \( x, y, z \). However, \((6.157)\) is not very explicit, since it contains a quadruple sum (a double sum in \((6.157)\) and a double sum in \((6.130)\)). Instead of partially summing \((6.157)\), in this section we shall find a polynomial realization directly (not relying on the correspondence with \( v_{\ell k j} \)) using the fact that the GWZ states are eigenvectors of the operators \( \hat{I}_z, \hat{Y}, \hat{I} \).

We shall proceed as follows. Let us denote (as in \((6.157)\)) the unknown polynomial function corresponding to \((m)\) by:

\[ \psi = \psi_{(m)}(x, y, z) \] \hspace{1cm} (6.160)

Naturally, \( \psi_{(m)} \) can differ from \( \phi_{(m)} \) in \((6.157)\) only by a multiplicative constant which we shall fix later.
In order to use effectively the fact that $\psi$ is an eigenfunction of $\tilde{I}_z$, $\tilde{Y}$, $\tilde{I}$ we use their explicit $q$-difference realization (6.112). We write:

\[
\tilde{I}_z \equiv \frac{1}{2} \Gamma_3(H_1) = \frac{1}{2} \left(2N_x - r_1 + N_z - N_y\right) \quad (6.161a)
\]
\[
\tilde{Y} \equiv \frac{1}{3} \Gamma_3(2H_2) = N_y + N_z - \frac{1}{3}(r_1 + 2r_2) \quad (6.161b)
\]
\[
\tilde{I}^2 \equiv \Gamma_3(E_{21})\Gamma_3(E_{12}) + \left[\tilde{I}_z\right]_q[\tilde{I}_z + 1]_q = 
\]
\[
[N_x + 1]_q[r_1 - N_x]_q q^{\frac{1}{2}(N_e - N_p)} + [N_z + 1]_q[N_y]_q q^{\frac{1}{2}(r_1 - 2N_x) +}
\]
\[
+ \frac{z}{xy} [N_x]_q[N_y]_q q^{\frac{1}{2}(r_1 - 2N_x + N_z - N_y) +} + [\tilde{I}_z]_q[\tilde{I}_z + 1]_q 
\]

The eigenfunction conditions satisfied by $\psi$ are:

\[
\tilde{I}_z \psi = I_z \psi = \left(m_{11} - \frac{1}{2}(m_{12} + m_{22})\right) \psi \quad (6.162a)
\]
\[
\tilde{Y} \psi = Y \psi = \left(m_{12} + m_{22} - \frac{2}{3}(r + r_1)\right) \psi \quad (6.162b)
\]
\[
\tilde{I}^2 \psi = [I]_q[I + 1]_q \psi = 
\]
\[
\left[ \frac{m_{12} - m_{22}}{2} \right]_q \left[ \frac{m_{12} - m_{22}}{2} + 1 \right]_q \psi \quad (6.162c)
\]

Next we consider the operators $\tilde{I}_z + \frac{1}{2} \tilde{Y}$, $\tilde{Y}$, from which we obtain the following homogeneity conditions:

\[
(N_x + N_z) \psi = \left(\tilde{I}_z + \frac{1}{2} \tilde{Y} + \frac{1}{3}(r + r_1)\right) \psi = m_{11} \psi \quad (6.163a)
\]
\[
(N_y + N_z) \psi = \left(\tilde{Y} + \frac{1}{3}(r - r_1)\right) = \kappa \psi , \quad (6.163b)
\]

\[
\kappa \equiv m_{12} + m_{22} - r_1
\]

From these homogeneity conditions and the explicit form of (6.162c) we are prompted to make the following change of variables:

\[
x' = x, \quad y' = y, \quad \zeta = \frac{z}{xy} \quad (6.164)
\]

from which follows:

\[
N_x = N_{x'}, \quad N_y = N_{y'}, \quad N_z = N_{z'} \quad (6.165)
\]

Thus, the homogeneity conditions (6.163) simplify to:

\[
N_{x'} \psi = m_{11} \psi , \quad N_{y'} \psi = \kappa \psi , \quad (6.166)
\]
that is, our polynomials actually have the form:

\[ \psi = \psi_{(m)} = x^{m_{11}} y^{r_1} \hat{\psi}(\zeta) \]  \hspace{1cm} (6.167)

Actually from this expression we can deduce that \( \hat{\psi} \) is a polynomial in \( \zeta \) of degree at most \( n_0 = \min (m_{11}, \kappa) \). Indeed, if \( \hat{\psi} \) is a polynomial in \( \zeta \) of higher degree, then \( \psi \) would not be a polynomial in \( x \) or \( y \) or both, contradicting our starting assumption.

Substituting now (6.167) in (6.162c) and taking into account (6.166) we obtain the following equation for \( \hat{\psi} \):

\[
\left( [m_{11} + 1 - N_\zeta]q [r_1 - m_{11} + N_\zeta]q^{N_\zeta - \frac{1}{2} \kappa} + \\
+ [1 + N_\zeta]q [\kappa - N_\zeta]q^{\frac{1}{2} (r_1 - \kappa) + \frac{1}{2} (1 - m_{11}) + N_\zeta} + \\
+ \zeta [m_{11} - N_\zeta]q [\kappa - N_\zeta]q^{\frac{1}{2} (r_1 - \kappa) + \frac{1}{2} (1 - m_{11}) + N_\zeta} + \\
+ \zeta^{-1} [N_\zeta]q [r_1 - m_{11} + N_\zeta]q^{\frac{1}{2} (r_1 - \kappa) + \frac{1}{2} (1 + m_{11}) + N_\zeta} + \\
+ [m_{11} - m_{12}]q [m_{11} - m_{22} + 1]q \right) \hat{\psi}(\zeta) = 0 \]  \hspace{1cm} (6.168)

The unique (up to nonzero multiple) polynomial solution of (6.168) is given by \( q \)-Jacobi or, equivalently, by \( q \)-hypergeometric polynomials. In particular, if \( \beta = r_1 - m_{11} + 1 \notin \mathbb{Z}_- \) then such a solution is:

\[
\hat{\psi}_1(\zeta) = \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} a \\ s \end{array} \right)_q^s \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} b \\ s \end{array} \right)_q^s \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} c \\ s \end{array} \right)_q^s \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} \beta \\ s \end{array} \right)_q^s q^{s^2} \right) q^{s^2 + s \zeta} \]  \hspace{1cm} (6.169)

where \( 2F_1 \) is a \( q \)-hypergeometric polynomial:

\[
2F_1(a, b; c; \zeta) = \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} a \\ s \end{array} \right)_q^s \left( \begin{array}{c} b \\ s \end{array} \right)_q^s \left( \begin{array}{c} c \\ s \end{array} \right)_q^s \left( \begin{array}{c} \beta \\ s \end{array} \right)_q^s q^{s^2} \]  \hspace{1cm} (6.170)

\( 1F_0 \) is a degenerate \( q \)-hypergeometric polynomial:

\[
1F_0(a; \zeta) = \sum_{s \in \mathbb{Z}_+} \left( \begin{array}{c} a \\ s \end{array} \right)_q^s \zeta^s = 2F_1(a, b; b; \zeta) \]  \hspace{1cm} (6.171)
and \((v)_s^q\) is the \(q\)-Pochhammer symbol:

\[
(v)_s^q \equiv [v + s - 1]_q[v + s - 2]_q \ldots [v]_q = \frac{\Gamma_q(v + s)}{\Gamma_q(v)}
\]  

(6.172)

Note that (6.172) ensures that (6.170) and (6.171) are polynomials of degree \(\min(-a, -b), -a\), respectively, when \(a, b \in \mathbb{Z}_-\), as is in our case. Note that for \(q = 1\) (6.170) goes into the standard hypergeometric polynomial, while (6.171) becomes just the binomial \((1 - \zeta)^{m_{22}}\).

If \(\beta = r_1 - m_{11} + 1 \in \mathbb{Z}_-\) then the polynomial solution of (6.168) is given by:

\[
\tilde{\psi}_2(\zeta) = c^{m_{11} - r_1} F^q_0(-m_{22}; q^{\frac{1}{4} (m_{22} - m_{12} - 2)} \zeta) \times \\
\times 2 F^q_1(m_{22} - r_1, m_{11} - m_{12}; m_{11} - r_1 + 1; q^{\frac{1}{4} (r_1 + \kappa)} \zeta)
\]  

(6.173)

In order to relate (6.169) and (6.173) it is enough to replace in (6.169)

\[
2 F^q_1(m_{22} - m_{11}, r_1 - m_{12}; r_1 - m_{11} + 1; q^{\frac{1}{4} (r_1 + \kappa)} \zeta) \mapsto \\
\mapsto \frac{1}{\Gamma_q(r_1 - m_{11} + 1)} F^q_1(m_{22} - m_{11}, r_1 - m_{12}; r_1 - m_{11} + 1; q^{\frac{1}{4} (r_1 + \kappa)} \zeta)
\]  

(6.174)

Then this expression is valid for arbitrary \(r_1 - m_{11} + 1\), and up to some multiplicative constant is equal to (6.173) when \(r_1 - m_{11} + 1 \in \mathbb{Z}_-\). Thus, finally we shall write the polynomial solution of (6.168) as:

\[
\tilde{\psi}(\zeta) = \frac{1}{\Gamma_q(r_1 - m_{11} + 1)} F^q_0(-m_{22}; q^{\frac{1}{4} (m_{22} - m_{12} - 2)} \zeta) \times \\
\times 2 F^q_1(m_{22} - m_{11}, r_1 - m_{12}; r_1 - m_{11} + 1; q^{\frac{1}{4} (r_1 + \kappa)} \zeta)
\]  

(6.175)

For the lowest-weight state \((m_{12} = r_1, m_{11} = m_{22} = 0)\) we get:

\[
\tilde{\psi}_{\text{hws}} = \frac{1}{[r_1]_q!}
\]  

(6.176)

For the highest-weight state \((m_{12} = m_{11} = r, m_{22} = r_1)\) we get:

\[
\tilde{\psi}_{\text{hws}}(\zeta) = q^{\frac{1}{4} (r^2 - r^1)} [r - r_1]_q! \zeta^{r - r_1} F^q_0(-r_1; q^{\frac{1}{4} (r_1 - r - 2)} \zeta)\]

\[
\psi_{\text{hws}}(x, y, z) = q^{\frac{1}{4} (r^2 - r^1)} [r - r_1]_q! x^{r_1} y^{r_1} z^{r - r_1} \times \\
\times 1 F^q_0(-r_1; q^{\frac{1}{4} (r_1 - r - 2)} \frac{z}{xy})
\]  

(6.177)

since from \(2 F^q_1\) survives only the term \(\zeta^{r - r_1} = \zeta^r\).
We shall write down the relation between the expressions (6.157) and (6.167) (with (6.175)) as:

\[ \phi_{(m)} = \mathcal{M}_{(m)} \psi_{(m)} \]  

(6.178)

For \( r_1 - m_{11} + 1 \not\in \mathbb{Z}_- \) we have \( \mathcal{M}_{(m)} = \mathcal{M}^+_{(m)} \) which we find by comparing

\[ \psi_{(m)} \big|_{z=0} = \frac{1}{\Gamma_q(r_1 - m_{11} + 1)} \chi^{m_{11}} y^{m_{12} + m_{22} - r_1} \]

with (6.158). Note that (6.159) is a partial case of (6.178). When \( r_1 - m_{11} + 1 \in \mathbb{Z}_- \) (i.e., \( m_{11} - r_1 \in \mathbb{N} \)), one has first to differentiate \( m_{11} - r_1 \) times w.r.t. \( z \) both \( \phi_{(m)} \) and \( \psi_{(m)} \) and then to set \( z = 0 \). In particular, for the highest-weight state we compare (6.131) and (6.177), since then \( \phi_{\text{hws}} = \nu_{0r_0} = \Gamma_3(E_{13})^j 1 \). Rewriting (6.131) as:

\[ \phi_{\text{hws}} = (-1)^{r_1} q^{\frac{1}{2}(r_1 - 1)} \frac{[r_1]_q!}{[r - r_1]_q!} \psi_{\text{hws}} \]

we get:

\[ \phi_{\text{hws}} = (-1)^{r_1} q^{\frac{1}{2}(2r_1 - r_2)} \frac{[r_1]_q!}{[r - r_1]_q!} \psi_{\text{hws}} \]  

(6.179)

6.4.3 Explicit Orthogonality of the GWZ Basis

For the orthogonality of the GWZ basis we shall use an adaptation of the so-called Shapovalov form [550]. This is a bilinear \( \mathbb{C} \)-valued form on Verma modules. The Verma module \( V^\Lambda \) of lowest-weight \( \Lambda \in \mathcal{H}^\ast \) is the lowest-weight module such that \( V^\Lambda = U_q(\mathcal{G}^+) \otimes v_0 \), where \( \mathcal{G}^+ \) is the subalgebra of the raising generators \( E_{jk}, j < k \), and \( v_0 \) is the lowest vector such that:

\[ E_{jk} v_0 = 0, \quad j > k, \quad H_k v_0 = \Lambda(H_k) v_0 \]  

(6.180)

The states in a Verma module correspond to the monomials of the Poincaré–Birkhoff–Witt basis of \( U_q(\mathcal{G}^+) \), namely:

\[ u_{\ell kj} \equiv p_{\ell kj} \otimes v_0, \]

\[ p_{\ell kj} \equiv (E_{23})^\ell(E_{13})^j(E_{12})^k, \quad \ell, k, j \in \mathbb{Z}_+ \]  

(6.181)

that is, this basis is one-to-one with the basis \( v_{\ell kj} \) for general \( r_k \). Further, for simplicity we shall omit the sign \( \otimes \); that is, we shall write: \( u_{\ell kj} = p_{\ell kj} v_0 \) or \( u = p v_0 \) for short. We need the involutive antiautomorphism of \( U_q(\mathcal{G}) \) such that:

\[ \omega(H_k) = H_k, \quad \omega(E_{jk}) = E_{kj}, \quad \omega(q) = q^{-1} \]  

(6.182)
Using the above conjugation the Shapovalov form can be defined as follows:

\[
\langle u, u' \rangle = \langle p v_0, p' v_0 \rangle = \langle v_0, \omega(p) p' v_0 \rangle =
\]

\[
= \langle \omega(p') p v_0, v_0 \rangle, \quad p, p' \in U_q(\mathfrak{g}^+), \quad u, u' \in V^A
\] (6.183)

supplemented by the normalization condition \((v_0, v_0) = 1\). More explicitly from (6.183) we have:

\[
\langle u_{\ell kj}, u_{\ell' k' j'} \rangle = \langle p_{\ell kj} v_0, p_{\ell' k' j'} v_0 \rangle =
\]

\[
= \langle v_0, \omega(p_{\ell kj}) p_{\ell' k' j'} v_0 \rangle = \langle \omega(p_{\ell' k' j'}) p_{\ell kj} v_0, v_0 \rangle =
\]

\[
= \langle v_0, (E_{21})^{\ell} (E_{31})^{k' k} (E_{32})^{\ell' j} (E_{13})^{k'} (E_{12})^{j} v_0 \rangle =
\]

\[
= \langle (E_{21})^{\ell'} (E_{31})^{k'} (E_{32})^{\ell' j} (E_{13})^{k} (E_{12})^{j} v_0, v_0 \rangle
\] (6.184)

Note that subspaces with different weights are orthogonal w.r.t. to this form:

\[
\langle u_{\ell kj}, u_{\ell' k' j'} \rangle \sim \delta_{\ell+k, \ell'+k'} \delta_{k+j, k'+j'}
\] (6.185)

To show (6.185) one uses (6.184b) when \(k + \ell > k' + \ell'\) and/or \(k + j > k' + j'\), while (6.184c) is used when \(k + \ell < k' + \ell'\) and/or \(k + j < k' + j'\).

We shall give a realization of the Shapovalov form in our setting in the following way. Using the one-to-one correspondence we replace \(u_{\ell kj}\) by \(v_{\ell kj}\) and the lowest-weight vector \(v_0\) by the lowest-weight vector \(\hat{1}\) of the abstract finite-dimensional irrep and by the function \(1\) in the polynomial realization. Namely, we shall use instead of (6.183) the following bilinear form:

\[
\langle u, u' \rangle_f = \langle p \hat{1}, p' \hat{1} \rangle_f \equiv \langle \Gamma_3(\omega(p)) \Gamma_3(p') 1 \rangle_{x=y=z=0}
\] (6.186)

More explicitly, we have:

\[
\langle u_{\ell kj}, u_{\ell' k' j'} \rangle_f = \langle p_{\ell kj} \hat{1}, p_{\ell' k' j'} \hat{1} \rangle_f =
\]

\[
= \langle \Gamma_3(\omega(p_{\ell kj})) \Gamma_3(p_{\ell' k' j'}) \hat{1} \rangle_{x=y=z=0} = \langle \hat{p}_{\ell kj} v_{\ell' k' j'} \rangle_{x=y=z=0},
\]

\[
\hat{p}_{\ell kj} = \langle \Gamma_3(\omega(p_{\ell kj})) (\Gamma_3(E_{21}))^{\ell} (\Gamma_3(E_{31}))^{k} (\Gamma_3(E_{32}))^{j} \rangle_f
\] (6.187)

Clearly, when \(k + \ell > k' + \ell'\) and/or \(k + j > k' + j'\) we have \(\hat{p}_{\ell kj} v_{\ell' k' j'} = 0\). When \(k + \ell < k' + \ell'\) and \(k + j < k' + j'\) the expression \(\hat{p}_{\ell kj} v_{\ell' k' j'}\) is not zero but a homogeneous polynomial of \(x, y, z\) which vanishes after the substitution \(x = y = z = 0\). Finally, when \(k + \ell = k' + \ell'\) and \(k + j = k' + j'\) the expression \(\hat{p}_{\ell kj} v_{\ell' k' j'}\) is a numerical one coinciding with \(\langle u_{\ell kj}, u_{\ell' k' j'} \rangle\) because of the automorphism.
We can further simplify (6.187) if we set \( x = y = z = 0 \) in \( \tilde{p}_{ekj} \) from the very beginning, namely, we replace \( \tilde{p}_{ekj} \) by:

\[
\tilde{p}_{ekj} = (\tilde{\Gamma}_3(E_{21}))^j (\tilde{\Gamma}_3(E_{31}))^k (\tilde{\Gamma}_3(E_{32}))^\ell
\]

Note that this operation affects only \( \tilde{\Gamma}_3(E_{21}) \) and it is easy to check that

\[
(\psi_{ekj} , \psi'_{ekj})_f \equiv (\tilde{p}_{ekj} \psi_{ekj}^{*})|_{x=y=z=0} (6.189)
\]

Further we note that:

\[
\tilde{p}_{ekj} = q^{\frac{1}{4}((\ell-k)N_x+(2k-j)N_y+j\ell+k(r_1-j))} \Gamma(x) (D_x)^j (D_y)^k (D_z)^\ell (6.190)
\]

We shall use the above to prove the main result in this section:

**Theorem 6.2.** Let \((\mathbf{m})\) and \((\mathbf{m'})\) be two GWZ patterns. Then we have:

\[
(\phi(\mathbf{m}), \phi(\mathbf{m'})) = (p(\mathbf{m}) \tilde{1}, p(\mathbf{m'}) \tilde{1})_f =
\]

\[
= \delta_{m_{11},m'_{11}} \delta_{m_{12},m'_{12}} \delta_{m_{22},m'_{22}} (-1)^{r_1} q^{\frac{1}{4}(m_{11}k-2r_1)} \times
\]

\[
\times \frac{[r]_q! [r-m_{12}]_q! [r-m_{22}+1]_q!}{[m_{12}-m_{22}+1]_q} \mathcal{N}(\mathbf{m}) (6.191)
\]

The proof is given in [244]. The appearance of the constant \( \mathcal{N}(\mathbf{m}) \) in (6.191) is due to the fact that in the derivation \( \phi(\mathbf{m'}) \) was substituted with \( \mathcal{N}(\mathbf{m}) \phi(\mathbf{m'}) \cdot \)

We can use the form (6.184) and (6.187) to define a scalar product if we consider our conjugation \( \omega \) as antilinear. Then we actually restrict to the real form \( U_q(su(3)) \) and \( q \) is restricted to be a phase \(|q| = 1 \) (cf. (6.182)). Then we define the scalar product of the functions \( \phi(\mathbf{m}) = \Gamma_3(p(\mathbf{m}) \tilde{1}) \) or \( \psi(\mathbf{m}) \)

\[
(\phi(\mathbf{m}), \phi(\mathbf{m'}))_p = (p(\mathbf{m}) \tilde{1}, p(\mathbf{m'}) \tilde{1})_f
\]

\[
(\psi(\mathbf{m}), \psi(\mathbf{m'}))_p = \frac{1}{|\mathcal{N}(\mathbf{m})|^2} (p(\mathbf{m}) \tilde{1}, p(\mathbf{m'}) \tilde{1})_f
\]
We note two partial cases:

\[
(\phi_{lws}, \phi_{lws})_p = (r_1 q^l)^4
\]

\[
(\psi_{lws}, \psi_{lws}) = \frac{1}{(r_1 q^l)^2}
\]

\[
(\phi_{hws}, \phi_{hws})_p = [r]_q! [r_1]_q!
\]

\[
(\psi_{hws}, \psi_{hws}) = \frac{[r]_q! ([r - r_1] q^l)^2}{[r_1]_q!}
\]  

(6.193a - 6.193b)

Using this scalar product we can introduce orthonormal GWZ polynomials by:

\[
\hat{\phi}_m = \frac{\psi_m}{|\langle \psi_m, \psi_m \rangle|^{1/2}}
\]  

(6.194)

so that

\[
\left( \hat{\phi}_m, \hat{\phi}_{m'} \right) = \delta_{m,m'}
\]  

(6.195)

In particular, we have:

\[
\hat{\phi}_{lws}(x, y, z) = [r_1]_q!
\]

\[
\hat{\phi}_{hws}(x, y, z) = \frac{1}{[r - r_1]_q!} \left( \frac{[r_1]_q!}{[r]_q!} \right)^{1/2} \psi_{hws}(x, y, z) =
\]

\[
= \left( \frac{[r]_q!}{[r_1]_q!} \right)^{1/2} q^{\frac{1}{4}(r^2 - r_1^2)} x^{r_1} y^{r_1} z^{r - r_1} \times
\]

\[
\times \, _1F_0^q \left( -r_1; \frac{1}{4}(r_1 - r - 2) \frac{z}{xy} \right)
\]  

(6.196a - 6.196b)

### 6.4.4 Normalized GWZ basis

#### 6.4.4.1 Action on the Unnormalized GWZ Bases and Relations between Them

Our aim now is to obtain normalized GWZ basis. To achieve this first we consider the action of the Chevalley generators \(X_j^\pm, j = 1, 2\), of \(U_q(sl(3))\) on the two realizations of the unnormalized GWZ basis introduced in above. After deriving the action we shall use it in order to find the proportionality constant between the two realizations.

First we consider the “operatorial” GWZ basis. We recall formulae (6.156). We rewrite (6.156b) differing by by an overall multiplicative constant:
\[ \tilde{\phi}(m) = (-1)^{r_1-m_{22}} q^{\frac{1}{2}(m_{22}-r_1)(m_{22}-1)} [r]_q [r - m_{22} + 1]_q [r - m_{12}]_q \times \]
\[ \times \sum_{t=0}^{r_{m_{12}}} \sum_{u \in \mathbb{Z}_+} (-1)^t \left( \frac{m_{12} - m_{11} + t}{u} \right)_q \times \]
\[ \times q^{\frac{1}{2}(r_1 - m_{12} - m_{22}) + \frac{u}{2}(u - 2m_{22} - m_{12} + m_{11} + r_1 + t)} [t]_q [m_{22} - t]_q [m_{12} - m_{22} + 1 + t]_q \times \]
\[ \times \left[ \frac{[t + r_1 - m_{22}]_q [m_{12} + m_{22} - m_{11} - u]_q}{[m_{11} - m_{12} - m_{22} + r_1 + u]_q [m_{12} + m_{22} - r_1 - u]_q} \right] \times \]
\[ \times (E_{23})^u (E_{13})^{m_{12} + m_{22} - r_1 - u} (E_{12})^{m_{11} - m_{12} - m_{22} + r_1 + u} \]  
(6.197)

The first result here is the explicit calculation of the action of the Chevalley generators on \( \tilde{\phi}(m) \) which we denote also by the variable numbers of the GWZ pattern:

\[ \tilde{\phi}(m_{12}, m_{11}, m_{22}) \equiv \tilde{\phi}(m) . \]  
(6.198)

We have:

\[ X^+_1 \tilde{\phi}(m) = [m_{12} - m_{11}]_q [m_{11} - m_{22} + 1]_q \tilde{\phi}(m_{12} - 1, m_{11}, m_{22}) \]  
(6.199a)

\[ X^+_1 \tilde{\phi}(m) = \tilde{\phi}(m_{12}, m_{11} - 1, m_{22}) \]  
(6.199b)

\[ X^+_2 \tilde{\phi}(m) = \frac{[r - m_{12}]_q [m_{12} - r_1 + 1]_q [m_{12} + 1]_q \tilde{\phi}(m_{12} + 1, m_{11}, m_{22})}{[m_{12} - m_{22} + 1]_q} + \]
\[ + \frac{[r - m_{22} + 1]_q [r_1 - m_{22}]_q [m_{22} + 1]_q \tilde{\phi}(m_{12}, m_{11}, m_{22} + 1)}{[m_{12} - m_{22} + 1]_q} \]  
(6.199c)

\[ X^+_2 \tilde{\phi}(m) = \frac{[m_{12} - m_{11}]_q \tilde{\phi}(m_{12} - 1, m_{11}, m_{22})}{[m_{12} - m_{22} + 1]_q} + \]
\[ + \frac{[m_{11} - m_{22} + 1]_q \tilde{\phi}(m_{12}, m_{11}, m_{22} - 1)}{[m_{12} - m_{22} + 1]_q} \]  
(6.199d)

In these calculations we use only (6.156a) and abstract algebra: the commutation relations between the Chevalley generators \( H_j, \ X^+_j, j = 1, 2 \), the definitions of \( \tilde{C} \) and \( E_{13} \), and the fact that \( \hat{1} \) is the lowest-weight vector.

Further, we shall use also the realization of \( U_q(sl(3)) \) given in (6.112) to obtain a polynomial in the variables \( x, y, z \) corresponding to the GWZ pattern \( m \). For this we define:

\[ \phi_{(\hat{m})}(x, y, z) \equiv (\Gamma_3(E_{21}))^{m_{12} - m_{11}} (\Gamma_3(\hat{C}))^{r_{1} - m_{12}} (\Gamma_3(E_{32}))^{m_{22} - m_{22}} (\Gamma_3(E_{13}))^{r_{1}} \]  
(6.200)
6.4 Application of the Gelfand–(Weyl)–Zetlin Basis

For this quantity will hold the same formulae (6.199) we derived above – this follows just because $\Gamma_3$ is a representation. On the other hand we should stress that this quantity is a polynomial in the variables $x, y, z$. We give two examples which we shall use below – of h.w.s. (using (6.175)) and l.w.s.:

\[
\phi_{(h.w.s.)} = (\Gamma_3(E_{13}))^r 1 = (-1)^r q^{\frac{1}{2}r_1(3-r_1)} [r]_q ! x^{r_1} y^{r_1} z^{r-r_1} \times
\]
\[
\times 1 F_0^q (-r_1; q^\frac{1}{2}(r_1-r_2) \frac{z}{xy})
\]

\[
(6.201a)
\]

\[
\phi_{(l.w.s.)} = (\Gamma_3(E_{21}))^r (\Gamma_3(\hat{E}))^{r_1} (\Gamma_3(E_{32}))^r_1 (\Gamma_3(E_{13}))^r 1 =
\]
\[
= (-1)^r q^{\frac{1}{2}} \frac{[r]_q ![r+1]_q ![r-r_1]_q ![r_1]_q ^l}{[r_1+1]_q}
\]

\[
(6.201b)
\]

Next we find the action on the realization of the unnormalized GWZ states via hypergeometric functions (cf. (6.167) and (6.175)):

\[
\psi_{(m)}(x, y, z) = \psi_{(m_{12}, m_{11}, m_{22})} = x^{m_{11}} y^{\kappa} \tilde{\psi}(\zeta),
\]
\[
\kappa = m_{12} + m_{22} - r_1, \quad \zeta = \frac{z}{xy}
\]

(6.202)

The second result here is the following action of the generators:

\[
\Gamma_3(X_1^+) \psi_{(m)} = q^{-\frac{1}{2} \kappa} \psi_{(m_{12}, m_{11}+1, m_{22})}
\]

(6.203)

\[
\Gamma_3(X_1^-) \psi_{(m)} = q^{\frac{1}{2} \kappa} [m_{12} - m_{11} + 1]_q [m_{11} - m_{22}]_q \psi_{(m_{12}, m_{11} - 1, m_{22})}
\]

\[
\Gamma_3(X_2^+) \psi_{(m)} = b_1^+ \psi_{(m_{12} + 1, m_{11}, m_{22})} + b_2^+ \psi_{(m_{12}, m_{11}, m_{22} + 1)}
\]

\[
\Gamma_3(X_2^-) \psi_{(m)} = b_1^- \psi_{(m_{12} - 1, m_{11}, m_{22})} + b_2^- \psi_{(m_{12}, m_{11}, m_{22} - 1)}
\]

where:

\[
b_1^+ = q^{-\frac{1}{2} m_{11}} \frac{[r - m_{12}]_q [m_{12} - m_{11} + 1]_q}{[m_{12} - m_{22} + 1]_q}
\]

(6.204)

\[
b_2^+ = q^{-\frac{1}{2} m_{11}} \frac{[r - m_{22} + 1]_q [m_{11} - m_{22}]_q}{[m_{12} - m_{22} + 1]_q}
\]

\[
b_1^- = q^{\frac{1}{2} m_{11}} \frac{[m_{12} - r_1]_q [m_{12} + 1]_q}{[m_{12} - m_{22} + 1]_q}
\]

\[
b_2^- = q^{\frac{1}{2} m_{11}} \frac{[r_1 - m_{22} + 1]_q [m_{22}]_q}{[m_{12} - m_{22} + 1]_q}
\]

We derive this action using only the explicit realization of $\Gamma_3(\cdot)$ given in (6.112) and using some relations between q-hypergeometric functions which are given in Appendix B of [230].
Now we use the action on the two unnormalized polynomial realizations of the
GWZ states $\phi_{\langle m \rangle}$ and $\psi_{\langle m \rangle}$ in order to derive the proportionality constant between them.
We set:

$$\phi_{\langle m \rangle} = N_{\langle m \rangle} \psi_{\langle m \rangle} \quad (6.205)$$

Before proceeding we note the two cases in which we already know this constant:

$$N_{\langle h,w,s \rangle} = (1)^{n_1} q^{n^2(2n_1-2)} \frac{[r]_q!}{[r-r_1]_q!} \quad (6.206a)$$

$$N_{\langle l,w,s \rangle} = (1)^{n_1} q^{n^2} \frac{[r]_q! [r + 1]_q! [r-r_1]_q! ([r_1]_q)!^2}{[r_1 + 1]_q} \quad (6.206b)$$

where (6.206a) is obtained by using (6.177), while (6.206b) is obtained by using (6.176).
The idea is as follows (on the example of $X_1$): On one hand we have:

$$\Gamma_3(X_1) \phi_{\langle m \rangle} = \phi_{\langle m_{12},m_{11}-1,m_{22} \rangle} = N_{\langle m_{12},m_{11}-1,m_{22} \rangle} \psi_{\langle m_{12},m_{11}-1,m_{22} \rangle} \quad (6.207)$$

On the other hand (using (6.203b)):

$$\Gamma_3(X_1) \phi_{\langle m \rangle} = N_{\langle m \rangle} X_1 \psi_{\langle m \rangle} = \quad (6.208)$$

$$= N_{\langle m \rangle} q^{\frac{n}{2}} \frac{1}{[m_{12} - m_{11} + 1]_q} [m_{11} - m_{22}]_q \psi_{\langle m_{12},m_{11}-1,m_{22} \rangle}$$

If $m_{11} > m_{22}$ comparing (6.207) and (6.208) we get a relation expressing $N_{\langle m \rangle}$ in terms of $N_{\langle m_{12},m_{11}-1,m_{22} \rangle}$. Besides the above there are five more relations, expressing $N_{\langle m \rangle}$ through $N_{\langle m' \rangle}$ with $m_{11} \rightarrow m_{11} + 1$, $m_{12} \rightarrow m_{12} \pm 1$, $m_{22} \rightarrow m_{22} \pm 1$. It is enough to use the three relations which decrease $m_{ij}$, each relation affecting only a single $m_{ij}$:

$$N_{\langle m \rangle} = q^{n} \frac{1}{[m_{12} - m_{11} + 1]_q} [m_{11} - m_{22}]_q \quad (6.209a)$$

$$N_{\langle m \rangle} = q^{\frac{1}{2}} m_{11} \frac{[m_{12} - m_{11}]_q}{[m_{12} - r_1]_q [m_{12} + 1]_q} \quad (6.209b)$$

$$N_{\langle m \rangle} = q^{\frac{1}{2}} m_{11} \frac{[m_{11} - m_{22} + 1]_q}{[r_1 - m_{22} + 1]_q [m_{22}]_q} \quad (6.209c)$$

(For the three relations which increase $m_{ij}$ we refer to [230].)

Now, we use relation (6.209c) until on the RHS we get $N_{\langle m_{12},m_{11},0 \rangle}$, then we use (6.209a) until on the RHS we get $N_{\langle m_{12},0,0 \rangle}$, finally we use (6.209b) until on the RHS we get $N_{\langle 1,0,0 \rangle} = N_{\langle l,w,s \rangle}$; that is, we get:

$$N_{\langle m \rangle} = q^{\frac{1}{2}} m_{11} \frac{[r_1 + 1]_q [r_1 - m_{22}]_q ![m_{12} - m_{11}]_q!}{[r_1]_q ![m_{11} - m_{22}]_q ![m_{12} + 1]_q ![m_{22}]_q} \quad (6.210)$$
and using (6.206b) we finally obtain:

\[
\mathcal{N}(\tilde{m}) = (-1)^{n} \frac{q^{\frac{1}{4}(2n_{1} + m_{12} + m_{22} - r_{1})}}{[r_{1}]_{q} [r + 1]_{q} [r - r_{1}]_{q} [r_{1}]_{q}} \times
\frac{[r_{1} - m_{22}]_{q}! [m_{12} - m_{11}]_{q}!}{[m_{12} - r_{1}]_{q}! [m_{11} - m_{22}]_{q}! [m_{12} + 1]_{q}! [m_{22}]_{q}!}
\]

(6.211)

From (6.211) follows also:

\[
\mathcal{N}(\tilde{h}.w.s.) = q^{-\frac{1}{4} r^{2}} \frac{[r_{1} + 1]_{q}}{([r_{1}]_{q} [r - r_{1}]_{q})^{2} [r + 1]_{q}!}
\]

(6.212)

which is consistent with (6.206).

### 6.4.5 Scalar Product and Normalized GWZ States

By (6.192) we have defined a scalar product in terms of the constant \( \mathcal{N}(\tilde{m}) \). Now that we know this constant we can fix the scalar product completely, that is, we have:

\[
\left( \phi_{(m)}, \phi_{(m')} \right)_{p} = \delta_{m_{11}, m'_{11}} \delta_{m_{12}, m'_{12}} \delta_{m_{22}, m'_{22}} \frac{[r_{1} + 1]_{q}}{[r_{1}]_{q} [r - r_{1}]_{q} [r_{1}]_{q}} [r - m_{12}]_{q}! [r - m_{22} + 1]_{q}! \times
\frac{[m_{12} - r_{1}]_{q}! [m_{11} - m_{22}]_{q}! [m_{12} + 1]_{q}!}{[m_{22}]_{q}! [m_{12} - m_{22} + 1]_{q}!}
\]

\[
\times \left( \phi_{(m)}, \phi_{(m')} \right)_{p} = \delta_{m_{11}, m'_{11}} \delta_{m_{12}, m'_{12}} \delta_{m_{22}, m'_{22}} \times
\frac{[r + 1]_{q}! [r - r_{1}]_{q}! [r_{1}]_{q}! [m_{12} - m_{22} + 1]_{q}!}{[r_{1} - m_{22}]_{q}! [m_{12} - m_{11}]_{q}!}
\]

(6.213)

or in terms of \( \psi_{(m)} \):

\[
\left( \psi_{(m)}, \psi_{(m')} \right) = \frac{1}{|\mathcal{N}(\tilde{m})|^{2}} \left( \phi_{(m)}, \phi_{(m')} \right)_{p} = \delta_{m_{11}, m'_{11}} \delta_{m_{12}, m'_{12}} \delta_{m_{22}, m'_{22}} \times
\frac{[r + 1]_{q}! [r - r_{1}]_{q}! [r_{1}]_{q}! [m_{12} - m_{22} + 1]_{q}!}{[r_{1} - m_{22}]_{q}! [m_{12} - m_{11}]_{q}!}
\]

(6.214)

Further, we complete our program of finding explicit polynomial realizations of the normalized GWZ states. We set:

\[
\hat{\phi}_{(m)} = \frac{\phi_{(m)}}{|\mathcal{N}_{\phi}(m)|^{2}} = \frac{1}{N_{\phi}(m)} \left( \Gamma_{3}(E_{21}) \right)^{m_{12} - m_{11}} \times
\left( \Gamma_{3}^{(C)} \right)^{r - m_{12}} \left( \Gamma_{3}(E_{32}) \right)^{r - m_{22}} \left( \Gamma_{3}(E_{13}) \right)^{r} \times
\]

(6.215)
\[ N_q(\hat{m}) = \sqrt{\langle \hat{\phi}(\hat{m}), \hat{\phi}(\hat{m}) \rangle_p} = \]
\[ = [r]_q! \sqrt{\frac{[r + 1]_q! [r - r_1]_q! [r - 1]_q! [m_{12}]_q! [m_{11} - m_{22}]_q! [m_{12} + 1]_q!}{[m_{12} - r_1]_q! [m_{11} - m_{22}]_q! [m_{12} + 1]_q!}} \times \]
\[ \times \sqrt{\frac{[r - m_{22} + 1]_q! [m_{12} - m_{11}]_q! [r_1 - m_{22}]_q!}{[m_{22}]_q! [m_{12} - m_{22} + 1]_q}} \]

Analogously, we set:
\[ \tilde{\psi}(\hat{m}) = \frac{\psi(\hat{m})}{N_q(\hat{m})} \]
\[ = \frac{1}{N_q(\hat{m})} x^{m_{11}} y^{m_{12} + m_{22} - r_1} \sum_{r=0}^{\infty} \frac{q^{\binom{m_{12} - m_{22} - 2}{2}} z}{\Gamma_q(r_1 - m_{11} + 1)} \times \]
\[ \times \sqrt{\frac{[r]_q! [r - m_{11}]_q! [r - m_{22} + 1]_q! [m_{12} - r_1]_q!}{[r + 1]_q! [r - r_1]_q! [r_1]_q!}} \]
\[ \times \sqrt{\frac{[m_{11} - m_{22}]_q! [m_{12} + 1]_q! [m_{22}]_q!}{[m_{12} - m_{22} + 1]_q [r_1 - m_{22}]_q! [m_{12} - m_{11}]_q!}} \]

Finally, we calculate the action of the Chevalley generators on our normalized GWZ states. We get:
\[ X_1^+ \hat{\phi}(\hat{m}) = \sqrt{[m_{12} - m_{11}]_q [m_{11} - m_{22} + 1]_q} \hat{\phi}(m_{12} - m_{11}, m_{22}) \]
\[ X_1^- \hat{\phi}(\hat{m}) = \sqrt{[m_{12} - m_{11} + 1]_q [m_{11} - m_{22}]_q} \hat{\phi}(m_{12}, m_{11} - m_{22}) \]
\[ X_2^+ \hat{\phi}(\hat{m}) = a_1^+ \hat{\phi}(m_{12} + 1, m_{11}, m_{22}) + a_2^+ \hat{\phi}(m_{12}, m_{11}, m_{22} + 1) \]
\[ X_2^- \hat{\phi}(\hat{m}) = a_1^- \hat{\phi}(m_{12} - 1, m_{11}, m_{22}) + a_2^- \hat{\phi}(m_{12}, m_{11}, m_{22} - 1) \]

\[ H_1 \hat{\phi}(\hat{m}) = (2m_{11} - m_{12} - m_{22}) \hat{\phi}(\hat{m}) \]
\[ H_2 \hat{\phi}(\hat{m}) = (2(m_{12} + m_{22}) - m_{11} - r - r_1) \hat{\phi}(\hat{m}) \]

\[ a_1^+ = \sqrt{\frac{[r - m_{12}]_q [m_{12} - r_1 + 1]_q [m_{12} + 2]_q [m_{12} - m_{11} + 1]_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22} + 2]_q}} \]
6.4 Application of the Gelfand–(Weyl)–Zetlin Basis

\[ a_2^+ = \sqrt{\frac{[r - m_{22} + 1]_q [m_{22} + 1]_q (m_{11} - m_{22})_q}{[m_{12} - m_{22}]_q [m_{12} - m_{22} + 1]_q}} \] (6.221)

\[ a_1^- = \sqrt{\frac{[r - m_{12} + 1]_q [m_{12} - r_1]_q [m_{12} + 1]_q (m_{11} - m_{12})_q}{[m_{12} - m_{22}]_q [m_{12} - m_{22} + 1]_q}} \]

\[ a_2^- = \sqrt{\frac{[r - m_{22} + 2]_q [m_{22} + 1]_q (m_{11} - m_{22})_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22} + 2]_q}} \]

Of course, the action of the Cartan generators (6.220) is the same for normalized and for unnormalized GWZ states. We note now that in (6.219) we have recovered the standard transformation rules which until now were written without derivation – for \( q = 1 \) in [65, 315], and for \( q \neq 1 \) in [143]. In fact, since the only restriction on the transformation rules were the commutation relations of \( U_q(sl(3)) \) later it was shown [47], that this restriction was very weak and one can generalize the above formulae by replacing the square roots, that is, the powers 1/2, in the matrix elements in (6.219) by the powers 0 and 1. There is no such freedom in our case. The only freedom we have is in phase factors, like the one relating \( \hat{\psi}(m) \) and \( \hat{\psi}(m) \). Indeed, the transformation rules for \( \hat{\psi}(m) \) are the same as (6.219) except for the \( q^{-\frac{1}{2}} \) factors which are the same as in (6.203) and (6.204).

6.4.6 Summation Formulae

In this section we derive summation formulae using formula (6.211) for the constant \( \mathcal{N}(m) \) and another independent expression for \( \mathcal{N}(m) \). To find the latter we use formulae (6.130),(6.132), then we recall the polynomial \( \phi_{(m)} \) at \( z = 0 \) using (6.158) Next we note the value of \( \psi_{(m)} \) at \( z = 0 \) (using (6.170)):

\[ \left( \psi_{(m)} \right)_{z=0} = \frac{x^{m_{11}y^{m_{12}+m_{22}-r_1}}}{\Gamma_q(r_1 - m_{11} + 1)} \] (6.222)

Now we compare (6.205), (6.158a), and (6.222), and conclude that:

\[ \mathcal{N}(m) = \mathcal{N}^+_m, \quad r_1 \geq m_{11} \] (6.223)

From the latter using (6.158b) and (6.211) we get the following summation formula:

\[ \sum_{t=0}^{r-m_{12}} \sum_{u \in \mathbb{Z}_+} (-1)^{t+u} \binom{m_{12} - m_{11} + t}{u}_q \times \]

\[ q^{\frac{t}{2}r \left( r_1 - m_{12} - m_{22} \right) + \frac{u}{2} \left( t + m_{12} - r - 1 \right)} \binom{r_1 - m_{22} + t}{t}_q \times \]

\[ \frac{[t]_q [m_{22} - t]_q [m_{12} - m_{22} + 1]_q}{[t]_q [m_{22} - t]_q [m_{12} - m_{22} + 1 + t]_q} \times \]
For the better comparison with the literature on q-summation formulae we rewrite our

\[ \frac{[r + m_{11} - m_{12} - m_{22} + u]_q! [m_{12} + m_{22} - m_{11} - u]_q!}{[r + m_{11} - m_{12} - m_{22} + u]_q! [m_{12} + m_{22} - r - u]_q!} = \]

\[ = \frac{[r + m_{11} - m_{12} - m_{22}]_q! [r_1 - m_{22}]_q! [m_{12} + m_{22} - m_{11}]_q!}{[m_{12} + 1 - m_{22}]_q! [m_{12} + m_{22} - r_1]_q! [m_{22}]_q!} \times \]

\[ \times \sum_{t=0}^{r-m_{12}} q^t (r_1 - m_{12} - m_{22}) \frac{(-m_{22})_t^q (r_1 - m_{22} + 1)_t^q}{[t]_q! (m_{12} - m_{22} + 2)_t^q} \times \]

\[ \times \frac{3F_2^q (r + m_{11} - m_{12} - m_{22} + 1, r_1 - m_{12} - m_{22}, m_{11} - m_{12} - t;}{r_1 + m_{11} - m_{12} - m_{22} + 1, m_{11} - m_{12} - m_{22}; q^{1/2(r + m_{12} - r - 1)}} = \]

\[ = (-1)^{m_{12} + r_1} q^r \frac{1}{[r_1 + (m_{11} + r - m_{12}) (r_1 - m_{12} - m_{22}) + m_{12} (m_{22} - 1)]_q} \times \]

\[ \times \frac{[r + 1]_q! [r_1]_q! [r + m_{11} - m_{12} - m_{22}]_q!}{[r + m_{12}]_q! [r_1]_q! [m_{12} - r_1]_q!} \times \]

\[ \times \frac{[r_1 - m_{22}]_q! [m_{12} - m_{11}]_q!}{[m_{12} + 1]_q! [m_{11} - m_{22}]_q! [m_{22}]_q!} \] (6.224)

In order to show better the properties of the above formula, we will rewrite it in representation independent parameters:

\[ b_1 = r_1 - m_{12} + m_{11} - m_{22} + 1, \quad b_2 = m_{11} - m_{12} - m_{22} \]

\[ m_1 = m_{12} - r_1, \quad m_2 = m_{22}, \quad N = r - m_{12} \] (6.225)

Now we rewrite (6.224) in the new variables using also the q-Pochhammer symbol:

\[ \frac{1}{\Gamma_q(b_1)} \sum_{t=0}^{N} q^{-\frac{1}{2}(m_1 + m_2)} \frac{(-m_{22})_t^q (b_1 - b_2 - m_{22})_t^q}{(b_1 - b_2 + m_1 - m_2 + 1)_t^q (1)_t^q} \times \]

\[ \times 3F_2^q (-m_1 + m_2), b_1 + m_1 + N, b_2 + m_2 - t; b_1, b_2; q^{1/2(t - N - 1)} = \]

\[ = (-1)^{m_1 + m_2} q^{-\frac{1}{2} \left( m_1 (m_1 + b_1 + N) + m_2 (m_2 + b_2 + N) + m_1 m_2 \right)} \times \]

\[ \times \frac{(b_1 - b_2 + m_1 + 1)_N^q (m_1 + 1)_N^q}{(b_1 - b_2 + m_1 - m_2 + 1)_N^q (1)_N^q [b_1 + m_1 - 1]_q! (b_2)_m^q} \] (6.226)

For the better comparison with the literature on q-summation formulae we rewrite our formula using notation from Gasper–Rahman [311]. We shall use the definition (1.2.15) for the q-shifted factorial:
\[ (a; q)_n = \begin{cases} 
1, & n = 0 \\
(1 - a)(1 - aq) \ldots (1 - aq^{n-1}), & n = 1, 2, \ldots 
\end{cases} \quad (6.227) \]

and (1.2.22) for the basic hypergeometric series:

\[
\sum_{n=0}^{\infty} \frac{(a_1; q)_n \ldots (a_r; q)_n}{(b_1; q)_n \ldots (b_s; q)_n} \left[ (-1)^n q^{n(n-1)/2} \right] z^n = \fn{r}{\phi}{s}(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \ldots (a_r; q)_n}{(b_1; q)_n \ldots (b_s; q)_n} \left[ (-1)^n q^{n(n-1)/2} \right] z^n
\quad (6.228)\]

For completeness we mention also the relation between our \(q\)-Pochhammer symbol and the notation of [311]:

\[
(a)_n^q = (-1)^{-n} q^{(n-1)n/2} q^{-na/2} (a; q)_n, \quad \lambda \equiv q^{1/2} - q^{-1/2} \quad (6.229)
\]

Now our summation formula (6.224) or (6.226) can be rewritten as (when \(b_1 > 0\)):

\[
\sum_{t=0}^{N} q^t \frac{(q^{-m_2}_t; q)_t (q^{b_1-b_2-m_2}_t; q)_t}{(q^{b_1-b_2+m_1-m_2+1}; q)_t (q; q)_t} \times \\frac{3\phi_2(q^{(m_1+m_2)_t}, q^{b_1+m_1+N}, q^{b_2-m_2-t}; q^{b_1}; q^{b_2}; q; q^{-N})}{(q^{b_1-b_2+m_1+1}; q)_N (q^{m_1+1}; q)_N (q; q)_m (q^{b_1}; q^{b_2}; q; q)_m} = (-1)^{m_1+m_2} q^{-1/2 (m_1+m_2)(m_1+m_2+1+2N)} \times \frac{(q^{b_1-b_2+m_1+1}; q)_N (q^{m_1+1}; q)_N (q; q)_m (q^{b_1}; q^{b_2}; q; q)_m}{(q^{b_1-b_2+m_1-m_2+1}; q)_N (q; q)_m (q^{b_1}; q^{b_2}; q; q)_m} \quad (6.230)
\]

This new summation formula seems unknown also for the classical case \(q = 1\). Partial cases can be found in the literature. For instance, the case \(N = 0\); that is, \(m_{12} = r\), reduces to a \(q\)-Karlsson–Minton formula (cf. (1.9.8) of [311]):

\[
3\phi_2(q^{-(m_1+m_2)}, q^{b_1+m_1}, q^{b_2+m_2}; q^{b_1}, q^{b_2}; q; 1) = (-1)^{m_1+m_2} q^{-1/2 (m_1+m_2)(m_1+m_2+1)} \frac{(q; q)^{m_1+m_2}}{(q^{b_1}; q)^{m_1} (q^{b_2}; q)^{m_2}} \quad (6.231)
\]

It corresponds to a 0-balanced \(3\phi_2\) [311].

### 6.4.7 Weight Pyramid of the SU(3) UIRs

#### 6.4.7.1 Geometrical Construction of the Weight Pyramid

First let us recall some well-known facts about the UIRs of \(SU(3)\) which hold also for the (anti)holomorphic representations of \(SL(3)\), also for the Lie algebras and quantum groups. Fix such a representation, that is, the non-negative integers \(r_1, r_2\), so that we have a representation of dimension \(d_{r_1+1,r_2+1}\). It is customary to depict
the weight lattice of every such irrep in the \((I_z, Y)\) plane. We recall that the notation comes from the popular application in which \(I_z\) is the third component of isospin, and \(Y\) is the hypercharge. The points of the weight diagram form a hexagon, the sides of the hexagon containing alternatively \(r_1 + 1\), \(r_2 + 1\) points. (Thus, the hexagon degenerates into a triangle if \(r_1r_2 = 0\).) Each point of the weight diagram represents all states with the same weight and differing only by the values of isospin \(I\), for which the corresponding \(I_z\) is admissible. It is also customary to connect all points with the same multiplicity. Then the resulting figure consists of nested hexagons if \(r_1r_2 \neq 0\), the most outward one containing the states with multiplicity one, the next inwards – the states with multiplicity two, and so on. When \(r_1r_2 = 0\) the resulting figure consists of nested triangles; moreover, each weight has multiplicity one and that is why such representations are called flat representations.

Now for our purposes we shall replace this customary weight diagram with a hexagonal pyramid (when \(r_1r_2 \neq 0\)) in the following way. We consider now a three-dimensional picture adding also the direction perpendicular to the \((I_z, Y)\) plane. The points in that plane have coordinates, say \((i_z, y, 0)\). Next we replace each point of the weight lattice of multiplicity \(m\) and coordinate \((i_z, y, 0)\) by \(m\) equally spaced points in direction perpendicular to the \((I_z, Y)\) plane which points have coordinates: \((i_z, y, k)\), \(k = 0, 1, \ldots, m - 1\). We consider now each point of the so formed pyramid as one state; that is, each point has also fixed value of isospin \(I\) and there is no multiplicity. From the algebraic formulae given in next section we shall see that for fixed \((I_z, Y)\) the value of isospin \(I\) diminishes as \(k\) increases.

Thus, we obtain a pyramid of height \(r_0 \equiv \min(r_1, r_2)\).

Consider now the states with coordinates \((i_z, y, k)\) for a fixed \(k\). We shall say that these states form a layer. We note now that by construction each such layer is actually a weight diagram in the \(I_z\) and \(Y\) axis and has the form of a hexagon. Moreover, this hexagon has exactly the form of a standard \(SU(3)\) weight diagram – the difference is that we put only one GWZ state at each site. Of course, it is important how we distribute the states with the same weight and this is what we explain next.

Let us agree, in order to save space, to omit the first row of the standard \(SU(3)\) GWZ pattern \((m)\) since we shall work with fixed representation parameters \(r_1, r_2\). Namely, we set:

\[
\begin{bmatrix}
m_{12} & m_{22} \\
m_{11}
\end{bmatrix} \equiv \begin{pmatrix}
r & r_1 & 0 \\
m_{12} & m_{22} & m_{11}
\end{pmatrix}
\]

(6.232)

We place the GWZ states on our pyramid in the following manner. The bottom, or zeroth, layer contains both the lowest-weight state and the highest-weight state of our representation. Overall it contains the following states:
The lowest-weight state \([ r_1 0 0 ]\) is in the bottom-left corner, the highest-weight state \([ r 0 0 ]\) is in the top-right corner, of this hexagon. (Of course, these states and the others on the edges of this initial hexagon are with no multiplicity, so their placement is more or less standard.) Analogously, we put the following states on the \(k\)-th layer, \(k \leq r_1, r_2\):

\[
\begin{bmatrix}
  r-k & r_1 \\
  r_1 & r_1+1 \\
  \vdots & \vdots \\
  r_1 & r_1+1 \\
  r_1+1 & 0 \\
  r_1+1 & 0 \\
  \vdots & \vdots \\
  r_1 & 0 \\
  \end{bmatrix}
\]

\[
\begin{bmatrix}
  r-k & r_1 \\
  r_1 & r_1+1 \\
  \vdots & \vdots \\
  r_1 & r_1+1 \\
  r_0 & 0 \\
  r_0 & 0 \\
  \vdots & \vdots \\
  r_0 & 0 \\
  \end{bmatrix}
\]

\[
(6.233)
\]

(Note that \(k = 0\) represents the bottom layer.) Clearly, there are \(r_1 - k + 1\) states on the bottom row of the above hexagon, \(r_1 - k + 2\) states on the next row, and so on, and \(r - 2k + 1\) states on the middle (longest) row, then the number of states decreases by one, the top row having \(r - k - r_1 + 1 = r_2 - k + 1\) states. If we sum these we obtain that the number of states in the \(k\)-th layer presented in (6.234) is:

\[
N^k_{r_1, r_2} = \frac{1}{2} (r + 1)(r + 2) + r_1 r_2 + 3k^2 - 3k(r + 1)
\]

\[
(6.235)
\]

From this it is easy to see that the number of states on the first \(k\) layers is:

\[
\sum_{s=0}^{k-1} N^s_{r_1, r_2} = \frac{k}{2} \left( r^2 + 6r + 2r_1 r_2 + 2k^2 - 3k(r + 2) \right)
\]

\[
(6.236)
\]
We make now the observation that the latter number is equal to the difference of two $SU(3)$ dimensions:

$$\sum_{s=0}^{k-1} N_{r_1,r_2}^s = d_{r_1+1,r_2+1} - d_{r_1+1-k,r_2+1-k}$$

(6.237)

that is, the dimension of the irrep we are considering minus the dimension of an irrep with each representation parameter $r_i$ decreased by $k$. This seems natural since the latter representation has a weight pyramid with bottom layer the $(k+1)$-th layer of our pyramid.

### 6.4.7.2 Algebraic Description of the Weight Pyramid

Now we explain the placement of the GWZ states on our pyramid. This is related to a procedure to obtain all GWZ states starting from the lowest weight state. (A similar procedure starting from the highest-weight state was used in the previous section.) To derive the necessary for the procedure relations between the GWZ states, up to normalization constants, it is enough to use only the fact that the GWZ states are eigenvectors of the operators $\hat{I}_z$, $\hat{Y}$, $\hat{I}^2$. Note that $\hat{I}^2$ is the Casimir of the $U_q(sl(2))$ quantum subgroup generated by $X_1^\pm$, $H_1$. We recall the relation of these eigenvalues to the parameters of the GWZ pattern:

$$\begin{bmatrix} m_{12} & m_{22} \\ m_{11} \end{bmatrix} = \begin{bmatrix} I + \frac{1}{2}Y + \frac{1}{3}(r + r_1) & -I + \frac{1}{2}Y + \frac{1}{3}(r + r_1) \\ I_z + \frac{1}{2}Y + \frac{1}{3}(r + r_1) \end{bmatrix}$$

(6.238)

with $I_z$, $Y$, $I$ denoting the eigenvalues of the corresponding operators.

Before giving the explicit formulae we mention some general facts: The states on a fixed row of a fixed layer (6.234) are states with the same value of $Y$ and $I$, while $I_z$ varies between $-I$ and $I$. On a fixed layer the value of $Y$ increases by 1 from the bottom to the top row. The states which have the same weight and differ only by the value of $I$ are one above the other in the pyramid, the value $I$ decreasing from the bottom up.

First, we describe the states on a fixed layer (hexagon), say, the $k$-th one.

Starting from the state in low-left corner of the hexagon, that is, $r_1 \ k$, we first obtain the states on the south-west edge of the hexagon:

$$\begin{bmatrix} r_1 \ k \end{bmatrix} = \mathcal{A}_2(s,k) \begin{bmatrix} r_1 + s \ k \end{bmatrix}, s = 0, 1, \ldots r_2 - k,$$

$$\mathcal{A}_2(s,k) = \left( \frac{[s]_q! [r_1 + 1 + s]_q! [r_2]_q! [r_1 + 1 - k]_q!}{[r_1 + 1]_q! [r_2 - s]_q! [r_1 + 1 - k + s]_q!} \right)^{1/2}$$

(6.239)

Now we prove the following lemma which is our main technical tool for the procedure.
Lemma: Let $\psi$ be an eigenstate of $\hat{I}^2$, $\hat{I}_z$ and $\eta$ with eigenvalues $\mu(\mu + 1)$, $-\mu$ and $\kappa$, respectively, and let

$$\psi^+ = \mathcal{C}\psi$$

where $\mathcal{C}$ is the following operator:

$$\mathcal{C} \equiv X_2^+ [H_1]_q + X_2^+ X_1^+ q^{{H_1}/2} = X_2^+ X_1^+ [H_1]_q + X_2^+ X_1^+ [1 - H_1]_q$$  \hspace{1cm} (6.240)

Then either $\psi^+ = 0$ or $\psi^+$ is an eigenstate of $\hat{I}^2$, $\hat{I}_z$ and $\eta$ with eigenvalues $(\mu - 1/2)(\mu + 1/2)$, $-\mu + 1/2$ and $\kappa + 1$, respectively. In terms of GWZ pattern: if $\psi \leftrightarrow \left[ \begin{array}{c} m_1 \\\ k \end{array} \right]$ then $\psi^+ \leftrightarrow \left[ \begin{array}{c} m_1 + 1 \\\ k + 1 \end{array} \right]$ unless $k = r_1$.

The proof is given in [245].

Using the above lemma we obtain the states on the north-west edge of the hexagon:

$$\mathcal{C}^t(X_2^+)_{r_2-k} \left[ \begin{array}{c} r_1 \\\ k \end{array} \right] = \mathcal{N}_2(r_2-k,k) \mathcal{C}^t \left[ \begin{array}{c} r-k \\\ k \end{array} \right] = \mathcal{N}_2(r_2-k,k) \mathcal{N}_3(t) \left[ \begin{array}{c} r-k + t \\\ k + t \end{array} \right], \quad t = 0, 1, \ldots r_1 - k,$$

$$\mathcal{N}_3(t) = \left( \frac{[r-k+1]_q! [r-2k+1]_q! [r_1-k]_q! [k+t]_q!}{[r-k+1+t]_q! [r-2k+1+t]_q! [r_1-k-t]_q! [k]_q!} \right)^{1/2}$$

Now all other states of the $k$-th layer are obtained by the action of the operator $X_1^+$ to the states on the edges (6.239),(6.241):

$$\mathcal{(X_1^+)^u}(X_2^+)_{r_2-k} \left[ \begin{array}{c} r_1 \\\ k \end{array} \right] = \mathcal{N}_2(s,k)(X_1^+)^u \left[ \begin{array}{c} r_1 + s \\\ k \end{array} \right] = \mathcal{N}_1(u,s,k) \mathcal{N}_2(s,k) \left[ \begin{array}{c} r_1 + s + k \\\ k + u \end{array} \right], \quad s = 0, 1, \ldots r_2 - k, \quad u = 0, 1, \ldots r_1 - k + s$$

$$\mathcal{N}_1(u,s,k) = \left( \frac{[r_1 + s - k]_q! [u]_q!}{[r_1 + s - k - u]_q!} \right)^{1/2}$$

$$\mathcal{(X_1^+)^u}\mathcal{C}^t(X_2^+)_{r_2-k} \left[ \begin{array}{c} r_1 \\\ k \end{array} \right] = \mathcal{N}_2(r_2-k,k) \mathcal{N}_3(t)(X_1^+)^u \left[ \begin{array}{c} r-k + t \\\ k + t \end{array} \right] = \mathcal{N}_2(r_2-k,k) \mathcal{N}_3(t) \left[ \begin{array}{c} r-k + t \\\ k + t \end{array} \right]$$
Invariant $q$-Difference Operators Related to $GL_q(n)$

$$= \mathcal{M}_1'(u) \mathcal{M}_2'(r_2 - k, k) \mathcal{M}_3'(t) \begin{bmatrix} r - k & k + t \\ k + t + u & \end{bmatrix},$$

$$t = 0, 1, \ldots r_1 - k, \quad u = 0, 1, \ldots r - 2k - t,$$

$$\mathcal{M}_1'(u) = \left( \frac{[r - 2k - t]_q! [u]_q!}{[r - 2k - t - u]_q!} \right)^{1/2}$$

Finally we explain how to obtain the lower-left-corner states $\begin{bmatrix} r_1 & k \\ 0 & 0 \end{bmatrix}$ starting from the lowest-weight state $\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix}$. This is achieved by using again the lemma above:

$$\tilde{C} \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{M}_3'(k, r) \begin{bmatrix} r_1 & k \\ k & 0 \end{bmatrix}, \quad k = 0, 1, \ldots, r_0 = \min(r_1, r_2)$$

$$\mathcal{M}_3'(k, r) = \left( \frac{[r + 1]_q! [r_1 + 1]_q! [r_1]_q! [k]_q!}{[r + 1 - k]_q! [r_1 + 1 - k]_q! [r_1 - k]_q!} \right)^{1/2}$$

(6.243)

For further use we note that relation (6.243) may be rewritten in two alternative ways:

$$\mathcal{M}_3'(k, r) \begin{bmatrix} r_1 & k \\ k & 0 \end{bmatrix} = \prod_{s=1}^k \tilde{C}_s \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} = \sum_{j=0}^k (-1)^{k-j} \frac{q^{j(r_1-s-1)}}{q^j} \left( \begin{array}{c} k \\ j \end{array} \right)_q [r_1 - j]q! [r_1 - k]q! \times$$

$$\times (X_2^+)^j (X_3^+)^{k-j} (X_1^+)^j \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix}$$

(6.244a)

$$= X_3^+ [s - 1 - r_1]_q + X_2^+ X_1^+ q^{1/2(s-1-r_1)} =$$

$$= X_3^+ X_2^+ [s - 1 - r_1]_q + X_2^+ X_1^+ [r_1 - s + 2]_q$$

(6.244b)

The proof of (6.244) is given in [245].

We should mention that similar formulae to (6.239) and (6.242a) for the relation between GWZ states may be found in the literature (cf., e.g., [47, 65, 143]). However, at the time we could not find in the literature formulae involving the operator $\tilde{C}$.

In this subsection we have not specified any realization of $U_q(sl(3))$. If we want to have the GWZ states realized as polynomials then we first identify the lowest-weight state $\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix}$ with the function 1 and then use the representation (6.112).

Finally we note the similarity of formula (6.244b) with the formula giving the singular vector in (2.37) for $A_2$. It is this similarity that will be exploited in the next section in order to prove the explicit realization of the irregular irreps in terms of GWZ states.

6.4.8 The Irregular Irreps in Terms of GWZ States

In the present section we combine the results of the previous sections to derive our main result. We set $q = e^{2\pi i/N}$, so that $[x]_q = \sin(\pi x/N)/\sin(\pi N)$. We consider the irregular representations characterized by (2.179), and we restrict the representation parameters $r_i = m_i - 1$ as needed in the current situation:
\[ 1 < r_1 + 1, r_2 + 1 < N < r_1 + r_2 + 2 = r + 2 < 2N \] (6.245)

With this the relevant singular vectors are (cf. (2.39) and (2.61) for \( A_2 \)):

\[ v_i = (X_i^+)^{r_i+1} v_0, \quad i = 1, 2, \] (6.246)

\[ \hat{v}_s^m = \mathcal{P}_s^m (X_1^+, X_2^+, X_3^+) v_0, \]

\[ \mathcal{P}_s^m = \sum_{j=0}^{\hat{m}} (-1)^{m-m_1} q^{\frac{1}{2}(j-r_1-1)} \left( \begin{array}{c} \hat{m} \\ j \end{array} \right) q^j \frac{[r_1 + 1 - \hat{m}] q^j}{[r_1 + 1 - j] q^j} \times \]

\[ (X_2^+)^j (X_3^+)^{m-j} (X_1^+)^j, \]

\[ \hat{m} = r + 2 - N. \]

As we know to obtain an irreducible representations we have to factor out the Verma submodule built on these singular vectors, or, in a function space realization of the lowest-weight representations, impose corresponding vanishing conditions using the corresponding invariant differential operators. In the GWZ basis the lowest-weight vector is \( \left[ \begin{array}{c} r_1 \\ 0 \\ 0 \end{array} \right] \), while the vanishing conditions following from above are:

\[ (X_1^+)^{r_1+1} \left[ \begin{array}{c} r_1 \\ 0 \\ 0 \end{array} \right] = 0 \] (6.247)

\[ \mathcal{P}_s^m (X_1^+, X_2^+, X_3^+) \left[ \begin{array}{c} r_1 \\ 0 \\ 0 \end{array} \right] = 0 \] (6.248)

Actually, the restrictions from (6.247) are valid in the GWZ basis by construction, e.g., from (6.239) one would obtain:

\[ (X_2^+)^{r_2+1} \left[ \begin{array}{c} r_1 \\ 0 \\ 0 \end{array} \right] = \mathcal{N}_2(s, 0) |_{s=r_2+1} \left[ \begin{array}{c} r + 1 \\ 0 \\ 0 \end{array} \right] = \mathcal{N}_2(s, 0) |_{s=r_2+1} \begin{pmatrix} r & r_1 & 0 \\ r + 1 & 0 & 0 \end{pmatrix} = 0 \] (6.249)

since the latter is an impossible GWZ state (the betweenness constraint (6.155) is violated), and \( \mathcal{N}_2(s, 0) |_{s=r_2+1} \sim (\Gamma_q (r_2 + 1 - s))^{-1} |_{s=r_2+1} = 0. \) Analogously from (6.242a) one would obtain:

\[ (X_1^+)^{r_1+1} \left[ \begin{array}{c} r_1 \\ 0 \\ 0 \end{array} \right] = \mathcal{N}_1(u, 0, 0) |_{u=r_1+1} \left[ \begin{array}{c} r_1 \\ 0 \\ r_1 + 1 \end{array} \right] = \mathcal{N}_1(u, 0, 0) |_{u=r_1+1} \begin{pmatrix} r & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix} = 0 \] (6.250)
again the latter is an impossible GWZ state, and $\mathcal{M}(u, 0, 0)|_{u=r_{1}+1} \sim \sim (\Gamma_q(r_1 + 1 - u))^{-1}|_{u=r_{1}+1} = 0$.

Thus the only new condition is (6.248). Indeed, it means that the lower-left-corner state $[r_1, \bar{m}]$ on the $\bar{m}$-th layer of our pyramid decouples from the irrep. This is clear from (6.244b) since the expression giving $[r_1, \bar{m}]$ is just $\varphi^\bar{m}(X_1^+, X_2^+, X_3^+)$. The decoupling of this state follows also from the explicit normalization factor in (6.243) with $k = \bar{m}$ and $r = N + \bar{m} - 2$ since:

$$\mathcal{M}_3'(\bar{m}, N + \bar{m} - 2) = 0 \tag{6.251}$$

which follows from:

$$\frac{[r + 1]_q!}{[r + 1 - k]_q!}|_{r=N+\bar{m}-2}^{k=\bar{m}} = \frac{[N + \bar{m} - 1]_q!}{[N - 1]_q!} = [N + \bar{m} - 1]_q [N + \bar{m} - 2]_q \ldots [N]_q = 0 \tag{6.252}$$

since $[N]_q = \sin(\pi N/N) / \sin(\pi/N) = 0$ when $q = e^{2\pi i/N}$.

The decoupling of the state $[r_1, \bar{m}]$ implies the decoupling of the lower-left-corner states on the higher layers, that is, the states $[r_1, k]$ with $k > \bar{m}$. This follows by noting that because of the factorization formula (6.244b) can be written also as:

$$\mathcal{M}_3''(k, \bar{m}) = \prod_{s=\bar{m}+1}^{k} \tilde{\mathcal{C}}_s \left[ r_1 \bar{m} \atop k \bar{m} \right] \tag{6.253}$$

that is, these states are descendants of $[r_1, \bar{m}]$. For consistency we note also that:

$$\mathcal{M}_3'(k, N + \bar{m} - 2) \begin{cases} \neq 0 & \text{for } k < \bar{m} \\ 0 & \text{for } k \geq \bar{m} \end{cases} \tag{6.254}$$

Clearly, together with the lower-left-corner states decouple also the states on their layers; that is, all states on layers $k = \bar{m}, \bar{m} + 1, \ldots, r_0$. Thus, we are left with the states on the first $\bar{m}$ layers. Their number is given by (6.236) and (6.237), with $k = \bar{m}$.

Thus, we have obtained the explicit description of the irregular representations of $U_q(sl(3))$ in terms of the GWZ basis. These are the states displayed in (6.234) for $k = 0, 1, \ldots, \bar{m} - 1 = r + 1 - N$.

We note that in the case when $\bar{m} = 1$; that is, $N = r + 1$, the irregular irrep is flat.
Finally, we discuss the representation action of $U_q(sl(3))$ in our irregular irreps. First we stress that when we consider unnormalized GWZ states the $U_q(sl(3))$ action is given straightforwardly as action on a truncated Verma module basis and there is no need even to display it explicitly. A little more care is needed when we consider the normalized GWZ basis. First we recall the standard action of $U_q(sl(3))$ on the normalized GWZ basis [47, 65], when $q$ is not a nontrivial root of 1:

$$H_1 \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = (2m_{11} - m_{12} - m_{22}) \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right]$$ (6.254a)

$$X_1^+ \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = \left( [m_{12} - m_{11}]_q \right) ^{\xi} \left( [m_{11} - m_{22} + 1]_q \right) ^{\xi'} \times \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} + 1 & \end{array} \right]$$ (6.254b)

$$X_1^- \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = \left( [m_{12} - m_{11}]_q \right) ^{1 - \xi} \left( [m_{11} - m_{22}]_q \right) ^{1 - \xi'} \times \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} - 1 & \end{array} \right]$$ (6.254c)

$$H_2 \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = (2(m_{12} + m_{22}) - m_{11} - r - r_1) \times$$

$$\times \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right]$$ (6.254d)

$$X_2^+ \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = a_1^+ \left[ \begin{array}{cc} m_{12} + 1 & m_{22} \\ m_{11} & \end{array} \right] +$$

$$+ a_2^+ \left[ \begin{array}{cc} m_{12} & m_{22} + 1 \\ m_{11} & \end{array} \right]$$ (6.254e)

$$X_2^- \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & \end{array} \right] = a_1^- \left[ \begin{array}{cc} m_{12} - 1 & m_{22} \\ m_{11} & \end{array} \right] +$$

$$+ a_2^- \left[ \begin{array}{cc} m_{12} & m_{22} - 1 \\ m_{11} & \end{array} \right]$$ (6.254f)

$$a_1^+ = \frac{([r - m_{12}]_q)^{\eta_1} ([m_{12} - r_1 + 1]_q)^{\eta_2}}{[m_{12} - m_{22} + 1]^{1/2}_q} \times$$

$$\times \frac{([m_{12} + 2]_q)^{\eta_3} ([m_{12} - m_{11} + 1]_q)^{1 - \xi}}{[m_{12} - m_{22} + 2]^{1/2}_q} ,$$

$$a_2^+ = \frac{([r - m_{22} + 1]_q)^{\xi_1} ([r_1 - m_{22}]_q)^{\xi_2}}{[m_{12} - m_{22}]^{1/2}_q} \times$$

$$\times \frac{([m_{22} + 1]_q)^{\xi_3} ([m_{11} - m_{22}]_q)^{1 - \xi'}}{[m_{12} - m_{22} + 1]^{1/2}_q} ,$$

$$a_1^- = \frac{([r - m_{12}]_q)^{\xi_1} ([m_{12} - r_1 + 1]_q)^{\xi_2}}{[m_{12} - m_{22} + 1]^{1/2}_q} \times$$

$$\times \frac{([m_{12} + 2]_q)^{\xi_3} ([m_{12} - m_{11} + 1]_q)^{1 - \xi}}{[m_{12} - m_{22} + 2]^{1/2}_q} ,$$

$$a_2^- = \frac{([r - m_{22} + 1]_q)^{\eta_1} ([r_1 - m_{22}]_q)^{\eta_2}}{[m_{12} - m_{22}]^{1/2}_q} \times$$

$$\times \frac{([m_{22} + 1]_q)^{\eta_3} ([m_{11} - m_{22}]_q)^{1 - \eta'}}{[m_{12} - m_{22} + 1]^{1/2}_q} ,$$
\[ a_1^\pm = \frac{(r - m_{12} + 1)_q^{1 - \eta_1} (m_{12} - r_1)_q^{1 - \eta_2}}{[m_{12} - m_{22}]^{1/2}_q} \times \]
\[ \times \frac{(m_{12} + 1)_q^{1 - \eta_3} (m_{12} - m_{11})_q^\xi}{[m_{12} - m_{22} + 1]^{1/2}_q}, \]
\[ a_2^\pm = \frac{(r - m_{22} + 2)_q^{1 - \zeta_1} (r_1 - m_{22} + 1)_q^{1 - \zeta_2}}{[m_{12} - m_{22} + 1]^{1/2}_q} \times \]
\[ \times \frac{(m_{22})_q^{1 - \zeta_3} (m_{11} - m_{22} + 1)_q^\zeta'}{[m_{12} - m_{22} + 2]^{1/2}_q}, \]
(6.255)

where the parameters \( \xi, \xi', \eta_1, \eta_2, \eta_3, \zeta_1, \zeta_2, \zeta_3 \) (introduced in [47]) take independently the values 0, \( \frac{1}{2} \), 1, the value \( \frac{1}{2} \) for all of them being the classical choice. Note, however, that some of the nonclassical choices have to be excluded if we want that the coefficients would automatically become zero for impossible GWZ states. Thus, there are the following exclusions: \( \xi \neq 0, \xi' \neq 1, \eta_1 \neq 0, \eta_2 \neq 1, \zeta_2 \neq 0, \zeta_3 \neq 1 \). Note that partial cases of (6.254) were actually used in the algebraic description of the pyramid above (with all extra parameters equal to \( \frac{1}{2} \)). Note also that for the unnormalized GWZ basis (6.254) would also hold, however, the coefficients \( a_1^\pm \) would be different; in particular, they will not contain any denominators.

For our purposes below we comment the action of the generators in relation to our pyramid structure (still in the generic \( q \) case). The action of the generators \( X_1^\pm \) is confined on fixed rows, which is expected since these rows form irreps of the \( U_q(sl(2)) \) quantum subgroup generated by \( X_1^\pm, H_1 \). The action of the generators \( X_2^\pm \) is more interesting. Consider the \( k \)-th layer. Then under the action the operator \( X_2^+ \) the states on the middle row (starting on the left with \( \left[ \begin{array}{c} r-k \\ k \end{array} \right] \)) and the rows above it are mapped into a state on the same layer (cf. the second term in (6.254e)) and a state on the layer \( k - 1 \) (cf. the first term in (6.254e)), while the states below the middle row are mapped into a state on the same layer (cf. the first term in (6.254e)) and a state on the layer \( k + 1 \) (cf. the second term in (6.254e)). Analogously, under the action the operator \( X_2^- \) the states on the middle row and the rows below it are mapped into a state on the same layer (cf. the first term in (6.254f)) and a state on the layer \( k - 1 \) (cf. the second term in (6.254f)), while the states above the middle row are mapped into a state on the same layer (cf. the second term in (6.254f)) and a state on the layer \( k + 1 \) (cf. the first term in (6.254f)). Certainly, in all cases the two resulting states are one above the other since they have the same weights (eigenvalues of \( H_1 \)). Note also that in some cases one of the two resulting states may miss when the initial state is on some of the sides or edges of the pyramid.

When \( q \) is a root of unity, as specified in the beginning of this section, there are two possible problems when using formulae (6.254). The first problem is that the action of the generators \( X_2^\pm \) is mixing in general neighbouring layers and thus we have to ensure
that formulae (6.254) will respect our factorization of the upper layers of the pyramid (which is so by construction if we use unnormalized GWZ states). This problem was cleared in [245].

The second possible problem, which is not specific for our approach and which was discussed in [4], is that there may arise zeros in the denominators of the coefficients (6.255). This necessitates modifications of (6.254) which were partially given in [4], and then in [245] where we also have checked that these modified formulae do not contradict our factorization. The modifications in (6.254) are as follows. First, we make the choice:

\[ \xi = 1, \quad \xi' = 0, \quad \eta_1 = 1, \quad \zeta_3 = 0 \]  

(6.256)

and then we set all remaining parameters equal to their classical value \( \frac{1}{2} \). Thus we have instead of (6.254b,c) and (6.255):

\[
X_1^+ \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = \begin{bmatrix} m_{12} - m_{11} \rangle_q & m_{12} \\ m_{11} + 1 & \end{bmatrix} \]  

(6.254b')

\[
X_1^- \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = \begin{bmatrix} m_{12} - m_{22} \rangle_q & m_{12} \\ m_{11} - 1 & \end{bmatrix} \]  

(6.254c')

\[
a_1^+ = [r - m_{12}]_q \left( \frac{[m_{12} - r_1 + 1]_q [m_{12} + 2]_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22} + 2]_q} \right)^{1/2} \]  

(6.255')

\[
a_2^+ = [m_{11} - m_{22}]_q \left( \frac{[r - m_{22} + 1]_q [r_1 - m_{22}]_q}{[m_{12} - m_{22}]_q [m_{12} - m_{22} + 1]_q} \right)^{1/2} \]  

\[
a_1^- = [m_{12} - m_{11}]_q \left( \frac{[m_{12} - r_1]_q [m_{12} + 1]_q}{[m_{12} - m_{22}]_q [m_{12} - m_{22} + 1]_q} \right)^{1/2} \]  

\[
a_2^- = [m_{22}]_q \left( \frac{[r - m_{22} + 2]_q [r_1 - m_{22} + 1]_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22} + 2]_q} \right)^{1/2} \]  

(6.255')

6.5 The Case of \( U_q(sl(4)) \)

6.5.1 Elementary Representations

In this section following [216, 217] we consider in more detail the case \( n = 4 \). It is convenient (also for the comparison with the \( q = 1 \) case) to make the following change of variables:

\[
Y_{31} = \tilde{Y}_{31} - q \tilde{Y}_{21} \tilde{Y}_{32}, \quad Y_{41} = \tilde{Y}_{41} - q \tilde{Y}_{21} \tilde{Y}_{42}, \\
Y_{21} = -q \tilde{Y}_{21}, \quad Y_{43} = q \tilde{Y}_{43}, \\
Y_{ij} = \tilde{Y}_{ij}, \quad \text{for } (ij) = (32), (42). \]  

(6.257)
Using (6.51) we have:

\[ Y_{ij} Y_{ij} = q^{1-2\delta_{ij}} Y_{ij} Y_{ij}, \quad 4 \geq i \neq j \geq 1, \quad (6.258a) \]

\[ Y_{kj} Y_{kj} = q^{1-2\delta_{kj}} Y_{kj} Y_{kj}, \quad 4 \geq k \neq j \geq 1, \quad (6.258b) \]

\[ Y_{41} Y_{32} = Y_{32} Y_{41} + \lambda Y_{31} Y_{42}, \quad (6.258c) \]

\[ Y_{4i} Y_{ji} = Y_{ji} Y_{4i}, \quad (ij) = (23), (32), \quad (6.258d) \]

\[ Y_{ki} Y_{ji} = q^{1-2\delta_{ki}} Y_{ki} Y_{ji} - (-1)^{\delta_{ki}} \lambda Y_{kj}, \quad 4 \geq k \neq j \geq 1 \quad (6.258e) \]

(each of (6.258a,b,e) has four cases). Note that (6.51g,h,i) holds also for \( Y_{ij} \) replacing \( Y_{ji} \).

Note that for \( q \) a phase (\(|q| = 1\)) the \( q \)-coset in the \( Y \) coordinates is invariant under the anti-linear anti-involution \( \omega \) acting as \( \omega \) (cf. (6.52)) with \( n = 4 \):

\[ \omega(Y_{ij}) = Y_{5-i,5-j}. \quad (6.259) \]

Thus it can be considered as a \( q \)-coset of the conformal quantum group \( SU_q(2,2) \).

The reduced functions for the \( \mathcal{W} \) action are (cf. (6.50)):

\[ \bar{\varphi}(\bar{Y}, \varphi) = \sum_{i,j,k,l,m,n \in \mathbb{Z}_+} \mu_{ijklmn} \bar{\varphi}_{ijklmn} \quad (6.260a) \]

\[ \bar{\varphi}_{ijklmn} = (Y_{21})^i (Y_{31})^j (Y_{32})^k (Y_{41})^l (Y_{42})^m (Y_{43})^n \times \]

\[ \times (\mathcal{R}_1)^{i1} (\mathcal{R}_2)^{j2} (\mathcal{R}_3)^{k3} \quad (6.260b) \]

Now the action of \( U_q(sl(4)) \) on (6.260) is given explicitly by:

\[ \tilde{\tau}_f(k') \bar{\varphi}_{ijklmn} = q^{i+k+l+m} [r_1 - i]_q \bar{\varphi}_{ijklmn} + \]

\[ + q^{i-k-1} [k]_q \bar{\varphi}_{i,j+1,k-1,l,m,n} + \]

\[ + q^{i-k-1} [k]_q \bar{\varphi}_{i,j,1,k-1,l,m,n} \]

\[ \tilde{\tau}_f(X'_1) \bar{\varphi}_{ijklmn} = q^{-1} [r_2 - j]_q \bar{\varphi}_{ijklmn} + \]

\[ + q^{-1} [r_2 - j + k + m - n]_q \bar{\varphi}_{i,j,k+l,m,n} + \]

\[ + q^{-1} [r_2 - j + k + m - n + r_2]_q \bar{\varphi}_{ijklmn} \]

\[ \tilde{\tau}_f(X'_2) \bar{\varphi}_{ijklmn} = q^{r_2+k-i} [r_2 - j + m - n]_q \bar{\varphi}_{i,j,k+l,m,n} + \]

\[ + q^{r_2+k-i} [r_2 - j + m - n + r_2]_q \bar{\varphi}_{ijklmn} + \]

\[ + q^{r_2+k-i} [r_2 - j + m - n + r_2]_q \bar{\varphi}_{ijklmn} \]

\[ \tilde{\tau}_f(X'_3) \bar{\varphi}_{ijklmn} = q^{r_2+k-i} [r_2 - j + m - n]_q \bar{\varphi}_{i,j,k+l,m,n} + \]

\[ + q^{r_2+k-i} [r_2 - j + m - n + r_2]_q \bar{\varphi}_{ijklmn} + \]

\[ + q^{r_2+k-i} [r_2 - j + m - n + r_2]_q \bar{\varphi}_{ijklmn} \]
\[
\hat{\eta}_r(X^\pm_i) \hat{\phi}_{ijklmn} = -q^{3-1-n+(j+k-\ell-m)/2} [j] q \hat{\phi}_{ijkl-1,k,\ell+1,mmn} - \\
-q^{3-1-n+(j+k-3\ell-m)/2} [k] q \hat{\phi}_{ijkl,k-1,\ell,m+1,n} + \\
+q^{-1-(j-k+\ell+m)/2} [n-r_3] q \hat{\phi}_{ijk\ell m,n+1}, \quad (6.263)
\]
\[
\hat{\eta}_r(X^\pm_i) \hat{\phi}_{ijklmn} = q^{1+(j-k-\ell-m)/2} [i] q \hat{\phi}_{i-1,jk\ell mnn} + \\
+q^{2+(j-k-\ell-m)/2} [j] q \hat{\phi}_{i1,j-1,k,\ell+1,mmn} + \\
+q^{2+(j-k-\ell-m)/2} [\ell] q \hat{\phi}_{ijk,\ell-1,m+1,n},
\]
\[
\hat{\eta}_r(X^\pm_i) \hat{\phi}_{ijklmn} = -q^{-(i+j-m+n)/2} [k] q \hat{\phi}_{ijk,k-1,\ell,mn}, \\
\hat{\eta}_r(X^\pm_i) \hat{\phi}_{ijklmn} = -q^{-n+(j-3k+\ell+3m)/2} [\ell] q \hat{\phi}_{ijkl,\ell-1,\ell-1,mn} - \\
-q^{-n+(j-k+\ell-m)/2} [m] q \hat{\phi}_{ijkl,k+1,\ell,m-1,n} - \\
-q^{1-(j-k+\ell+m)/2} [n] q \hat{\phi}_{ijk\ell m,n-1}. \quad (6.264)
\]

It is easy to check that \(\hat{\eta}_r(k_i), \hat{\eta}_r(X^\pm_i)\) satisfy (6.10).

From (6.263) and (6.264) one can easily write down the explicit action of the non-simple root generators. These are defined as follows [198, 360]:
\[
X^\pm_{ab} = \pm q^{1/2} \left( q^{1/2} X^\pm_a X^\pm_b - q^{1/2} X^\pm_b X^\pm_a \right), \quad (ab) = (12), (23),
\]
\[
X^\pm_{13} = \pm q^{1/2} \left( q^{1/2} X^\pm_1 X^\pm_3 - q^{1/2} X^\pm_3 X^\pm_1 \right) = \\
= \pm q^{1/2} \left( q^{1/2} X^\pm_{12} X^\pm_3 - q^{1/2} X^\pm_3 X^\pm_{12} \right). \quad (6.265)
\]

We give only the negative roots action, since these formulae will be used below:
\[
\hat{\eta}_r(X^\pm_2) \hat{\phi}_{ijklmn} = -q^{(i-k-\ell+n+3)/2} [j] q \hat{\phi}_{ijkl-1,k,\ell+1,mmn} + \\
+q^{(i-k-\ell+n+3)/2} [k] q \hat{\phi}_{ijkl,k-1,\ell-1,m+1,n},
\]
\[
\hat{\eta}_r(X^\pm_3) \hat{\phi}_{ijklmn} = -q^{-(i+j-k-n-3)/2} [m] q \hat{\phi}_{ijkl,\ell-1,m-1,n},
\]
\[
\hat{\eta}_r(X^\pm_1) \hat{\phi}_{ijklmn} = -q^{3+(i+j-m-n)/2} [\ell] q \hat{\phi}_{ijkl,\ell-1,mn}. \quad (6.266)
\]

Further we consider the restricted functions (cf. (6.59)):
\[
\hat{\phi}(\hat{Y}) = \sum_{i,j,k,\ell,m,n,\ell,n \in \mathbb{Z}^+} \mu_{ijklmn} \hat{\phi}_{ijklmn}, \quad (6.267)
\]
\[
\hat{\phi}_{ijklmn} = (Y_{21})^i (Y_{31})^j (Y_{32})^k (Y_{41})^l (Y_{42})^m (Y_{43})^n. \quad (6.268)
\]

As a consequence of the intertwining property (6.60), we obtain that \(\hat{\phi}_{ijklmn}\) obey the same transformation rules (6.261), (6.263), (6.264), and (6.266), as \(\hat{\phi}_{ijklmn}\).

Recall that we consider the representations \(\hat{\eta}_r\) for arbitrary complex \(r_i\) and we expect as in the \(q = 1\) case (cf. Section I.4.) that whenever some \(m_i = r_i + 1\) or \(m_{ij} = m_i + \cdots + m_j\) \((i < j)\) is a positive integer the representations are reducible and there exist invariant subspaces. We give now two simple examples.
Let $m_1 = r_1 + 1 \in \mathbb{N}$. Then it is clear that functions $\tilde{\phi}$ with $\mu_{ijk\ell mn} = 0$ if $i \geq m_1$ form an invariant subspace since:

$$
\hat{\pi}_1(X_1^+)\tilde{\phi}_{r_1,jk\ell mn} = q^{(i+1-m-\ell-2-k)/2}[k]q^{r_1,j+1,k-1,\ell mn} + q^{(i+\ell-k-2-m)/2}[m]q^{r_1,j,k+1,m-1,n}
$$

(6.268)

and all other operators in (6.261), (6.263) and (6.264) either preserve or lower the index $i$. The same is true for the functions $\hat{\phi}$. In particular, for $r_1 = 0$ the functions in the invariant subspace do not depend on the variable $Y_{21}$.

Analogously if $m_3 = r_3 + 1 \in \mathbb{N}$ the functions $\tilde{\phi}$ with $\mu_{ijk\ell mn} = 0$ if $n \geq m_3$ form an invariant subspace since:

$$
\hat{\pi}_1(X_3^+)\tilde{\phi}_{ijk\ell mn,r_3} = -q^{(k+j+m-\ell-2)/2}[j]q^{r_1,j+1,k,\ell+1,m,r_3} - q^{(k+\ell+j-3m-2)/2}[m]q^{r_1,j+1,k,m,r_3+1},
$$

(6.269)

and all other operators in (6.261), (6.263) and (6.264) either preserve or lower the index $n$, the same holding for the functions $\hat{\phi}$. In particular, for $r_3 = 0$ the functions in the invariant subspace do not depend on the variable $Y_{43}$.

It is an useful exercise to rewrite the transformation rules (6.261), (6.263), (6.264), and (6.266) for the functions $\tilde{\phi}$ using the operators (6.61), (6.62), and (6.85).

### 6.5.2 Intertwining Operators

The general prescription for finding the intertwining operators was already discussed in detail. In order to apply this procedure here we need the explicit action of $\pi_R(X_i^+)$ on our functions. First we have to calculate the action on the new basis $Y_{jk}$. We have instead of (6.67b):

$$
\pi_R(X_i^+)(Y_{jk})^n = (-1)^{\delta_{ii}}\delta_{\ell\ell}^\delta_{i+1,i}q^{n-1/2}[n]q^{(Y_{i+1})^n-1}Y_{i+1}^{2}Y_{i-1}^{2}, \quad i = 1, 3
$$

$$
\pi_R(X_2^+)(Y_{jk})^n = q^{-\delta_{\ell\ell}^\delta_{i+1,i}}q^{(n-2)(\ell-1)+1/2}[n]q^{Y_{2j}(Y_{jk})^n-1}Y_{2j}^{2}Y_{2j}^{2}, \quad i = 1, 3
$$

(6.270)

where we again use $\varepsilon_0 = \varepsilon_0 = Y_{jj} = 1$, $Y_{j\ell} = 0$ for $j < \ell$.

Using (6.270) and (6.67a) we obtain:

$$
\pi_R(X_1^-)\tilde{\phi}_{ijk\ell mn}^{r_1,r_2,r_3} = -q^{i+j-k-m+(r_1-1)/2}[i]q^{r_1,j+1,k-1,\ell mn} + q^{(r_1-1)/2}[r_1]q^{r_1,j+1,k-1,\ell mn}Z_{12},
$$

(6.271a)
\[ \pi_R(X^s) \varphi^{r_3} = q^{2k+m-n+(r_2-1)/2} [j]_q \varphi^{r_1+1,j-1,k \\text{even}} + q^{k+m-n+(r_2-3)/2} [k]_q \varphi^{r_1+1,j-1,k \\text{even}} + q^{k-j+2m-n+(r_2-3)/2} [\ell]_q \varphi^{r_1+1,r_2-2,r_3+1} + q^{m-n+(r_2-5)/2} [m]_q \varphi^{r_1+1,r_2-2,r_3+1} - q^{2m-n+(r_2-3)/2} \lambda(k)_q \varphi^{r_1+1,r_2-2,r_3+1} + q^{(r_2-1)/2} [r_2]_q \varphi^{r_1,r_2,r_3} Z_{23} , \tag{6.271b} \]

\[ \pi_R(X^{-s}) \varphi^{r_3} = q^{n+(r_3-1)/2} [n]_q \varphi^{r_1+1,r_2+1,r_3-2} + q^{(r_3-1)/2} [r_3]_q \varphi^{r_1,r_2,r_3} Z_{34} , \tag{6.271c} \]

where we have labelled the functions also with the representation parameters \( r_s \). As in the classical case [197] the right action is taking out from the representation space \( \mathcal{C}_r \), and while some of the terms are functions from other representation spaces (depending on which \( X^s \) is acting), there are terms involving the \( Z_{jk} \) variables which do not belong to any of our representation spaces. These terms vanish only when the respective \( r_s \) is equal to zero, and in these cases (6.271) describe three different intertwining operators corresponding to the simple roots of the root system of \( sl(4) \). If \( r_s \in \mathbb{N} \) then the terms with \( Z_{jk} \) vanish exactly when we take \((\pi_R(X^s))^{m_s} \) [197], [211], \( m_s = r_s + 1 \).

Indeed, we know from the general prescription that if \( m_s \in \mathbb{N} \) then there exist an intertwining operator \( \pi_R \) of the form \( (\pi_R(X^s))^{m_s} \). We have the following intertwining properties (cf. (6.73)):

\[ I^1_{m_1} \circ \pi_{m_1,m_2,m_3} = \pi_{-m_1,m_1+m_2,m_3} \circ I^1_{m_1}, \quad m_1 \in \mathbb{N} , \tag{6.272a} \]

\[ I^2_{m_2} \circ \pi_{m_1,m_2,m_3} = \pi_{m_1,-m_2+m_2,m_3} \circ I^2_{m_2}, \quad m_2 \in \mathbb{N} , \tag{6.272b} \]

\[ I^3_{m_3} \circ \pi_{m_1,m_2,m_3} = \pi_{m_1,m_2,-m_3+m_2} \circ I^3_{m_3}, \quad m_3 \in \mathbb{N} , \tag{6.272c} \]

where we label the representations with the numbers \( m_s \) instead of \( r_s = m_s - 1 \) to simplify the notation. The expressions for two of these operators (up to \( q^{-1} \) factors) are:

\[ (\pi_R(X^-))^m \varphi^{m_1,m_2,m_3} = (-1)^{m_1} \frac{[j]_q}{[i-m_1]_q} \varphi^{i-1,m_1,m_2,m_3} \]  

\[ \pi_R(X^3)^m \varphi^{m_1,m_2,m_3} = \frac{[n]_q}{[n-m_3]_q} \varphi^{m_1,m_2,-m_3} \]  

6.5 The Case of \( U_q(sl(4)) \)
It will be convenient to use also the following notation for the coordinates of the coset:

\[
\xi = Y_{21}, \quad x = Y_{31}, \quad u = Y_{32}, \quad w = Y_{41}, \quad y = Y_{42}, \quad \eta = Y_{43}.
\] (6.275)

Having in mind the preceding discussion let us introduce the following \(q\)-difference operators (using notation (6.61), (6.62), (6.85), and (6.275)):

\[
\hat{I}_1 \equiv -q^{(r_1-1)/2} \hat{\mathcal{D}}_{\xi} T_{\xi} T_{x} T_{w} (T_{u} T_{y})^{-1}
\] (6.276a)

\[
\hat{I}_2 \equiv q^{(r_2-3)/2} \left( q\hat{\mathcal{M}}_{\xi} \hat{\mathcal{D}}_{x} T_{u} + \hat{\mathcal{D}}_{u} T_{u} + \hat{\mathcal{M}}_{\eta} \hat{\mathcal{D}}_{w} T_{x}^{-1} T_{y} T_{u} + q^{-1} \hat{\mathcal{M}}_{\eta} \hat{\mathcal{D}}_{y} - \hat{\mathcal{M}}_{x} \hat{\mathcal{M}}_{\eta} \hat{\mathcal{D}}_{u} \hat{\mathcal{D}}_{w} T_{y} T_{u}^{-1} T_{y} T_{\eta}^{-1} \right)
\] (6.276b)

\[
\hat{I}_3 \equiv q^{(r_3-1)/2} \hat{\mathcal{D}}_{\eta} T_{\eta}
\] (6.276c)

It is not difficult to see that if \(m_s \in \mathbb{N}\) we have (cf. (6.76)):

\[
\hat{I}_s^{m_s} = I_s^{m_s} = (\pi_R(X_s^{-}))^{m_s}
\] (6.277)

We go back to the general situation. There are altogether six different operators corresponding to the positive roots of \(\Delta\) which exist when the respective number from the set \(m_1, m_2, m_3, m_{12}, m_{23}, m_{13}\) is a positive integer. We have considered the three simple roots. To obtain the remaining three operators it is enough to substitute in (6.74) the expressions in the \(sl(4)\) case given in (2.37) for the singular vectors corresponding to the three nonsimple roots \(\alpha_{12}, \alpha_{23}, \alpha_{13}\), realized when \(m_{12} \in \mathbb{N}, m_{23} \in \mathbb{N}, m_{13} \in \mathbb{N}\), respectively. We shall give explicitly the cases we need in the next chapter.