7 q-Maxwell Equations Hierarchies

Summary
In this chapter we start by using q-conformal invariance to propose a new q-Minkowski space–time and q-Maxwell equations. We are using an indexless formulation in which the spin properties are expressed not through Lorentz indices but through polynomial dependence on two conjugate variables, \( z, \bar{z} \). The proposed new q-Minkowski coordinates together with \( z, \bar{z} \) can be interpreted as the six local coordinates of a \( SU_q(2, 2) \) flag manifold. The new q-Maxwell equations are q-conformal invariant and are the first members of an infinite new hierarchy of q-difference equations parametrized by an integer \( n \in \mathbb{Z}_+ \). We also present a generalized q-Maxwell equations hierarchy indexed by two integers which includes the initial q-Maxwell equations hierarchy as a subfamily. Another subfamily of the generalized q-Maxwell equations hierarchy is the potential q-Maxwell equations hierarchy. Yet another subfamily of the generalized q-Maxwell equations hierarchy is the q-d’Alembert equations hierarchy with first member the q-d’Alembert equation. The latter hierarchy intersects the initial q-Maxwell equations hierarchy exactly with the q-Maxwell equations. Further, we present polynomial solutions and q-plane-wave solutions of the q-d’Alembert equation. Next, we present q-plane-wave solutions of the potential q-Maxwell hierarchy. Then we present q-plane-wave solutions of the full q-Maxwell equations. We also consider the q-Weyl gravity equations hierarchy and present q-plane-wave solutions of the lowest member which is q-deformation of linear conformal gravity. As a small detour we present a multiparameter deformation of quantum Minkowski space–time. This chapter is based mainly on [214, 215, 221, 226, 229, 237–240, 247].

7.1 Maxwell Equations Hierarchy

The present section follows mostly [214]. It is well known that Maxwell equations

\[
\begin{align*}
\partial^\mu F_{\mu\nu} &= J_\nu, \quad (7.1a) \\
\partial^\mu *F_{\mu\nu} &= 0, \quad (7.1b)
\end{align*}
\]

(where \( *F_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \), \( \epsilon_{\mu\nu\rho\sigma} \) being totally antisymmetric with \( \epsilon_{0123} = 1 \)), or, equivalently

\[
\begin{align*}
\partial_k E_k &= J_0 (= 4\pi p), \\
\partial_0 E_k - \epsilon_{k\ell m} \partial_\ell H_m &= J_k (= -4\pi j_k), \\
\partial_k H_k &= 0, \quad \partial_0 H_k + \epsilon_{k\ell m} \partial_\ell E_m = 0, \quad (7.2)
\end{align*}
\]

where \( E_k \equiv F_{k0}, H_k \equiv (1/2)\epsilon_{k\ell m} F^{\ell m}, \) can be rewritten in the following manner:

\[
\begin{align*}
\partial_k F^+_k &= J_0, \quad \partial_0 F^+_k \pm i \epsilon_{k\ell m} \partial_\ell F^+_m = J_k, \quad (7.3)
\end{align*}
\]

where

\[
F^+_k \equiv E_k \pm i H_k. \quad (7.4)
\]
Not so well known is the fact that the eight equations in (7.3) can be rewritten as two conjugate scalar equations in the following way:

\[ I^+ F^+ (z) = J(z, \bar{z}), \quad (7.5a) \]
\[ I^- F^- (\bar{z}) = J(z, \bar{z}), \quad (7.5b) \]

where

\[ I^+ = \bar{z} \partial_{x_+} + \partial_v - \frac{1}{2} \left( z \partial_{x_+} + z \partial_v + \bar{z} \partial_{\bar{v}} + \partial_- \right) \partial_z, \quad (7.6a) \]
\[ I^- = z \partial_{x_+} + \partial_{\bar{v}} - \frac{1}{2} \left( \bar{z} \partial_{x_+} + \bar{z} \partial_v + z \partial_{\bar{v}} + \partial_- \right) \partial_{\bar{z}}, \quad (7.6b) \]
\[ F^+ (z) \equiv z^2 F^+ - 2 z F^+ + (F^+ - i F^+), \quad (7.8a) \]
\[ F^- (\bar{z}) \equiv \bar{z}^2 F^- - 2 \bar{z} F^- + (F^- + i F^-), \quad (7.8b) \]
\[ J(z, \bar{z}) \equiv \bar{z} z (J_0 + J_3) + z (J_1 + i J_2) + \bar{z} (J_1 - i J_2) + (J_0 - J_3) = \quad (7.8c) \]
\[ \equiv \bar{z} z J_+ + z J_+ + \bar{z} J_+ + J_+ \]

where we continue to suppress the \( x_\mu \), respectively, \( x_\pm, v, \bar{v} \), dependence in \( F \) and \( J \).

(\text{The conjugation mentioned above is standard and in our terms it is: } I^+ \longleftrightarrow I^-, F^+ (z) \longleftrightarrow F^- (\bar{z}).)

It is easy to recover (7.3) from (7.5) – just note that both sides of each equation are first-order polynomials in each of the two variables \( z \) and \( \bar{z} \); then comparing the independent terms in (7.5) one gets at once (7.3).

Writing the Maxwell equations in the simple form (7.5) has also important conceptual meaning. The point is that each of the two scalar operators \( I^+, I^- \) is indeed a single object, namely, it is an intertwiner of the conformal group, while the individual components in (7.1)–(7.3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations \( T^\chi \) of the four-dimensional conformal algebra \( su(2, 2) \) may be labelled by \( \chi = [n_1, n_2; d] \), where \( n_1, n_2 \) are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being \( (n_1 + 1)(n_2 + 1) \)), and \( d \) is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by \( (j_1, j_2) = (n_1/2, n_2/2) \).) Then the intertwining properties of the operators in (7.6) are given by:
\[ I^+ : C^+ \longrightarrow C^0, \quad I^+ \circ T^+ = T^0 \circ I^+, \quad (7.9a) \]

\[ I^- : C^- \longrightarrow C^0, \quad I^- \circ T^- = T^0 \circ I^-, \quad (7.9b) \]

where \( T^a = T^a_\alpha, \alpha = 0, +, -, \), \( C^a = C^a_\mu \) are the representation spaces, and the signatures are given explicitly by:

\[ \chi^+ = [2, 0; 2], \quad \chi^- = [0, 2; 2], \quad \chi^0 = [1, 1; 3], \quad (7.10) \]

as anticipated. Indeed, \((n_1, n_2) = (1, 1)\) is the four-dimensional Lorentz representation (carried by \( J_\mu \) above), and \((n_1, n_2) = (2, 0), (0, 2)\) are the two conjugate three-dimensional Lorentz representations (carried by \( F^\pm_\kappa \) above), while the conformal dimensions are the canonical dimensions of a current \((d = 3)\), and of the Maxwell field \((d = 2)\). We see that the variables \( z, \bar{z} \) are related to the spin properties, and we shall call them “spin variables”. More explicitly, a Lorentz spin-tensor \( G(z, \bar{z}) \) with signature \((n_1, n_2)\) is a polynomial in \( z, \bar{z}\) of order \( n_1, n_2\), respectively.

Formulae (7.9) and (7.10) are part of an infinite hierarchy of couples of first-order intertwiners given already in [235] for the Euclidean conformal group \( SU^*(4)\), and then for the conformal group \( SU(2, 2)\) in [194, 503]. (Note that [235, 503] use a different approach, while [194] already uses the essential features of [197] in the context of the conformal group; see also Volume 1.)

Explicitly, instead of (7.9) and (7.10) we have [194]:

\[ I^+_n : C^+_n \longrightarrow C^0_n, \quad I^+_n \circ T^+_n = T^0_n \circ I^+_n, \quad (7.11a) \]

\[ I^-_n : C^-_n \longrightarrow C^0_n, \quad I^-_n \circ T^-_n = T^0_n \circ I^-_n, \quad (7.11b) \]

where \( T^a_n = T^a_\alpha, C^a_n = C^a_\mu \), and the signatures are:

\[ \chi^+_n = [n + 2, n; 2], \quad \chi^-_n = [n, n + 2; 2], \quad \chi^0_n = [n + 1, n + 1; 3], \quad n \in \mathbb{Z}_+, \quad (7.12) \]

while instead of (7.5) we have:

\[ I^+_n F^+_n(z, \bar{z}) = J_n(z, \bar{z}), \quad (7.13a) \]

\[ I^-_n F^-_n(z, \bar{z}) = J_n(z, \bar{z}), \quad (7.13b) \]

where

\[ I^+_n = \frac{n + 2}{2} \left( \bar{z} \partial_+ + \partial_v \right) - \frac{1}{2} \left( \bar{z} z \partial_+ + z \partial_v + \bar{z} \partial_v + \partial_- \right) \partial_{\bar{z}}, \quad (7.14a) \]

\[ I^-_n = \frac{n + 2}{2} \left( z \partial_+ + \partial_v \right) - \frac{1}{2} \left( \bar{z} z \partial_+ + z \partial_v + \bar{z} \partial_v + \partial_- \right) \partial_{\bar{z}}, \quad (7.14b) \]

\((n \in \mathbb{Z}_+), \) while \( F^+_n(z, \bar{z}), F^-_n(z, \bar{z}), J_n(z, \bar{z}), \) are polynomials in \( z, \bar{z}\) of degrees \((n + 2, n), (n, n + 2), (n + 1, n + 1), \) respectively, as explained above. If we want to use the notation
with indices as in (7.1), then \( F^+(z, \bar{z}) \) and \( F^-(z, \bar{z}) \) correspond to \( F_{\mu\nu, a_1, \ldots, a_n} \), which is antisymmetric in the indices \( \mu, \nu \), symmetric in \( a_1, \ldots, a_n \), and traceless in every pair of indices, while \( I_n(z, \bar{z}) \) corresponds to \( I_{\mu, a_1, \ldots, a_n} \), which is symmetric and traceless in every pair of indices. Note, however, that the analogues of (7.1) would be much more complicated if one wants to write explicitly all components. The crucial advantage of (7.13) is that the operators \( I^\pm_n \) are given just by a slight generalization of \( I^\pm_0 \).

We shall call the hierarchy of equations (7.13) the **Maxwell hierarchy**. The Maxwell equations are the zero member of this hierarchy.

To proceed further we rewrite (7.14) in the following form:

\[
I^+_n = \frac{1}{2} \left( (n + 2)I_1 I_2 - (n + 3)I_2 I_1 \right),
\]

\[
I^-_n = \frac{1}{2} \left( (n + 2)I_3 I_2 - (n + 3)I_2 I_3 \right),
\]

where

\[
I_1 = \partial_z, \quad I_2 = \bar{z}z \partial_+ + z \partial_v + \bar{z} \partial_\bar{\nu} + \partial_- \quad I_3 = \partial_\bar{z}.
\]

We note in passing that group-theoretically the operators \( I_a \) correspond to the three simple roots of the root system of \( sl(4) \), while the operators \( I^\pm_n \) correspond to the two nonsimple nonhighest roots [194, 197].

**Remark 7.1.** If we use induction from the ten-dimensional parabolic \( P_1 = M_1 A_1 N_1 \) (cf. [232]), the variables there are \( x_+, \nu, \bar{\nu}, z, \bar{z} \) (from \( N_1 \)), \( y \) (from \( M_1 \)). The relation between the variables of the \( P_0 \) (or \( P_2 \)) induction and \( P_1 \) induction is:

\[
\begin{align*}
    x_0^+ &= x_1^+ + y(z^1)^2, \quad x_0^- = y, \quad \nu^0 = \nu^1 + yz^1, \quad z^0 = z^1 \\
    x_1^+ &= x_0^+ - x_0^-(z^0)^2, \quad y = x_0^-, \quad \nu^1 = \nu^0 - x_0^0 z^0, \quad z^1 = z^0.
\end{align*}
\]

From this change of variables follow:

\[
\begin{align*}
    \partial_0^+ &= \partial_1^+, \partial_0^- = \partial_1^-, \partial_0^0 = \partial_\nu - z^1 \partial_\nu + \bar{z} \partial_\bar{\nu} - z \partial_\bar{\nu}, \\
    \partial_0^0 &= \partial_2^1 - yz^1 \partial_1^1 - y \partial_\nu^1, \partial_0^2 = \partial_2^1 - yz^1 \partial_1^1 - y \partial_\nu^1.
\end{align*}
\]

Correspondingly, the operators \( I_k \) from (7.16) in the \( P_1 \) variables are:

\[
I_1 = \partial_\nu - yz \partial_+ - y \partial_\bar{\nu}, I_2 = \partial_\bar{\nu}, I_3 = \partial_\bar{z} - yz \partial_+ - y \partial_\bar{\nu},
\]

where we have omitted the subscript 1. 

\[\Diamond\]
This is the form (7.15) that we generalize for the $q$-deformed case. In fact, we can write at once the general form, which follows from the expressions for the singular vectors corresponding to those nonsimple nonhighest roots given by (2.37) with $u = 1$, $m = 1$, $n_{i_1} = 1$, $q_{i_1} = 1$:

$$qT_n^+ = \frac{1}{2} \left( [n + 2]_q r_1^q r_2^q - [n + 3]_q r_1^q r_1^q \right), \quad (7.17a)$$

$$qT_n^- = \frac{1}{2} \left( [n + 2]_q r_3^q r_2^q - [n + 3]_q r_2^q r_3^q \right). \quad (7.17b)$$

It is our task (using the previous sections) to make this form explicit by first generalizing the variables and then the functions and the operators.

### 7.2 Quantum Minkowski Space–Time

#### 7.2.1 $q$-Minkowski Space–Time

The variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the coset $\mathcal{G} = SL(4)/B$, where $B$ is the Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. (Equally well one may take the coset $SL(4)/B^-$, where $B^-$ is the Borel subgroup of lower diagonal matrices.) Under the natural conjugation (cf. also below), this is also a coset of the conformal group $SU(2, 2)$.

We know from Section 4.5 what are the properties of the noncommutative coordinates on the $SLq(4)$ coset. We make the following identification (compare with (6.275)):

$$x_+ = w = Y_{41}, \quad x_- = u = Y_{32}$$

$$v = x = Y_{31}, \quad \bar{v} = y = Y_{42}$$

$$z = \xi = Y_{21}, \quad \bar{z} = \eta = Y_{43}$$

for the $q$-Minkowski space–time coordinates and for the spin coordinates, which we denote as their classical counterparts. Thus, we obtain for the commutation rules of the $q$-Minkowski space–time coordinates (cf. (6.258)):

$$x_+ v = q^{+1} v x_+, \quad x_+ \bar{v} = q^{+1} \bar{v} x_+,$$

$$x_+ x_- - x_- x_+ = \lambda v \bar{v}, \quad \bar{v} v = v \bar{v}. \quad (7.19)$$
It is easy to notice that these relations are as the $GL_q(2)$ commutation relations [462], if we identify our coordinates with the standard $a, b, c, d$ generators of $GL_q(2)$ as follows:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_+ & v \\ \tilde{v} & x_- \end{pmatrix}. \quad (7.20)$$

The $q$-Minkowski length is defined as the $GL_q(2)$ $q$-determinant:

$$\ell_q \doteq \det_q M = ad - qbc = x_+x_- - q\tilde{v}v, \quad (7.21)$$

and hence it commutes with the $q$-Minkowski coordinates. It has the correct classical limit $\ell_{q=1} = x_0^2 - \vec{x}^2$.

We know from (5.183) that for $q$ phase ($|q| = 1$) the commutation relations (7.19) are preserved by an antilinear anti-involution $\omega$ acting as (cf. (5.183)):

$$\omega(x_+) = x_-, \quad \omega(v) = \tilde{v}, \quad (7.22)$$

from which follows also that $\omega(\ell_q) = \ell_q$.

**Remark 7.2.** Note that relations (7.19) are different from the commutation relations of $q$-Minkowski space–time (with $q$ real) in [123, 455, 544], (cf. also [173, 174], and references therein). Later, Majid [456] has shown that the latter $q$-Minkowski space of [123, 455, 544] can be obtained by a quantum Wick rotation (twisting) from a $q$-Euclidean space. The latter is also related to $GL_q(2)$, as our $q$-Minkowski space; however, for $q$ real and under a different anti-linear anti-involution: $\hat{\omega}_E(a) = d$, $\hat{\omega}_E(b) = -q^{-1}c$; that is, for the matrix $M$ (cf. (7.20)) this is the unitary $^*$ while with our conjugation (7.22) $M$ is hermitian.

The commutation rules of the spin variables $\tilde{z}, z$ between themselves, with the $q$-Minkowski coordinates and with the $q$-Minkowski length are (cf. (6.258)):

\[
\begin{align*}
\tilde{z}z &= z\tilde{z}, \\
x_+z &= q^{-1}xz_+, & x_-z &= qzx_- - \lambda v, \\
vz &= q^{-1}zv, & \tilde{v}z &= q\tilde{v}z - \lambda x_+, \\
\tilde{z}x_+ &= qx_+\tilde{z}, & \tilde{z}x_- &= q^{-1}x_-\tilde{z} + \lambda \tilde{v}, \\
\tilde{z}v &= q^{-1}v\tilde{z} + \lambda x_+, & \tilde{z}\tilde{v} &= q\tilde{v}\tilde{z}, \\
z\ell_q &= \ell_qz, & \tilde{z}\ell_q &= \ell_q\tilde{z}. 
\end{align*}
\quad (7.23)
\]

Certainly, the commutation relations (7.23) are also preserved (for $q$ phase) by the conjugation $\omega$ – supplementing (7.22) by $\omega(z) = \tilde{z}$ (all follow from (5.183)). Thus, with this conjugation $Y_q$ becomes a coset of $SU_q(2, 2)$. 

\[\square\]
From (6.260) we know the normally ordered basis of the $q$-coset $\mathcal{Y}_q$ considered as an associative algebra:

$$\hat{\varphi}_{ijklmn} = z^i v_j x^k \hat{x}^l \hat{v}^m x^n, \quad i, j, k, \ell, m, n \in \mathbb{Z}_+. \quad (7.24)$$

Let us denote by $\mathcal{Z}$, $\mathcal{Z}_\bar{q}$, and $\mathcal{M}_q$ the associative algebras with unity generated by $z, \bar{z}$ and $x_+, v, \bar{v}$, respectively. These three algebras are subalgebras of $\mathcal{Y}_q$, and we notice the following structure of $\mathcal{Y}_q$:

$$\mathcal{Y}_q \cong \mathcal{Z} \otimes \mathcal{M}_q \supset \mathcal{Z}_\bar{q}, \quad (7.25)$$

where $A \otimes B$ denotes the tensor product of $A$ and $B$ with $A$ acting on $B$.

### 7.2.2 Multiparameter Quantum Minkowski Space–Time

In this subsection following [221], we shall present the multiparameter version of our quantum Minkowski space–time. We start from the case $n = 4$ of the multiparameter deformation $GL_{q,q}(n)$ of $GL(n)$, which we discussed in Section 4.5. The flag manifold $\mathcal{Y}_{q,q} = GL_{q,q}(n)/\mathcal{B}_{q,q}(n)$ depends on the same number of parameters $(n^2 - n + 2)/2$. For $n = 4$ we pass from the variables $Y_{ij}$ to the variables on the above coset in the manner of (7.18), keeping the same notation as in the one-parameter case of Section 6.5 and the previous subsection. Thus, we obtain the explicit multiparameter commutation relations (instead of (7.19) and (7.23), $\lambda = q - q^{-1}$):

\[
\begin{align*}
    x_+ v &= \frac{q_{23} q_{34}}{q_{24}} x_+ v, \\
    x_- v &= \frac{q_{13}}{q_{12} q_{23}} x_- v, \\
    \bar{v} v &= \frac{q_{13} q_{34}}{q_{12} q_{24}} \bar{v} v, \\
    x_+ x_- &= \frac{q_{12} q_{24}}{q} x_+ x_- + \lambda v \bar{v}, \\
    \bar{z} z &= \frac{q_{13} q_{24} z \bar{z}}{q_{14} q_{23}}, \\
    \bar{z} x_+ &= \frac{q_{13} q_{34} x_+ \bar{z}}{q_{14}}, \\
    \bar{z} \bar{v} &= \frac{q_{23} q_{34} \bar{v} \bar{z}}{q_{24}}, \\
    x_+ z &= \frac{q_{14}}{q_{12} q_{24}} x_+ z, \\
    x_- z &= \frac{q^2 q_{13}}{q_{12} q_{23}} x_- z - \lambda v, \\
    v z &= \frac{q_{13}}{q_{12} q_{23}} v z, \\
    \bar{v} z &= \frac{q^2 q_{14}}{q_{12} q_{24}} \bar{v} z - \lambda x_+.
\end{align*}
\]
Thus, in (7.26) we have the expected seven-parameter quantum Minkowski space–time.

We note that when all deformation parameter are phases; that is, \(|q| = 1, |q_{ij}| = 1\), and in addition hold the following relations:

\[
q_{13} = \frac{q_{12}q_{34}}{q_{34}}, \quad q_{14} = \frac{q_{12}q_{24}^2}{q_{23}q_{34}},
\]

then the commutation relations (7.26) and (7.27) are preserved by the antilinear anti-involution \(\omega\) acting as in the previous subsection.

Further, we recall from Section 4.5.5 that the dual quantum algebra \(U_{q,q}(\mathfrak{gl}(n))\) has the quantum algebra \(U_{q,q}(\mathfrak{sl}(n))\) as a commutation subalgebra but not as a co-subalgebra. In order to achieve the complete splitting of \(U_{q,q}(\mathfrak{sl}(n))\) we have to impose some relations between the parameters; thus, the genuine multiparameter deformation \(U_{q,q}(\mathfrak{sl}(n))\) depends on \((n^2 - 3n + 4)/2\) parameters. Using the same conditions we also ensure that we can restrict from \(GL_{q,q}(n)\) to \(SL_{q,q}(n)\).

Thus, in the case of \(n = 4\) for the genuine \(U_{q,q}(\mathfrak{sl}(4))\) we have four parameters. Explicitly, we achieve this by imposing that the parameters \(q_{i,i+1}\) are expressed through the rest as:

\[
q_{12} = \frac{q^3}{q_{13}q_{14}}, \quad q_{23} = \frac{q^4}{q_{13}q_{14}q_{24}}, \quad q_{34} = \frac{q^3}{q_{14}q_{24}}.
\]

Thus, the four-parameter quantum Minkowski space–time and the embedding quantum flag manifold \(\mathcal{Y}_{q,q}\) are given by (7.26) and (7.27) with (7.29) enforced.

If we would like to enforce also the conjugation \(\omega\), then there are more relations between the deformation parameters, namely, we get:

\[
q_{12} = q_{23} = q_{34} = \frac{q^2}{q_{14}}, \quad q_{13} = q_{24} = q,
\]

and all deformation parameter are phases.

Thus, in this case we have a two-parameter deformation and using the above relations (7.26) and (7.27) simplify as follows:

\[
\begin{align*}
x_+ \nu &= p \nu x_+, \quad \nu x_+ = p^{-1} x_+ \nu, \\
x_- \nu &= p^{-1} \nu x_-, \quad \nu x_- = p x_- \nu, \\
\dot{\nu} \nu &= \nu \dot{\nu}, \\
\frac{q}{p} x_+ x_- &= p \frac{q}{q} x_+ x_- + \lambda \nu \dot{\nu}, \\
\ddot{z} z &= z \ddot{z}, \\
\ddot{x}_+ &= p x_+ \ddot{z}, \\
\ddot{x}_- &= p \frac{q}{q} x_- \ddot{z} + \lambda \nu.
\end{align*}
\]
\[ \bar{z}v = p \bar{v}z, \quad zv = \frac{p}{q^2} vz + \lambda x_+ , \]
\[ x_+ z = p^{-1} z x_+ , \quad x_- z = \frac{q^2}{p} z x_- - \lambda v , \]
\[ vz = p^{-1} z v , \quad \bar{v}z = \frac{q^2}{p} z \bar{v} - \lambda x_+ , \]

where \( p = q^3/q_4^2 \).

### 7.3 \( q \)-Maxwell Equations Hierarchy

We return to the one-parameter setting of Section 7.2.1. We introduce now the representation spaces \( C_7^7 \), \( \chi = [n_1, n_2; d] \). The elements of \( C_7^7 \), which we shall call (abusing the notion) functions, are polynomials in \( z, \bar{z} \) of degrees \( n_1, n_2 \), respectively, and formal power series in the \( q \)-Minkowski variables. (In the general \( U_q(sl(n)) \) situation the signatures \( n_1, n_2 \) are complex numbers and the functions are formal power series in \( z, \bar{z} \) too, cf. (5.38b).) Namely, these functions are given by:

\[
\hat{\phi}_{n_1, n_2}(\bar{Y}) = \sum_{i, j, k, l, m, n \in \mathbb{Z}^+} \mu_{i j k l m n}^{n_1, n_2} \hat{\phi}_{i j k l m n} , \tag{7.33}
\]

where \( \bar{Y} \) denotes the set of the six coordinates on \( \mathcal{Y}_q \). Thus the analogues of \( F^\pm_n, J_n \), cf. (7.13), are:

\[
q F^+_n = \hat{\phi}_{n+2, n}(\bar{Y}) , \quad q F^-_n = \hat{\phi}_{n, n+2}(\bar{Y}) , \quad q J_n = \hat{\phi}_{n+1, n+1}(\bar{Y}) . \tag{7.34}
\]

Using the above we now present explicitly a \( q \) version of the Maxwell hierarchy of equations. We recall that the explicit form of the operators \( I_a \) in (7.16) is obtained by the infinitesimal right action of the three simple root generators of \( sl(4) \) on the coset \( \mathcal{Y} \) (cf. (5.150)). Adapting this to our notation we have for the \( q \)-analogues of \( I_a \) (cf. (6.276)):

\[
q I_1 = \hat{\mathcal{D}}_z T_z T_v T_+ (T_+ T_v)^{-1} \tag{7.35a}
\]
\[
q I_2 = \left( q \hat{M}_z \hat{\mathcal{D}}_v T^2_v + \hat{\mathcal{D}}_+ T_+ + \hat{M}_z \hat{M}_z \hat{\mathcal{D}}_+ T_v T_+ T^2_v + q^{-1} \hat{M}_z \hat{\mathcal{D}}_v - \lambda \hat{M}_z \hat{M}_z \hat{\mathcal{D}}_+ T_v T^2_v \right) \tag{7.35b}
\]
\[
q I_3 = \hat{\mathcal{D}}_z T^2_v . \tag{7.35c}
\]

With this we have now the \( q \)-Maxwell hierarchy of equations – it remains just to substitute the operators of (7.35) in (7.17). In fact, we can also rewrite these in the \( q \)-analog of (7.13). We have:
\[
q I^+_n = \frac{1}{2} \left( (q \hat{\nabla}_n + \hat{M}_z \hat{\nabla}_+ (T_- T_\nu)^{-1} T_\nu) [n + 2 - N_z] q - \right. \\
- q^{-n-2} \left( \hat{\nabla}_- T_\nu + q^{-1} \hat{M}_z \hat{\nabla}_\nu - \right. \\
- \lambda \hat{M}_\nu \hat{M}_z \hat{\nabla}_- \hat{\nabla}_+ T_\nu \right) T_+ T_- T_\nu T_z T_z^{-1}
\]

(7.36a)

\[
q I^-_n = \frac{1}{2} \left( \hat{\nabla}_\nu + q \hat{M}_z \hat{\nabla}_+ T_\nu T_- T_\nu^{-1} - \right. \\
- q \lambda \hat{M}_\nu \hat{\nabla}_- T_\nu \right) T_\nu [n + 2 - N_z] q - \\
- \frac{1}{2} q^{n+3} \left( \hat{\nabla}_- + q \hat{M}_z \hat{\nabla}_\nu T_\nu \right) \hat{\nabla}_- T_\nu T_\nu.
\]

(7.36b)

Clearly, for \( q = 1 \) the operators in (7.35) and (7.36) coincide with (7.15) and (7.16), respectively.

With this the final result for the \( q \)-Maxwell hierarchy of equations is (cf. (7.34)):

\[
q I^+_n q F^+_n = q I_n, \tag{7.37a}
\]

\[
q I^-_n q F^-_n = q I_n. \tag{7.37b}
\]

**Remark 7.3.** Note that our free \( q \)-Maxwell equations, obtained from (7.37) for \( n = 0 \), and \( q J_0 = 0 \), are different from the free \( q \)-Maxwell equations of [472, 508]. The advantages of our equations are (1) they have simple indexless form; (2) we have a whole hierarchy of equations; (3) we have the full equations, and not only their free counterparts; (4) our equations are \( q \)-conformal invariant, not only \( q \)-Lorentz [472], or \( q \)-Poincaré [508], invariant.

Formulae (7.13), (7.11), and (7.12) are part of a much more general classification scheme (mentioned above, cf. [194, 198]) involving also other intertwining operators, and of arbitrary order. A subset of this scheme are two infinite two-parameter families of representations which are intertwined by the same operators (7.14) (cf. [213]). The latter set was called generalized \( q \)-Maxwell hierarchy, the \( q \)-Maxwell hierarchy being just a one-parameter subhierarchy. Explicitly, instead of (7.11), (7.12) we have:

\[
I^+_{n_1, n_2} : C^+_{n_1, n_2} \longrightarrow C^{0+}_{n_1, n_2},
\]

\[
I^+_1 \circ T^+_{n_1, n_2} = T^+_{n_1, n_2} \circ I^+_{n_1, n_2},
\]

(7.38a)

\[
I^-_{n_1, n_2} : C^-_{n_1, n_2} \longrightarrow C^{0-}_{n_1, n_2},
\]

\[
I^-_{n_1, n_2} \circ T^-_{n_1, n_2} = T^-_{n_1, n_2} \circ I^-_{n_1, n_2},
\]

(7.38b)

where \( T^a_{n_1, n_2} = T^{a}_{n_1, n_2} \), \( C^a_{n_1, n_2} = C^{a}_{n_1, n_2} \), \( a = \pm \), or \( a = 0 \pm \), and
\[
\chi_n^{+}, n_2^{-} = [n_1^+ + n_2^+, \frac{n_1^+ - n_2^+}{2} + 1] \tag{7.39a}
\]
\[
\chi_n^{0+}, n_2^{-} = [n_1^+ - 1, n_2^+ + 1; \frac{n_1^+ - n_2^+}{2} + 2], \quad n_1^+ \in \mathbb{N}, n_2^+ \in \mathbb{Z}_+
\]
\[
\chi_n^{-}, n_2^0 = [n_1^-, n_2^-; \frac{n_2^- - n_1^-}{2} + 1] \tag{7.39b}
\]
\[
\chi_n^{0-}, n_2^{-} = [n_1^- + 1, n_2^- - 1; \frac{n_2^- - n_1^-}{2} + 2], \quad n_1^- \in \mathbb{Z}_+, n_2^- \in \mathbb{N},
\]

while instead of (7.13) in the \( q = 1 \) case and (7.37) in the \( q \)-deformed case, we have:

\[
qT_n^+ F_n^{+, n_2^+}(z, \tilde{z}) = J_n^{+}, n_2^+(z, \tilde{z}), \tag{7.40a}
\]
\[
qT_n^- F_n^{-, n_2^-}(z, \tilde{z}) = J_n^{-, n_2^-}(z, \tilde{z}), \tag{7.40b}
\]

where \( qT_n^+ \), \( qT_n^- \) are given by (7.36) (or (7.14) for \( q = 1 \)), while \( F_n^{+, n_2^+}(z, \tilde{z}), J_n^{+, n_2^+}(z, \tilde{z}) \) are polynomials in \( z, \tilde{z} \) of degrees \((n_1^+, n_2^+)\), \((n_1^-, n_2^-)\), \((n_1^+ \pm 1, n_2^\pm 1)\), respectively.

The crucial feature which unifies these representations is the form of the operators \( qT_n^\pm \), which is not generalized anymore in equations (7.40).

We call the hierarchy of equations (7.40) the **generalized \( q \)-Maxwell hierarchy**. The \( q \)-Maxwell hierarchy is obtained in the partial case when \( \chi_n^{0+} = \chi_n^{0-} = \chi_n^0 \) which fixes three of the four parameters: \( n_1^+ = 2 = n_2^- = n_2^- = n_2^+ = 2 = n \).

Another one-parameter subhierarchy of the generalized \( q \)-Maxwell hierarchy involves the two signatures of \( \chi_n^0 = [n + 2, n; 2], \chi_n^- = [n, n + 2; 2] \), and in addition

\[
\chi_n^{00} = [n + 1, n + 1; 1] = [n + 2, -1 - n, n + 2], \quad n \in \mathbb{Z}_+ \tag{7.41}
\]

The intertwining relations are:

\[
I_n^+ : C_n^{00} \rightarrow C_n^{-}, \quad I_n^+ \circ T_n^{00} = T_n^{-} \circ I_n^{+}, \tag{7.42a}
\]
\[
I_n^- : C_n^{00} \rightarrow C_n^{+}, \quad I_n^- \circ T_n^{00} = T_n^{+} \circ I_n^{-} \tag{7.42b}
\]

where \( T_n^{00} = T_n^{00} \), \( C_n^{00} = C_n^{00} \) : Thus, instead of (7.13) in the \( q = 1 \) case and (7.37) in the \( q \)-deformed case, we have:

\[
qI_{n-1}^+ A_n = qF_n^- \tag{7.43a}
\]
\[
qI_{n-1}^- A_n = qF_n^+ \tag{7.43b}
\]

where \( qI_n^\pm \) are given by (7.36) (or (7.14) for \( q = 1 \)), \( qA_n \) has the signature \( \chi_n^{00} \).

This hierarchy will be called the **potential \( q \)-Maxwell hierarchy**. The reason is that the lowest member obtained for \( n = 0 \) and \( q = 1 \) is just:

\[
\partial_{[\mu} A_{\nu]} = F_{\mu\nu}. \tag{7.44}
\]
Of course, as in the classical case these equations have auxiliary character w.r.t. (7.1). One of the reasons for their introduction is to make transparent the gauge invariance of the Maxwell equations. We recall that substituting (7.44) in (7.1b) gives an identity, while from (7.1a) one gets:

\[ \square A_\mu - \partial_\mu \partial^\sigma A_\sigma = J_\mu \] (7.45a)
\[ \square \equiv \partial^\sigma \partial_\sigma \] (7.45b)

Thus the eight equations (7.1) are reduced to the four equations (7.45). The lost equations are actually traded for gauge symmetry:

\[ A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \phi \] (7.46)

since the last substitution leave \( F_{\mu \nu} \) and (7.45) unchanged. One uses (7.46) to simplify (7.45) by setting:

\[ \partial^\sigma A_\sigma = 0 \] (7.47)

This is called the Lorentz gauge condition, and it is equivalent to find suitable \( \phi \). Indeed, if \( \partial^\sigma A_\sigma \neq 0 \), take \( \phi \) so that \( \square \phi = -\partial^\sigma A_\sigma \); then it follows that \( \partial^\sigma A'_\sigma = 0 \). Thus, one may assume that (7.47) holds, and then from (7.45) follows:

\[ \square A_\mu = J_\mu \] (7.48)

Further we note a special gauge symmetry now also of (7.47):

\[ A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \phi_0 \]
\[ \square \phi_0 = 0 \] (7.49)

This may be used to get rid of one component of \( A_\mu \), if the same component of \( J_\mu \) is zero; for example, if \( J_0 = 0 \), take \( \phi_0 \) so that \( \partial_0 \phi_0 = -A_0, A'_0 = 0 \). Thus in this case we have:

\[ \square A_k = J_k, \quad k = 1, 2, 3, A_0 = J_0 = 0 \] (7.50a)
\[ \partial_k A_k = 0. \] (7.50b)

The last condition (7.50)b is called Coulomb gauge condition. This gauge also used when \( \partial_0 A_0 = 0 \).

Let us see how these things are related to representation theory. The fact that using (7.44) Maxwell equations reduces to four equations is expressed group-theoretically for the whole hierarchy by the fact that the two possible composition maps intertwining the "potential" representations \( \chi_n^{00} \) and "current" representations \( \chi_n^0 \) (via \( \chi_n^+ \) or \( \chi_n^- \))
7.3 $q$-Maxwell Equations Hierarchy

Coincide; that is,

\[
q^I_n \circ q^{I-1}_n = q^I_n \circ q^{I-1}_n \equiv q^{\nabla}_n
\]

(7.51a)

\[
q^{\nabla}_n : C_0 \rightarrow C^0, \quad q^{\nabla}_n \circ q^{I0}_n = q^{I0}_n \circ q^{\nabla}_n.
\]

(7.51b)

And the equation is:

\[
q^{\nabla}_n qA_n = q^{I-1}_n \circ q^{I-1}_n qA_n = q^{I-1}_n q^F_n =
\]

\[
= q^{I0}_n \circ q^{I0}_n qA_n = q^{I0}_n q^F_n =
\]

\[
= qJ_n.
\]

(7.52)

Further, as an example we consider the Maxwell case; that is, $n = 0$, setting also $\nabla_0 = qA_0$. After a short calculation we find first for $q = 1$:

\[
\nabla A = \Box A - I_2 (\partial \cdot A) = J
\]

(7.53a)

\[
(\partial \cdot A) \equiv \frac{1}{2}(\partial_+ A_+ + \partial_- A_- - \partial_\nu A_\nu - \partial_\nu A_\nu) - \partial_\mu A_\mu
\]

(7.53b)

Thus, also in this language the suitable gauge condition is the Lorentz one (7.47), while for the elimination of one further degree of freedom here it is more convenient to set $A_+ = 0$ or $A_- = 0$. This is called a light-front gauge condition.

In the $q$-deformed case we have instead of (7.53):

\[
\nabla_q A = \Box_q A - I^2_2 (\partial \cdot A)_q = J
\]

(7.54a)

\[
\Box_q \equiv \left( \partial_+ A_+ - q \partial_- A_- + q \partial_\nu A_\nu \right) T_+ T_+ T_- T_-
\]

(7.54b)

\[
(\partial \cdot A)_q \equiv \frac{1}{2} \left( q^2 \partial_+ T_+ T_+ A_+ + q \partial_+ T_+ T_+ A_- - q^3 \partial_+ T_+ T_+ A_\nu - \partial_\nu T_+ T_+ A_\nu + + q \lambda \tilde{M}_\nu \partial_- \partial_+ T_+ T_+ A_\nu \right).
\]

(7.54c)

Further we consider the free equations, that is, $J = 0$, in the $q$-Lorentz gauge:

\[
\Box_q A = 0,
\]

(7.55a)

\[
(\partial \cdot A)_q = 0.
\]

(7.55b)

Since the first equation is valid component-wise, we can use its $A_\nu$ component to simplify the gauge condition. Thus, finally we have:

\[
\Box_q A = \left( \partial_+ A_+ - q \partial_- A_- + q \partial_\nu A_\nu \right) T_+ T_+ T_- T_-= 0
\]

\[
2(\partial \cdot A)_q = q^2 \partial_+ T_+ T_+ A_+ + q \partial_+ T_+ T_+ A_- - - q^2 \partial_+ T_+ T_+ A_\nu - \partial_\nu T_+ T_+ A_\nu = 0.
\]

(7.56)
7.4 *q*-d’Alembert Equations Hierarchy

Here we consider another one-parameter subhierarchy of the generalized *q*-Maxwell hierarchy which is obtained from (7.39) for $n_1^* = n_2^* = r \in \mathbb{N}$, $n_1^- = n_2^+ = 0$; that is,

\[
\begin{align*}
\chi_{r}^{d+} &= [r, 0; \frac{r}{2} + 1], \\
\chi_{r}^{d0+} &= [r - 1, 1; \frac{r}{2} + 2], \quad r \in \mathbb{N} \\
\chi_{r}^{d-} &= [0, r; \frac{r}{2} + 1], \\
\chi_{r}^{d0-} &= [1, r - 1; \frac{r}{2} + 2], \quad r \in \mathbb{N},
\end{align*}
\] (7.57a)

where the two conjugated equations follow from (7.40):

\[
\begin{align*}
qI^+F_{d}^{d+} &= J_{d}^{d+}, \quad (7.58a) \\
qI^-F_{d}^{d-} &= J_{d}^{d-}, \quad (7.58b)
\end{align*}
\]

where $qI_r^\pm$ is given by (7.36).

For the minimal possible value of the parameter $r = 1$, we obtain the two conjugate *q*-Weyl equations.

The case $r = 2$ gives the *q*-Maxwell equations (note that $J_2^{d+} = J_2^{d-}$). This is the only intersection of the present hierarchy with the *q*-Maxwell hierarchy.

We call this hierarchy *q*-d’Alembert hierarchy following the classical case (cf. [215] and Volume 1), due to the following. We consider the representations $\chi_{a}^{d\pm}$ for the excluded above value $r = 0$, when they coincide. Thus, we set: $\chi^{d} \equiv \chi_{0}^{d\pm} = [0, 0; 1], \quad F^{d} \equiv F_{0}^{d\pm}$. Furthermore, the relevant equation is the *q*-d’Alembert equation [215]:

\[
\Box_{q} F^{d} = J^{d}
\] (7.59)

where the signature of $J^{d}$ is $\chi^{d0} = [0, 0; 3]$, and $\Box_{q}$ is as in (7.56).

Finally, we recall [215] that the solutions of the free equations (7.58) satisfy also the *q*-d’Alembert equation.

### 7.4.1 Solutions of the *q*-d’Alembert Equation

Here and in the next Subsection we follow [226] to find solutions of the *q*-d’Alembert equation (7.59) with trivial RHS:

\[
\Box_{q} F^{d} = 0.
\] (7.60)

We recall that the elements of our representation spaces are formal power series in the variables $x_{\pm}, v, \bar{v}, z, \bar{z}$ of the coset $\mathcal{Y}$. But here there is no dependence on the spin.
variables $z, \bar{z}$ and our solutions will be power series in the $q$-Minkowski variables $x_+, \nu, \bar{\nu}$:

$$\hat{\varphi} = \sum_{j, n, \ell, m \in \mathbb{Z}^+} j \ell m \hat{\varphi}_{j n \ell m}, \quad \hat{\varphi}_{j n \ell m} = \nu^j x_+^n \bar{\nu}^m. \quad (7.61)$$

We substitute the above in $\square_q \hat{\varphi} = 0$ to obtain:

$$\square_q \hat{\varphi} = \sum_{j, n, \ell, m \in \mathbb{Z}^+} j \ell m \square_q \hat{\varphi}_{j n \ell m} = 0, \quad (7.62a)$$

$$\square_q \hat{\varphi}_{j n \ell m} = q^{1+2n+2j+\ell} [n]_q [\ell]_q \hat{\varphi}_{j, n-1, \ell-1, m} - q^{n+j+\ell+m} [j]_q [m]_q \hat{\varphi}_{j-1, n, \ell, m-1}. \quad (7.62b)$$

We first show two polynomial solutions (with $a, b, c, d \in \mathbb{Z}_+$):

$$\hat{\varphi} = \sum_{n=0}^{n_{a,b}} q^{n(c+d+n)} (-a)_n^q (-b)_n^q \nu^{n+d} x_+^{a-n} \bar{\nu}^{c+n}, \quad (7.63)$$

where $(a)_n^q = \Gamma_q(a+n)/\Gamma_q(a)$ is the $q$-Pochhammer symbol,

$$\hat{\varphi}_{a,b,c} = \sum_{n=0}^{n_{a,b}} q^{n(c+n)} (-a)_n^q (-b)_n^q \nu^n x_+^{a-n} \bar{\nu}^{c+n}, \quad (7.64)$$

where $n_{a,b} = \min(a, b)$.

### 7.4.2 $q$-Plane-Wave Solutions

Next we look for solutions of the $q$-d’Alembert equation in terms of a $q$-deformation of the classical plane wave $\exp(k \cdot x)$, where

$$(k \cdot x) = k^\mu x_\mu = \frac{1}{2}(k_+ x_- + k_- x_+ - k_0 \bar{\nu} - k_0 \nu), \quad (7.65)$$

and $(k_+, k_-, k_0, k_\bar{\nu})$ are related to the components $k_\mu$ of the four-momentum as the variables $(\nu, x_-, x_+, \bar{\nu})$ are related to $x_\mu$. Clearly, the natural $q$-deformation of the plane wave is:

$$(\exp(k \cdot x))_q = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} f_s(\nu, x_-, x_+, \bar{\nu}), \quad (7.66)$$

where $f_s$ is a homogeneous polynomial of degrees in both sets of variables $(k_+, k_-, k_0, k_\bar{\nu})$ and $(\nu, x_-, x_+, \bar{\nu})$, such that $(f_s)_q = (k \cdot x)^s$. Thus, we set $f_0 = 1$. One may expect that $f_s$ for $s > 1$ would be equal or at least proportional
to \( (f_1)^s \), but it turns out that this is not the case. In order to proceed systematically, we have to impose the conditions of \( q \)-Lorentz covariance and the \( q \)-d’Alembert equation.

The complexification of the \( q \)-Lorentz subalgebra of the \( q \)-conformal algebra is generated by \( k_j^\pm, X_j^\pm, j = 1, 3 \). Using (6.263a,c) and (6.264a,c) it is easy to check that:

\[
\pi(X_j^\pm) \mathcal{L}_q = 0, \quad \Rightarrow \quad \pi(X_j^\pm) (\mathcal{L}_q)^s = 0, \quad j = 1, 3. \quad (7.67)
\]

Since \( (k \cdot x)^s \) is a scalar as \( (\mathcal{L}_q)^s \), then also the \( q \)-deformations \( f_s \) should be scalars, and thus also should obey (7.67). In order to implement this we suppose that the \( q \)-Minkowski coordinates, and that they commute with the coordinates. Also the ordering of the momentum basis will be the same for the coordinates. Taking all this into account we can see that a natural expression for \( f_s \) is:

\[
f_s = \sum_{a,b,n \in \mathbb{Z}_+} \beta^s_{a,b,n} \frac{(-1)^{s-a-b}}{\Gamma_q(a-n+1) \Gamma_q(b-n+1) [n]_q} \times \frac{k_y^{s-a-b+n} k_-^{b-n} k_+^{a-n} k_0^n v^n x_-^{a-n} x_+^{b-n} \bar{v}^{s-a-b+n}}{\Gamma_q(s-a-b+n+1)}, \quad (7.68)
\]

where we have introduced some factors that are obvious from the correspondence with the case \( q = 1 \). (The expression in (7.68) does not involve terms that would vanish for \( q = 1 \). Actually, we shall see that such expressions would lead to noncovariant momenta light cone.) In order to implement \( q \)-Lorentz covariance we impose the conditions:

\[
\pi(X_j^\pm) f_s = 0, \quad j = 1, 3. \quad (7.69)
\]

For this calculation we suppose that the \( q \)-Lorentz action on the noncommutative momenta is given by (6.79a,c), (6.263a,c), and (6.264a,c). We also have to use the twisted derivation rule which here is:

\[
\pi(X_j^\pm) \psi \cdot \psi' = \pi(X_j^\pm) \psi \cdot \pi(k_j^{-1}) \psi' + \pi(k_j) \psi \cdot \pi(X_j^\pm) \psi', \quad (7.70)
\]

\[
\psi = k_y^{s-a-b+n} k_-^{b-n} k_+^{a-n} k_0^n, \quad \psi' = v^n x_-^{a-n} x_+^{b-n} \bar{v}^{s-a-b+n}.
\]

The four conditions (7.69) bring eight relations between the coefficients \( \beta \); however, only three are independent, namely, the relations:

\[
\beta^s_{a,b,n} = q^{s-2n+a+2b} \beta^s_{a,b-1,n}, \quad (7.71a)
\]
\[
\beta^s_{a,b,n} = q^{s-2n-2a+b} \beta^s_{a-1,b,n}, \quad (7.71b)
\]
\[
\beta^s_{a,b,n} = q^{s+4n-2a-2b-2} \beta^s_{a,b,n-1}, \quad (7.71c)
\]
solving which we find the following solution:

\[ \beta_{a,b,n}^s = q^{n(s-2a-2b+2n)} + a(s-a-1) + b(-s+a+b+1) \beta_{0,0,0}^s, \quad (7.72) \]

that is, for each \( s \geq 1 \) only one constant remains to be fixed.

Next we impose the \( q \)-d’Alembert equation on \( f_s \):

\[ \Box_q f_s = 0, \quad (7.73) \]

which holds trivially for \( s = 0, 1 \). For \( s \geq 2 \) we substitute (7.68) to obtain (for details see [226]):

\[ \Box_q f_s = (q k_+ - k_+ k_+) \times \]

\[ \times \sum_{a,b,n \in \mathbb{Z}_+} \frac{(-1)^{s-a-b}}{\Gamma_q(a-n) \Gamma_q(b-n) \Gamma_q(s-a-b+n+1)} [n]_q! \times \]

\[ \times q^{s-a-b+n} k_+^{b-n-1} k_+^{a-n-1} k_+^n \phi_{n,a-n-1,b-n-1,s-a-b+n} = \]

\[ = (k_+ - q^{-1} k_+ k_+) q^{2s} \beta_{0,0,0}^s f_{s-2}. \quad (7.74) \]

If (7.73) holds then for every \( s \geq 2 \) we obtain (as for \( q=1 \)) the condition that the momentum operators are on the \( q \)-Lorentz covariant \( q \)-light cone (cf. (7.21)):

\[ \mathcal{L}_q^{sk} = k_+ k_+ - q^{-1} k_+ k_+ = 0. \quad (7.75) \]

Now it remains only to fix the coefficient \( \beta_{0,0,0}^s \). We note that for \( q=1 \) it holds:

\[ (k \cdot x)|_{k \to x} = (x \cdot x) = \mathcal{L}, \quad (7.76) \]

and thus we shall impose the conditions:

\[ (f_s)|_{k \to x} = (\mathcal{L}_q^s)^s. \quad (7.77) \]

Next we note that:

\[ (\mathcal{L}_q^s)^s = \sum_{n=0}^{s} (-1)^n \binom{s}{n} q^{n(n-s-1)} \sum_n \frac{s-n}{n+n} \mathcal{L}_q^{s-n} \mathcal{L}_q^{s-n} \mathcal{L}_q^{n}. \quad (7.78) \]

A tedious calculation shows that:

\[ (f_s)|_{k \to x} = \beta_{0,0,0}^s \mathcal{L}_q^s \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q! [s-p]_q!}, \quad (7.79) \]
and comparing (7.79) with (7.77) we finally obtain:

\[
(\beta_{0,0,0}^s)^{-1} = \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q! [s-p]_q!}.
\]  

(7.80)

Note that \((\beta_{0,0,0}^s)^{-1}|_{q=1} = 2^s/s!\), as expected.

Finally, we note that our \(fs\) for \(s > 1\) is not equal, and not even proportional, to \((f_1)^s\). Actually, imposing the \(q\)-d’Alembert equation on \((f_1)^s\) will bring a \(s\)-dependent relation between the momenta, which is not \(q\)-Lorentz covariant. For instance, for \(s = 2\) imposing: \(\Box_q (f_1)^2 = 0\) results in the following condition on the momenta: \([2]_q k_+ k_- = (3 - q^2) k_v k_0\) instead of (7.75) (cf. more details in [226]).

Thus, though our \(q\)-plane wave has some properties analogous to the classical one, it is not an exponent or \(q\)-exponent. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter.

### 7.4.3 \(q\)-Plane-Wave Solutions for Non-Zero Spin

Here we follow [239] looking for solutions of the free equations (7.58).

\[
\begin{align*}
q_I^r \hat{\varphi} &= 0, \quad (7.81a) \\
q_I^{\hat{r}} \hat{\varphi} &= 0. \quad (7.81b)
\end{align*}
\]

We start with (7.81b). As we know from [215] since it depends only on one spin-variable \(\hat{z}\) that equation becomes a couple of equations:

\[
\begin{align*}
([r - N_z]_q \hat{\varphi} + T_v T_v^{-1} - q^{r+1} \hat{\varphi} \hat{z} T_0^{-1} T_v) T_0 \hat{\varphi} &= 0, \quad (7.82a) \\
([r - N_z]_q \hat{\varphi} - q^{r+1} \hat{\varphi} \hat{z} T_0^2 T_0^{-1} T_v) T_v \hat{\varphi} &= 0. \quad (7.82b)
\end{align*}
\]

The spin dependence is encoded in the spin variable \(\hat{z}\) in which the solutions depend polynomially of degree \(r \in \mathbb{N}\). As it was shown in [215], if a function satisfies (7.82) then it satisfies also the \(q\)-d’Alambert equation (7.60). Thus, it is justified to look for solutions in terms of \(q\)-deformation of the plane wave:

\[
\exp_q(k \cdot x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} h_s,
\]

(7.83)

\[
\begin{align*}
h_s = \beta^s \sum_{a,b,n \in \mathbb{Z}_+} (-1)^{s-a-b} q^{n(s-2a-2b+2n) + a(s-a-1) + b(-s+a+b+1)} \Gamma_q(a - n + 1) \Gamma_q(b - n + 1) \Gamma_q(s - a - b + n + 1) [n]_q! \\
\times k_v^{s-a-b+n} k_+^{b-n} k_-^{a-n} k_0^{n} x_v^{s-a-b+n} x_0^{s-a-b+n}.
\end{align*}
\]

(7.84)
This deformation of the plane wave generalizes the one from the previous subsection. To obtain the latter one has to replace \( P_s(\alpha, b) \) by 0. Each \( h_s \) satisfies the \( q \)-d’Alembert equation (7.60) on the momentum \( q \)-cone (7.75).

We look for the solutions of (7.82) in a form analogous to (7.83):

\[
\hat{\varphi} = \sum_{s=0}^{\infty} \frac{1}{|s|!} \hat{\varphi}_s . \tag{7.85}
\]

The solutions are constructed component-wise; that is, we solve (7.82) separately for each \( \hat{\varphi}_s \) and we find that

\[
\hat{\varphi}_s = \sum_{m=0}^{r} \hat{\gamma}_m^{rs} \left( \prod_{i=-r+1}^{m} (k_+ - q^{-A_s} k_v \bar{z}) \right) \times \left( \prod_{j=-m+1}^{0} (k_v - q^{-A_s} k \_ \bar{z}) \right) h_s , \tag{7.86a}
\]

\[
P_s(\alpha, b) = A_s a + P_s(b), \tag{7.86b}
\]

where \( \hat{\gamma}_m^{rs} \) are \( r + 1 \) independent constants, \( A_s \) is an arbitrary constant, and \( P_s(b) \) is an arbitrary polynomial in \( b \).

In order to be able to write the general solution of the system (7.82) in terms of the deformed plane wave we have to suppose that the \( \hat{\gamma}_m^{rs} \) and \( A_s \) for different \( s \) coincide: \( \hat{\gamma}_m^{rs} = \hat{\gamma}_m^r, A_s = A \). Then we have:

\[
\hat{\varphi} = \sum_{m=0}^{r} \hat{\gamma}_m^r \left( \prod_{i=-r+1}^{m} (k_+ - q^{-A} k_v \bar{z}) \right) \times \left( \prod_{j=-m+1}^{0} (k_v - q^{-A} k \_ \bar{z}) \right) \exp_q(k \cdot x) . \tag{7.87}
\]

We pass now to equation (7.81a). As in the first case it produces a couple of equations:

\[
\left( [r - N_z]_q D_v - q^T D_z D_\_ T_\_ \right) T_v \dot{\varphi} = 0 , \tag{7.88a}
\]

\[
\left( [r - N_z]_q D_+ T_{\_}^{-1} - q^T D_z D_\_ T_\_ T_v \right) T_\_ \dot{\varphi} = 0 . \tag{7.88b}
\]

As found in [247] for these equations we need to use a basis conjugate to the basis in (7.61); that is,

\[
\tilde{\varphi} = \sum_{j,n,\ell,m \in \mathbb{Z}_+} \mu_{jn\ell\ell} \tilde{\varphi}_{jn\ell\ell} , \tag{7.89}
\]

\[
\tilde{\varphi}_{jn\ell\ell} = v_J^m \_m \, x_\_ x_\_ v_\_ = \omega(\tilde{\varphi}_{jn\ell\ell}) .
\]
We also recall that here the \( q \)-d’Alembert equation is slightly different [239]:

\[
\left( \tilde{\mathcal{D}} - \tilde{\mathcal{D}} + q \tilde{\mathcal{D}} \right) T_+ \tilde{\varphi} = 0
\]

though it coincides with (7.56) when \( q = 1 \).

Analogously to the first case we use the expansion

\[
\tilde{\varphi} = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \tilde{\varphi}_s,
\]

and we solve it again component-wise. Here we shall use another deformation of the plane wave:

\[
\hat{\exp}_q(k \cdot x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{h}_s,
\]

where \( Q_s(a, b) \) are arbitrary polynomials. The \( \hat{h}_s \) has the same properties as the \( h_s \), but the conjugated basis is used; in particular, they satisfy the \( q \)-d’Alembert equation (7.90) on the momentum \( q \)-cone (7.75). The solutions of (7.88) are polynomials of degree \( r \) in the spin variable \( z \). Explicitly they are given by:

\[
\tilde{\varphi}_s = \sum_{m=0}^{r} \tilde{\gamma}_m^{rs} \left( \prod_{i=r+2}^{m+1} (k_+ - q^i B_s k_+ z) \right) \times
\]

\[
\times \left( \prod_{j=m+2}^{m+1} (k_+ - q^j B_s k_+ z) \right) \tilde{h}_s,
\]

\[
Q_s(a, b) = Q_s(a) + B_s b,
\]

where \( \tilde{\gamma}_m^{rs} \) are \( r + 1 \) independent constants, \( Q_s(a) \) is an arbitrary polynomial in \( a \), and \( B_s \) is an arbitrary constant. In order to be able to write the general solution of the system (7.88) in terms of the deformed plane wave, we have to suppose that the \( \tilde{\gamma}_m^{rs} \) and \( A_s \) for different \( s \) coincide: \( \gamma_m^{rs} = \gamma_m^r \) and \( A_s = B \). Then we have:
\[ \Phi = \sum_{m=0}^{r} \tilde{q}_{m} \left( \prod_{i=-r+2}^{m+1} \left( k_{i} - q^{iB} k_{i} z \right) \right) \times \left( \prod_{j=-m-2}^{1} \left( k_{j} - q^{jB} k_{j} z \right) \right) e^{\Phi q_{i}(k \cdot \chi)}. \] (7.95)

### 7.5 q-Plane-Wave Solutions of the Potential q-Maxwell Hierarchy

Here we use results of [240]. We mentioned that the q-d’Alembert hierarchy for \( r = 2 \) intersects with the q-Maxwell hierarchy for \( n = 0 \). Thus, we shall identify \( \Phi, \varphi \) at \( r = 2 \) from the previous subsection with \( q F_{0}^{z} \)

\[ \hat{\Phi}_{r=2} = q F_{0}^{0}, \quad \varphi_{r=2} = q F_{0}^{i} \] (7.96)

Accordingly, we would like to use the solutions for \( \hat{\Phi}, \varphi \) in equations (7.43):

\[ \begin{align*}
q F_{-1} q A_{0}^{0} &= q F_{0}^{0} = \Phi_{r=2}, \\
q F_{-1} q A_{0}^{i} &= q F_{0}^{i} = \varphi_{r=2}.
\end{align*} \] (7.97a, 7.97b)

We start with solving (7.97a) for \( q A_{0}^{0} \), with \( q F_{0}^{0} = \Phi \) given by (7.87). We write:

\[ \begin{align*}
q A_{0}^{0} &= \ddot{z} A_{+} + z A_{+} + \ddot{z} A_{+} + A_{-} = \sum_{s=0}^{\infty} \frac{1}{[s]_{q}!} q A_{s}^{0} h_{s+1}^{+} \\
A_{x} &= A_{x}(k, \chi) = \sum_{s=0}^{\infty} \frac{1}{[s]_{q}!} A_{x}^{s}(k) h_{s+1}^{\pm}, \quad \chi = \pm, \nu, \tilde{\nu}.
\end{align*} \] (7.98, 7.99)

Substituting, we take into account that the action of \( q F_{-1}^{+} \) converts \( h_{s+1}^{+} \) into \( h_{s}^{+} \), but this requires \( P_{s+1}^{-}(a, b) = P_{s}^{-}(a, b) = P^{-}(a, b) = Ca + Q(b) = Ca + Bb \) (in the last step we use the fact that \( Q(b) \) has to be linear in \( b \) and any constant term would be absorbed in the constant \( \tilde{\gamma} \)). Then comparing the coefficients of 1, \( \ddot{z} \), \( \ddot{z}^{2} \) we obtain, respectively:

\[ \begin{align*}
(q s^{+} A_{c}^{s}(k) k_{\nu} + q^{-1} A_{c}^{s}(k) k_{\nu} ) h_{s}^{+} &= \\
= -2d_{s} (q s^{+} k_{1} + \tilde{\gamma}_{1}^{s} k_{1} k_{\nu} + \tilde{\gamma}_{2}^{s} k_{1}^{2} ) h_{s}^{+}, \\
(q s^{+} s^{+} A_{c}^{s}(k) k_{-} - q s^{+} s^{+} s^{+} A_{c}^{s}(k) k_{-} - q^{-2} A_{c}^{s}(k) k_{-} + q^{-1} A_{c}^{s}(k) k_{-} ) h_{s}^{+} &= \\
= -2d_{s} [2q q^{-C} (q^{s+1} k_{1} k_{1} + \tilde{\gamma}_{1}^{s} k_{1} k_{\nu} + \tilde{\gamma}_{2}^{s} k_{1} k_{\nu} ) h_{s}^{+} \\
(q s^{+} s^{+} A_{c}^{s}(k) k_{-} - q^{-2} A_{c}^{s}(k) k_{-} ) h_{s}^{+} &= \\
2d_{s} q^{-2C} (q^{s+1} k_{1} + \tilde{\gamma}_{1}^{s} k_{1} k_{\nu} + \tilde{\gamma}_{2}^{s} k_{2} ) h_{s}^{+} \quad (7.100)
\end{align*} \]

Note, however, that only two of these three equations are independent when they are compatible (see below). Furthermore, we see that \( A_{x}^{s}(k) \) should be linear in \( k \) and in fact should be given as follows:

\[ \begin{align*}
\end{align*} \]
\[ A^s_\pm(k) = \pm \lambda^s_\pm k_\pm + v^s_\pm k_\mp, \quad A^s_\mp(k) = \pm \lambda^s_\mp k_\pm + v^s_\pm k_\mp, \quad (7.101) \]

where for the constants we have:

\[ \begin{align*}
\lambda^s_v &= -2d_s q^{1+B} v_0^{s-}, \\
v^s_v &= -q^{s+6+B} v_0^{s-}, \\
\lambda^s_+ &= 2d_s q^{1-2C} v_0^{s-}, \\
v^s_+ &= -q^{s+6+B} v_0^{s+} + 2d_s q^{2-2C} v_0^{s-}, \\
C &= -B,
\end{align*} \quad (7.102)\]

where the last condition arises from compatibility between the equations (7.100).

Now we substitute this result for \( q^A \) in (7.97b). It turns out that we obtain a result compatible with the general solution above only when \( B = C = 0 \). Thus, in fact \( q^A \) is given in terms of the original components \( f_{s+1} \) (cf. (7.84)). Furthermore, the action of \( q^{A-1} \) converts \( f_{s+1} \) into \( h^+_s \), with \( P^s_s(a, b) = -2b, B_s = 2 + s \). The result is:

\[ \Phi_s = (qF^s_0)_s = q^{A-1} A^s_0 = \]

\[ = -q^{s+1} \left( \frac{v^s_0 + v^s_1}{2d_s} \right) (k_+ - q^{s+2} z k_0)(k_0 - q^{s+3} z k_-) h^+_s, \]

which is a special case of the general solution (7.94), with \( \gamma^2_0 = \gamma^2_2 = 0, \gamma^2_1 = -q^{s+1}(v^s_0 + v^s_1)/2d_s \). Thus, the resulting \( q^F^0 \) is not given in terms of the \( q \)-plane wave (only componentwise).

Let us now repeat the calculations in the other order, namely, we solve (7.97b) for \( q^A \) with \( q^F \) given by (7.94), but since we want this to be compatible with what we obtained above we take: \( P^s_s(a, b) = -2b \). We use again the decomposition (7.98) but with \( f_{s+1} \) instead of \( h_{s+1} \). Substituting and comparing the coefficients of 1, \( z \), \( z^2 \), we obtain, respectively:

\[ \begin{align*}
(q^{s+1} A^s_0(k) k_0 + q^{s+2} A^s_v(k) k_0) h^+_s &= \\
= -2d_s (\gamma^{s+}_0 k_0^2 + \gamma^{s+}_1 k_0 k_1 + \gamma^{s+}_2 k_0^2) h^+_s, \\
(q^s A^s_0(k) k_0 + q^{s+1} A^s_v(k) k_0 - q^s A^s(v) k_0 - q^{-1} A^s_0(k) k_0) h^+_s &= \\
= -2d_s 2q (\gamma^{s+}_0 k_0 k_1 + \gamma^{s+}_1 k_0 k_0 + \gamma^{s+}_2 k_0 k_0) h^+_s, \\
(q^{-2} A^s_0(k) k_0 + q^{-3} A^s_v(k) k_0) h^+_s &= \\
= 2d_s (\gamma^{s+}_0 k_0^2 + \gamma^{s+}_1 k_0 k_0 + \gamma^{s+}_2 k_0^2) h^+_s.
\end{align*} \]

(7.104)

Now instead of (7.101) we have:

\[ \begin{align*}
A^s_\pm(k) &= \mu^s_\pm k_\pm + v^s_\pm k_\mp, \\
A^s_\mp(k) &= \mu^s_\mp k_\pm + v^s_\pm k_\mp,
\end{align*} \quad (7.105)\]
where from the constants $\mu^s, \nu^s$ only six can be determined (due to the gauge freedom). Making some choice we find:

\[
\begin{align*}
\mu^s &= -2d_s q^{-s-1} \gamma_0^s, \quad \mu^s_v = -2d_s q^{-s-2} \gamma_1^s, \\
\nu^s_v &= - \nu^s - 2d_s q^{-s-2} \gamma_1^s, \\
\mu^s &= 2d_s q^3 \gamma_0^s, \quad \mu^s_s = 2d_s q^2 \gamma_2^s, \\
\nu^s &= - \nu^s + 2d_s q^2 \gamma_2^s.
\end{align*}
\]  

(7.106)

Now we can substitute this result for $qA^0$ in (7.97a). The action of $qI^+ \rightarrow 1$ converts $f_{s+1}$ into $f_s$, and we obtain for the components:

\[
\hat{F}^+_s = - \frac{(\nu^s q^{-2} + \nu^s q^{s+2})}{2d_s} (k_+ - q^{-1} k_\nu \hat{z})(k_+ - k_\nu \hat{z}) f_s, 
\]

(7.108)

which is consistent with the solution (7.86), with $\gamma_0^2 = \gamma_2^2 = 0, \gamma_1^2 = -(\nu^s q^{-2} + \nu^s q^{s+2})/2d_s$. Thus, the resulting $qF_0^+$ is not given in terms of the $q$-plane wave (only componentwise).

Finally, we impose that we use the same $qA^0$ for $qF^+_0 = \tilde{\varphi}_{r=2}$ and $qF^-_0 = \tilde{\varphi}_{r=2}$. Then instead of (7.101) and (7.105) we have:

\[
A^s_+ = \nu^s_+ k_-, \quad A^s_- = \nu^s_- k_+ , \quad A^s_v = \nu^s_v k_v, \quad A^s_\nu = \nu^s_\nu k_\nu, 
\]

(7.109)

where from the four constants in (7.109) only three can be determined since their sum is zero:

\[
\nu^s_+ + \nu^s_- + \nu^s_v + \nu^s_\nu = 0
\]

(7.110)

and using (7.102) and (7.106) we have:

\[
\begin{align*}
\nu^s &= -q^{s+4} \nu^s_+ - 2d_s q^2 \gamma_1^s, \quad \nu^s_v = -\nu^s_+ - 2d_s q^{-s-2} \gamma_1^s, \\
\nu^s &= q^{s+4} \nu^s_- + 2d_s q^2 (\gamma_0^s + \gamma_1^s).
\end{align*}
\]  

(7.111)

The disappearance of the constants $\lambda^s, \mu^s$ is consistent with $\gamma_0^{s+} = \gamma_2^{s+} = 0$. Substituting (7.106) in (7.103) and (7.108) we obtain, respectively:

\[
\hat{F}^+_s = \gamma_1^s q^{-1} (k_+ - q^{s+2} z k_\nu)(k_+ - q^{s+3} z k_-) f_s^+, 
\]

(7.112)

\[
\hat{F}^-_s = \gamma_1^s (k_+ - q^{-1} k_\nu \hat{z})(k_+ - k_\nu \hat{z}) f_s. 
\]

(7.113)

We stress that for each $s$ there are only three independent constants: $\gamma_0^{s+}, \nu^s$, the latter entering only the expressions for the $q$-potentials (7.109) and being a manifestation of the gauge freedom. We can eliminate the $A_-$ components by setting $\nu^s_- = 0$ and/or the $A_+$ components by setting $\gamma_1^{s+} = -\gamma_1^{s-} - q^{s+2} \nu^s /2d_s$. 


Finally we note that we can write $qF_0^-$ in terms of $\hat{\exp}_q(k, x)$ but not $qF_0^+$ because of the $s$ dependence in the prefactors. If we use the basis (7.89) the roles of $qF_0^-$ and $qF_0^+$ would be exchanged.

If we want $qF_0^+$ on an equal footing then one should consider $qF_0^-$ on the basis (7.61) and $qF_0^+$ on the basis (7.89). However, then one should use two different $q$-potentials and furthermore should ensure that the two are not mixing because of the equations (7.40); that is, the $q$-potential obtained from solving from one of the equations (7.40) should give zero contribution after substitution in the other. This is easy to ensure through the gauge-freedom constants in the $q$-potentials, for example, setting $\hat{v}^s + v^s = 0$ we obtain that $\hat{F}_s^+ = 0$ in (7.103). Thus, the fields $qF_0^+$ and $qF_0^-$ may be seen as living on different copies of $q$-Minkowski space–time, similarly to the two four-dimensional sheets in the Connes–Lott model [153].

### 7.6 $q$-Plane-Wave Solutions of the Full $q$-Maxwell Equations

Here we use results from [237, 238]. First we shall use the basis (7.61). The general solutions of (7.37) for $n = 0$ in the homogeneous case ($J = 0$) are:

$$F^{h\pm} = (qF_0^\pm)_{f=0} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q^1} \hat{F}^{h\pm}_{ms}(k) f_s, \quad (7.114)$$

$$\hat{F}^{h\pm}_{ms}(k) = \sum_{i=0}^{m} \left( \sum_{j=0}^{m-i} \hat{p}_{ij}^{ms1} k^i k^{m-i-j} k^j (k_v - q^{s+6} z k_\perp)(k_v - q^{s+3} z k_\parallel) + \right.\right.$$  

$$+ \left. \hat{p}_{ij}^{ms2} k^i k^j (k_v - q^{s+6} z k_\perp)(k_v - q^{s+3} z k_\parallel) + \right.$$  

$$+ \left. \sum_{j=0}^{m-i} \hat{p}_{ij}^{ms3} k^i k^{m-i-j} k^j (k_v - q^{s+6} z k_\perp)(k_v - q^{s+3} z k_\parallel) \right), \quad (7.115)$$

$$\hat{F}^{h-}_{ms}(k) = \sum_{i=0}^{m} \left( \sum_{j=0}^{m-i} \hat{r}_{ij}^{ms1} k^i k^{m-i-j} k^j (k_v - q^{-1} z k_\perp)(k_v - \bar{z} k_\perp) + \right.\right.$$  

$$+ \left. \hat{r}_{ij}^{ms2} k^i k^j (k_v - q^{-1} z k_\perp)(k_v - \bar{z} k_\perp) + \right.$$  

$$+ \left. \sum_{j=0}^{m-i} \hat{r}_{ij}^{ms3} k^i k^{m-i-j} k^j (k_v - q^{-1} z k_\perp)(k_v - \bar{z} k_\perp) \right), \quad (7.116)$$

where $\hat{p}_{ij}^{msa}$, $\hat{r}_{ij}^{msa}$ are independent constants. The check that these are solutions is done as in the previous sections. Actually, the solution for $\hat{F}_{r=2}$ given in (7.86) is obtained here for $m = 0$. As for (7.86) going to (7.87) the solution (7.117) can be written in terms of the deformed plane wave if we suppose that the $\hat{r}_{ij}^{msa}$ for different $s$ coincide: $\hat{r}_{ij}^{msa} = \hat{r}_{ij}^{msa}$. Then we have:
\[ \tilde{F}^{h-} = \sum_{m=0}^{\infty} \tilde{F}^{h-}_m(k) \exp_q(k, x), \quad \tilde{F}^{h-}_m(k) = \tilde{F}^{h-}_m(k). \] (7.117)

Also as before the solution (7.115) cannot be written in terms of the same deformed plane wave.

In the inhomogeneous case the solutions of (7.37) for \( n = 0 \) are:

\[ q^0 = 2z\tilde{j}_+ + z\tilde{j}_v + z\tilde{j}_q + \tilde{j}_- , \] (7.118)

\[ \tilde{j}_\kappa = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \tilde{J}_\kappa^m(k)f_{s-1}, \quad \kappa = \pm, v, \nu , \] (7.119)

\[ \tilde{j}_+^m(k) = -\tilde{K}_m^s(k)k_-, \] (7.120)

\[ \tilde{j}_-^m(k) = -q^{-2s} \tilde{K}_m^s(k)k_+, \] (7.121)

\[ \tilde{j}_v^m(k) = \tilde{K}_v^s(k)k_v, \] (7.122)

\[ \tilde{j}_\nu^m(k) = q^{-2s} \tilde{K}_v^s(k)k_\nu, \] (7.123)

\[ \tilde{K}_m^s(k) = \gamma_v^s k_v^{m+1} + \gamma_k^{s} k_\nu^{m+1} + \gamma_k^s k_\nu^{m+1} + \gamma_k^s k_\nu^{m+1}, \] (7.124)

\[ q F_0^\pm = \tilde{F}^\pm + \tilde{F}^{h-} , \] (7.125)

\[ \tilde{F}^{\pm} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \tilde{F}_m^\pm(k)f_s, \] (7.126)

\[ \tilde{F}_m^+(k) = 2d_s q^{-s} ((q^{-s-5} \gamma_v^s k_\nu^{m+1} + z\gamma_v^s k_v^{m+1} )(k_v - q^{s+3} z k_\nu) + \) \]

\[ + (q^{-s-5} \gamma_v^s k_\nu^{m+1} + z\gamma_v^s k_v^{m+1} )(k_v - q^{s+3} z k_\nu)), \] (7.127)

\[ \tilde{F}_m^-(k) = 2d_s q^{-2s-2} ((\gamma_v^s k_\nu^{m+1} + q^{-2s-2} \gamma_v^s k_v^{m+1} )(k_v - z k_\nu) + \) \]

\[ + (\gamma_v^s k_v^{m+1} + q^{-2s-2} \gamma_v^s k_v^{m+1} )(k_v - z k_\nu), \] (7.128)

where \( d_s = \beta^s/\beta^{s+1} \). As in the homogeneous case we can make \( \tilde{F}_m^-(k) \) independent of \( s \) by choosing \( \gamma_v^s \sim q^{-2s} d_s^{-1} \), but we cannot make \( \tilde{F}_m^+(k) \) or \( \tilde{j}_\kappa^m(k) \) independent of \( s \).

Since we work with the full Maxwell equations, we have also to check the \( q \)-deformation of the current conservation \( \partial^\nu j_\nu = 0 \):

\[ I_{13} = 0 , \] (7.129)

\[ I_{13} = q^3 [N_v - 1]q T_z \hat{d}_z \hat{d}_v T_v T_+ + q \hat{d}_z T_z \hat{d}_v T_v T_+ + \] \[ + q [N_v - 1]q T_z [N_v - 1]q \hat{d}_+ T_v T_+ + \] \[ + q^{-1} [N_v - 1]q \hat{d}_z T_z \hat{d}_v T_v T_- - \] \[ - \lambda \hat{M}_v [N_v - 1]q \hat{d}_z T_z \hat{d}_v T_v T_- T_+ T_+ \] (7.130)
Substituting (7.118 and 7.119) in the above we get:

\[ q f^i_j(k_\pm, k_v^\pm, k_v^\mp) + q f^i_j(k_v^\pm, k_v^\mp, k_v^\mp) = 0. \]  

(7.125)

The latter is fulfilled by the explicit expressions in (7.120), but we should note that these expressions fulfil also the following splittings of (7.125):

\[ q f^i_j(k_\pm, k_v^\pm, k_v^\mp) + q f^i_j(k_v^\pm, k_v^\mp, k_v^\mp) = 0, \quad q f^i_j(k_v^\pm, k_v^\mp, k_v^\mp) = 0, \]  

(7.126)

Furthermore the expressions from (7.120) fulfil also:

\[ q f^i_j(k_\pm, k_v^\pm, k_v^\mp) = 0, \quad q f^i_j(k_\pm, k_v^\pm, k_v^\mp) = 0, \]  

(7.127)

Now we shall use the basis (7.89). Then solutions of (7.37) for \( n = 0 \) in the homogeneous case \( (J = 0) \) are:

\[
F^{h+} \equiv (q F^i_0)_{j=0} = \sum_{m, s=0}^{\infty} \frac{1}{[s] q} F^{h+}_{ms}(k) \tilde{h}_s, \tag{7.128}
\]

\[
F^{h+}_{ms}(k) = \sum_{i=0}^{m-i} \left( \sum_{j=0}^{m-i} P^{ms}_{ij} k_i^j k_v^m-k_v^i-j (k_\pm - z k_v) (k_v - q z k_v) +
\right.

\left. + P^{ms}_{ij} k_v^i k_v^m-k_v^i-j (k_\pm - z k_v) (k_\pm - q z k_v) +
\right)

\left. + \sum_{j=0}^{m-i} P^{ms}_{ij} k_v^i k_v^m-k_v^i-j (k_\pm - z k_v) (k_\pm - q z k_v) \right), \tag{7.129}

\[
F^{h-}_{ms}(k) = \sum_{i=0}^{m-i} \left( \sum_{j=0}^{m-i} \text{tr}^{ms}_{ij} k_i^j k_v^m-k_v^i-j (k_\pm - q z k_v) (k_\pm - q z k_v) +
\right.

\left. + \text{tr}^{ms}_{ij} k_v^i k_v^m-k_v^i-j (k_\pm - q z k_v) (k_\pm - q z k_v) +
\right)

\left. + \sum_{j=0}^{m-i} \text{tr}^{ms}_{ij} k_v^i k_v^m-k_v^i-j (k_\pm - q z k_v) (k_\pm - q z k_v) \right), \tag{7.130}
\]

where \( P^{ms}_{ij}, \text{tr}^{ms}_{ij} \) are independent constants, \( Q_s(a, b) = 0 \) in \( \tilde{h}_s \). (The solution for \( \tilde{\varphi}_{r>2} \) given in (7.94) is obtained here for \( m = 0 \).) The solution (7.129) can be written in terms of the deformed plane wave if we suppose that the \( P^{ms}_{ij} \) for different \( s \) coincide: \( P^{ms}_{ij} = P^{ms}_{ij} \). Then we have:
\[ \tilde{F}_{m+} = \sum_{m=0}^{\infty} \tilde{F}_{m+}^{h}(k) \exp_q(k, x), \quad \tilde{F}_{m}^{h}(k) = \tilde{F}_{ms}^{h}(k). \]  

(7.131)

In the inhomogeneous case the solutions of (7.37) for \( n = 0 \) are:

\[ q f^0 = 2z \tilde{J}_+ + z \tilde{J}_v + z \tilde{J}_r + \tilde{J}_- , \]  

(7.132)

\[ \tilde{J}_k = \sum_{m,s=0}^{\infty} \frac{1}{[s]q^1} \tilde{J}_{ms}^{k}(k) \tilde{h}_{s-1}, \quad \kappa = \pm, v, \tilde{v}, \]  

(7.133)

\[ \tilde{J}_{ms}^{+}(k) = -q^{s+1} \tilde{K}_m^{s}(k) k_-, \]  

(7.134)

\[ \tilde{J}_{ms}^{-}(k) = -q^{-1} \tilde{K}_m^{s}(k) k_+, \]  

\[ \tilde{J}_{ms}^{v}(k) = \tilde{K}_m^{s}(k) k_v, \]  

\[ \tilde{J}_{ms}^{\tilde{v}}(k) = q^s \tilde{K}_m^{s}(k) k_{\tilde{v}}, \]  

\[ \tilde{K}_m^{s}(k) = \tilde{K}_s^{m+1} k_v^s + \tilde{K}_s^{m-1} k_v^{-s} + \tilde{K}_s^{m+1} k_v^{s+1} + \tilde{K}_s^{m-1} k_v^{-s-1}, \]  

\[ q F_0^{\pm} = \tilde{F}^{\pm} + \tilde{F}_{ms}^{h}, \]  

(7.135)

\[ \tilde{F}^{\pm} = \sum_{m,s=0}^{\infty} \frac{1}{[s]q^1} \tilde{F}_{ms}^{\pm}(k) \tilde{h}_s, \]  

(7.136)

\[ \tilde{F}_{ms}^{\pm}(k) = 2d_s q^{s-2} \left( (\tilde{K}_s^{m+1} k_v^s + q^{-1} z \tilde{K}_s^{m-1} k_v^{-s})(k_v - qz k_+) + \right) \\
+ (\tilde{K}_s^{m+1} k_v^s + q^{-1} z \tilde{K}_s^{m-1} k_v^{-s})(k_v - qz k_+), \]  

\[ \tilde{F}_{ms}^{-}(k) = 2d_s \left( (q^{-s-3} \tilde{K}_s^{m+1} k_v^s + q z \tilde{K}_s^{m-1} k_v^{-s})(k_v - q^{-s-2} z k_+) + \right) \\
+ (q^{-s-3} \tilde{K}_s^{m+1} k_v^s + q z \tilde{K}_s^{m-1} k_v^{-s})(k_v - q^{-s-2} z k_+), \]  

where \( \tilde{d}_s = b^s / b^{s+1} \), \( Q_s(a, b) = 0 \) in \( \tilde{h}_s \). We can make \( \tilde{F}_{ms}(k) \) independent of \( s \) by choosing \( \tilde{y}_s^{\pm} \sim q^{-s} \tilde{d}_s \), but we cannot make \( \tilde{F}_{ms}^{-}(k) \) or \( \tilde{J}_{ms}^{\pm}(k) \) independent of \( s \).

Also here we shall check whether the \( q \)-deformation of the current conservation (7.123) is fulfilled. The analog of (7.124) in the basis (7.89) is:

\[ I_{13} = [N_z - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_v T_v T_+ T_-^{-1} + q [N_z - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_v T_v T_+ \]  

\[ + q [N_z - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_v T_v T_+ T_- \]  

\[ + q^2 [N_z - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_v T_v T_- T_+ - \]  

\[ - \lambda q \hat{M}_v [N_z - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_\tilde{v} T_+ T_- T_+ . \]  

(7.137)
Then the analog of (7.125) is:

\[
J^s_\pm(k) k_\pm + q^s J^s_\pm(k) k_\mp + q^s f^s_\pm(k) k_\mp = 0. \tag{7.138}
\]

The latter is fulfilled by the explicit expressions in (7.134), but we should note that these expressions fulfil also the following splittings of (7.138):

\[
\begin{align*}
J^s_+(k) k_+ + q^s J^s_+(k) k_\mp & = 0, & J^s_-(k) k_+ + q^s f^s_-(k) k_\mp & = 0, \\
J^s_+(k) k_+ + J^s_+(k) k_\mp & = 0, & J^s_-(k) k_+ + J^s_-(k) k_\mp & = 0. \tag{7.139}
\end{align*}
\]

Furthermore the expressions from (7.134) fulfil also:

\[
\begin{align*}
J^s_+(k) k_\mp + q^s J^s_+(k) k_\mp & = 0, & J^s_-(k) k_\mp + q^s f^s_-(k) k_\mp & = 0, \\
J^s_+(k) k_\mp + J^s_+(k) k_\mp & = 0, & J^s_-(k) k_\mp + J^s_-(k) k_\mp & = 0. \tag{7.140}
\end{align*}
\]

Summarizing, we have given solutions of the full \( q \)-Maxwell equations in two conjugated bases (7.61) and (7.89). The solutions of the homogeneous equations are also more general than the solutions for \( \hat{F}_+ \) and \( \hat{F}_- \) for general \( r \). As before we see that the roles of the solutions \( F^+ \) and \( F^- \) are exchanged in the two conjugated bases. We note also that the current components are different: \( J^m_{\kappa s} \neq J^m_{\kappa s} \) (for \( q \neq 1, \kappa \neq \nu \)), and in both cases they cannot be made independent of \( s \). Thus, there is no advantage of choosing either of the bases (7.61) or (7.89). It may be also possible to use both in a Connes-Lott type model [153].

### 7.7 \( q \)-Weyl Gravity Equations Hierarchy

In this section we follow [229, 238]. Here we study another hierarchy which is given as follows:

\[
\begin{array}{c}
C_m^+ \\
\downarrow \\
C_m^h \\
\downarrow \\
C_m^T \\
\downarrow \\
C_m^- \\
\end{array}
\]

(7.141)

where \( m \in \mathbb{N} \), and the corresponding signatures are:

\[
\begin{align*}
\chi^+_m &= [2m, 0; 2], & \chi^-_m &= [0, 2m; 2], \\
\chi^h_m &= [m, m; 2 - m], & \chi^T_m &= [m, m; 2 + m]. \tag{7.142}
\end{align*}
\]
For future reference we also give the Dynkin labels \( \chi = \{ m_1, m_2, m_3 \} \) of these representations:

\[
\begin{align*}
\chi_m^+ &= \{ 2m + 1, -m - 1, 1 \}, & \chi_m^- &= \{ 1, -m - 1, 2m + 1 \}, \\
\chi_m^h &= \{ m + 1, -1, m + 1 \}, & \chi_m^T &= \{ m + 1, -2m - 1, m + 1 \}.
\end{align*}
\] (7.143)

The arrows on (7.141) represent invariant differential operators of order \( m \). It is a partial case of the general conformal scheme parametrized by three natural numbers \( p, \nu, n \) (cf. formula (6.170) and figure (6.171) of Volume 1), setting here: \( \nu = 1, p = n = m \). This hierarchy intersects with the Maxwell hierarchy for the lowest value \( m = 1 \). Here we consider the linear Weyl gravity which is obtained for \( m = 2 \).

### 7.7.1 Linear Conformal Gravity

We start with the \( q = 1 \) situation, and we first write the linear conformal gravity equations, or Weyl gravity equations in our indexless formulation, trading the indices for two conjugate variables \( z, \bar{z} \).

Weyl gravity is governed by the Weyl tensor \( C_{\mu \nu \alpha \tau} \), which is given in terms of the Riemann curvature tensor \( R_{\mu \nu \alpha \tau} \), Ricci curvature tensor \( R_{\mu \nu} \), scalar curvature \( R \):

\[
C_{\mu \nu \alpha \tau} = R_{\mu \nu \alpha \tau} - \frac{1}{2} (g_{\mu \alpha} R_{\nu \tau} + g_{\nu \tau} R_{\mu \alpha} - g_{\mu \tau} R_{\nu \alpha} - g_{\nu \alpha} R_{\mu \tau}) + \frac{1}{6} (g_{\mu \alpha} g_{\nu \tau} - g_{\mu \tau} g_{\nu \alpha}) R,
\] (7.144)

where \( g_{\mu \nu} \) is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where \( \eta_{\mu \nu} \) is the flat Minkowski metric, \( h_{\mu \nu} \) are small so that all quadratic and higher-order terms are neglected. In particular: \( R_{\mu \nu \alpha \tau} = \frac{1}{2} \left( \partial_{\mu} \partial_{\tau} h_{\nu \alpha} + \partial_{\nu} \partial_{\alpha} h_{\mu \tau} - \partial_{\mu} \partial_{\alpha} h_{\nu \tau} - \partial_{\nu} \partial_{\tau} h_{\mu \alpha} \right) \). The equations of linear conformal gravity are:

\[
\partial^n \partial^T C_{\mu \nu \alpha \tau} = T_{\mu \alpha} ,
\] (7.145)

where \( T_{\mu \nu} \) is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

\[
\begin{align*}
C_0 &= C_{0123}, & C_1 &= C_{2121}, & C_2 &= C_{0202}, & C_3 &= C_{3012}, \\
C_4 &= C_{2021}, & C_5 &= C_{1012}, & C_6 &= C_{2023}, \\
C_7 &= C_{3132}, & C_8 &= C_{2123}, & C_9 &= C_{1213}.
\end{align*}
\] (7.146)

Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as \( C^\pm \) (cf. (7.142) for \( m = 2 \)). The tensors \( T_{\mu \nu} \) and \( h_{\mu \nu} \) are symmetric and traceless with nine independent components.

Further, we shall use again the fact that a Lorentz irrep (spin-tensor) with signature \( (n_1, n_2) \) may be represented by a polynomial \( G(z, \bar{z}) \) in \( z, \bar{z} \) of order \( n_1, n_2 \).
respectively. More explicitly, for the Weyl gravity representations mentioned above we use:

\[
C^+(z) = z^4 C_4^+ + z^3 C_3^+ + z^2 C_2^+ + zC_1^+ + C_0^+ ,
\]

\[
C^-(\bar{z}) = z^4 C_4^- + z^3 C_3^- + z^2 C_2^- + \bar{z}C_1^- + C_0^- ,
\]

\[
T(z, \bar{z}) = z^2 z^2 T_{22}^' + z^2 \bar{z} T_{21}^' + z^2 T_{20}^' +
\]
\[+ z\bar{z}^2 T_{12}^' + \bar{z}z T_{11}^' + z T_{10}^' +
\]
\[+ \bar{z}^2 T_{02}^' + \bar{z} T_{01}^' + T_{00}^' ,
\]

\[
h(z, \bar{z}) = z^2 z^2 h_{22}^' + z^2 \bar{z} h_{21}^' + z^2 h_{20}^' +
\]
\[+ z\bar{z}^2 h_{12}^' + \bar{z}z h_{11}^' + z h_{10}^' +
\]
\[+ \bar{z}^2 h_{02}^' + \bar{z} h_{01}^' + h_{00}^' ,
\]

where the indices on the RHS are not Lorentz-covariance indices, they just indicate the powers of \(z, \bar{z}\). The components \(C^\pm_k\) are given in terms of the Weyl tensor components as follows:

\[
C_0^+ = C_2 - \frac{1}{2} C_1 - C_6 + i(C_0 + \frac{1}{2} C_3 + C_7)
\]
\[
C_1^+ = 2(C_6 - C_8 + i(C_9 - C_5))
\]
\[
C_2^+ = 3(C_1 - iC_3)
\]
\[
C_3^+ = 8(C_6 + C_8 + i(C_9 + C_5))
\]
\[
C_4^+ = C_2 - \frac{1}{2} C_1 + C_6 + i(C_0 + \frac{1}{2} C_3 - C_7)
\]
\[
C_5^+ = C_2 - \frac{1}{2} C_1 - C_6 - i(C_0 + \frac{1}{2} C_3 + C_7)
\]
\[
C_6^+ = 2(C_6 - C_8 - i(C_9 - C_5))
\]
\[
C_7^+ = 3(C_1 + iC_3)
\]
\[
C_8^+ = 2(C_6 + C_8 - i(C_9 + C_5))
\]
\[
C_9^+ = C_2 - \frac{1}{2} C_1 + C_6 - i(C_0 + \frac{1}{2} C_3 - C_7).
\]

while the components \(T_{ij}^\prime\) are given in terms of \(T_{\mu\nu}\) as follows:

\[
T_{22}^\prime = T_{00} + 2T_{03} + T_{33}
\]
\[
T_{11}^\prime = T_{00} - T_{33}
\]
\[
T_{00}^\prime = T_{00} - 2T_{03} + T_{33}
\]
\[
T_{21}^\prime = T_{01} + iT_{02} + T_{13} + iT_{23}
\]
\[
T_{12}^\prime = T_{01} - iT_{02} + T_{13} - iT_{23}
\]
\[
T_{10}^\prime = T_{01} + iT_{02} - T_{13} - iT_{23}
\]
\[
T_{01}^\prime = T_{01} - iT_{02} - T_{13} + iT_{23}
\]
\[
T_{20}^\prime = T_{11} + 2iT_{12} - T_{22}
\]
\[
T_{02}^\prime = T_{11} - 2iT_{12} - T_{22}
\]

and similarly for \(h_{ij}^\prime\) in terms of \(h_{\mu\nu}\).

In these terms all linear conformal Weyl gravity equations (7.145) (cf. also (7.141)) may be written in compact form as the following pair of equations:
\[ I^+ C^+(z) = T(z, \bar{z}), \quad I^- C^-(z) = T(z, \bar{z}), \] (7.152)

where the operators \( I^\pm \) are given as follows:

\[
I^+ = \left( z^2 \partial_z^2 + z^2 \partial_p^2 + z^2 \partial_{\bar{p}}^2 + \partial_z^2 + \partial_p^2 + \partial_{\bar{p}}^2 \right) + 2z^2 z\partial_z \partial_p + 2zz^2 \partial_z \partial_{\bar{p}} + 2zz \partial_z \partial_{\bar{p}} + 2z\partial_z \partial_{\bar{p}} + 2z \partial_z \partial_{\bar{p}} + 6(zz^2 \partial_z^2 + z\partial_z^2 + 2zz \partial_z \partial_p + 2z \partial_z \partial_{\bar{p}} + z \partial_z \partial_{\bar{p}} + z \partial_z \partial_{\bar{p}}) \partial_z + 12 \left( z^2 \partial_z^2 + \partial_p^2 + 2z \partial_p \partial_{\bar{p}} \right),
\]

\[
I^- = \left( z^2 \partial_z^2 + z^2 \partial_p^2 + z^2 \partial_{\bar{p}}^2 + \partial_z^2 + \partial_p^2 + \partial_{\bar{p}}^2 \right) + 2z^2 z\partial_z \partial_p + 2zz^2 \partial_z \partial_{\bar{p}} + 2zz \partial_z \partial_{\bar{p}} + 2z \partial_z \partial_{\bar{p}} + 2z \partial_z \partial_{\bar{p}} + 6(zz^2 \partial_z^2 + z\partial_z^2 + 2zz \partial_z \partial_p + 2z \partial_z \partial_{\bar{p}} + z \partial_z \partial_{\bar{p}} + z \partial_z \partial_{\bar{p}}) \partial_z + 12 \left( z^2 \partial_z^2 + \partial_p^2 + 2z \partial_p \partial_{\bar{p}} \right).
\]

using the Minkowski conformal variables. We recall that in terms of these variables the d’Alembert equation is:

\[
\Box \varphi = (\partial_+ \partial_+ - \partial_p \partial_{\bar{p}}) \varphi = 0. \quad (7.154)
\]

To make more transparent the origin of (7.152) and in the same time to derive the quantum group deformation of (7.152) and (7.153) we first introduce the following parameter-dependent operators:

\[
I_n^+ = \frac{1}{2} \left( n(n-1)I_1^2 I_2^2 - 2(n^2 - 1)I_1 I_2 I_1 + n(n + 1)I_3^2 I_4^2 \right),
\]

\[
I_n^- = \frac{1}{2} \left( n(n-1)I_1^2 I_2^2 - 2(n^2 - 1)I_3 I_2 I_3 + n(n + 1)I_4^2 I_3^2 \right),
\]

where \( I_1 = \partial_z, \ I_2 = z\partial_z + z\partial_p + z\partial_{\bar{p}} + \partial_-, \) and \( I_3 = \partial_{\bar{z}} \) are from (7.16). We recall that group-theoretically the operators \( I_n \) correspond to the three simple roots of the root system of \( sl(4) \), while the operators \( I_n^\pm \) correspond to the singular vectors for the two nonsimple nonhighest roots. More precisely, the operator \( I_n^+ \) is obtained from the \( sl(4) \) formula for the singular vector given by (2.37) of weight \( m_{12} \alpha_{12} = 2\alpha_{12} \). Analogously, the operator \( I_n^- \) is obtained from the same formula for weight \( m_{23} \alpha_{23} = 2\alpha_{23} \). The parameter \( n = \max(2j_1, 2j_2) \).
It is easy to check that we have the following relation:

\[ I^\pm = I_4^\pm, \quad (7.156) \]

that is, \((7.152)\) are written as:

\[ I_4^+ C^+(z) = T(z, \bar{z}), \quad I_4^- C^-(\bar{z}) = T(z, \bar{z}). \quad (7.157) \]

This is the form that is immediately generalizable to the \(q\)-deformed case in next subsection.

Using the same operators we can write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

\[ I_2^+ h(z, \bar{z}) = C^-(\bar{z}), \quad I_2^- h(z, \bar{z}) = C^+(z). \quad (7.158) \]

We stress again the advantage of the indexless formalism due to which two different pairs of equations – \((7.157)\) and \((7.158)\) – may be written using the same parameter-dependent operator expressions by just specializing the values of the parameter.

### 7.7.2 \(q\)-Plane-Wave Solutions of \(q\)-Weyl Gravity

We consider now the \(q\)-deformed setting supposing that \(q\) is not a nontrivial root of unity.

Using the \(U_q(\mathfrak{sl}(4))\) formula for the singular vector given in \((2.37)\), we obtain for the \(q\)-analogue of \((7.16)\):

\[
\begin{align*}
qI_n^+ &= \frac{1}{2} \left( \left[ n \right]_q \left[ n - 1 \right]_q qI_3^2 \left[ n \right]_q qI_3 qI_1 + \right. \\
&\quad \left. + \left[ n + 1 \right]_q qI_2^2 qI_1 \right), \\
qI_n^- &= \frac{1}{2} \left( \left[ n \right]_q \left[ n - 1 \right]_q qI_3^2 qI_2 \right. \\
&\quad \left. + \left[ n + 1 \right]_q \left[ n \right]_q qI_2^2 qI_3 \right),
\end{align*}
\quad (7.159)
\]

where the \(q\)-deformed versions \(qI_n\) of \((7.16)\) are given in \((7.35)\).

Then the \(q\)-Weyl gravity equations are (cf. \((7.157)\)):

\[ qI_n^+ C^+(z) = T(z, \bar{z}), \quad qI_n^- C^-(\bar{z}) = T(z, \bar{z}), \quad (7.160) \]

while \(q\)-analogues of \((7.158)\) are:

\[ qI_2^+ h(z, \bar{z}) = C^-(\bar{z}), \quad qI_2^- h(z, \bar{z}) = C^+(z). \quad (7.161) \]
For the solutions we shall use the basis (7.61). The solutions of the first equation in (7.160) in the homogeneous case \((T = 0)\) are:

\[
q C^+_0 = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{c}^+_s, 
\]

(7.162)

\[
\hat{C}^+_s = \sum_{m=0}^{4} \hat{y}^+_m \left( \prod_{i=0}^{-m+3} (k_i - q^{i+B_s+s+4} k_i z) \right) \times \\
\times \left( \prod_{j=-m+4}^{3} (k_j - q^{j+B_s+s+4} k_j z) \right) \hat{h}^+_s, 
\]

(7.163)

where \(\hat{h}^+_s\) is \(h_s\) with:

\[
P_s(a, b) = P^+_s(a, b) = R_s(a) + B_s b, 
\]

(7.164)

\(\hat{y}^+_m, B_s\) are arbitrary constants and \(R_s(a)\) is an arbitrary polynomial in \(a\). In order to be able to write the above solution in terms of the deformed plane wave, we have to suppose that the \(\hat{y}^+_m, B_s + s\) for different \(s\) coincide: \(\hat{y}^+_m = \gamma^+_m\), for example, we can make the choice \(B_s = B^t - s - 4\). Then we have:

\[
q C^+_0 = \sum_{m=0}^{4} \hat{y}^+_m \left( \prod_{i=0}^{-m+3} (k_i - q^{i+B_s+s+4} k_i z) \right) \times \\
\times \left( \prod_{j=-m+4}^{3} (k_j - q^{j+B_s+s+4} k_j z) \right) \exp^+_q(k, x), 
\]

(7.165)

where \(\exp^+_q(k, x)\) is \(\exp^+_q(k, x)\) with the choice (7.164).

The solutions of the second equation in (7.160) are:

\[
q C^-_0 = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{c}^-_s 
\]

(7.166)

\[
\hat{C}^-_s = \sum_{m=0}^{\hat{A}} \hat{y}^-_m \left( \prod_{i=-1}^{-m+2} (k_i - q^{i-D_s} k_i z) \right) \times \\
\times \left( \prod_{j=-m+3}^{2} (k_j - q^{j-D_s} k_j z) \right) \hat{h}^-_s 
\]

(7.167)

where \(\hat{h}^-_s\) is \(h_s\) with:

\[
P_s(a, b) = P^-_s(a, b) = D_s a + Q_s(b), 
\]

(7.168)
\(\hat{\gamma}^{-}_m, D_s\) are arbitrary constants and \(Q_s(b)\) is an arbitrary polynomial. In order to be able to write this solution in terms of the deformed plane wave, we have to suppose that the \(\hat{\gamma}^{-}_m, D_s\) for different \(s\) coincide: \(\hat{\gamma}^{-}_m = \hat{\gamma}^{-}_m, D_s = D_s\). Then we have:

\[
q C_0 = \sum_{m=0}^{4} \hat{\gamma}^{-}_m \left( \prod_{i=-1}^{m+2} (k_{v} - q^{i-D} k_{v} \hat{z}) \right) \times \\
\times \left( \prod_{j=\pm m+3}^{\pm 2} (k_{v} - q^{i-D} k_{v} \hat{z}) \right) \exp^{-}_q(k, x),
\]

(7.169)

where \(\exp^{-}_q(k, x)\) is \(\exp_q(k, x)\) with the choice (7.168).