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4 Nonlinear Fracture Mechanics of Concrete

Ageing sewers may sustain varying degrees of material and structural damage, such as the general cracking of concrete due to the occurrence of numerous small cracks in the sewer structure, the localised fracturing of structural members due to a few large cracks, shear fractures at the joint interfaces of the top/bottom plates and the sidewalls, and the frequent loss of concrete cover (also as a result of cracking) and the corrosion of rebars. Clearly, the fracture mechanics of concrete plays an important role in the structural analysis and renovation design of ageing sewers, providing the theoretical basis for crack analysis in sewerage structures.

The theory of fracture mechanics of concrete, with its governing law for crack propagation based on the inelastic material behaviour exhibited in an extensive fracture process zone (FPZ) ahead of an open crack, is largely developed from the theory of linear elastic fracture mechanics (LEFM). This chapter introduces the fundamental concepts of LEFM and nonlinear fracture mechanics (NLFM) of concrete, and presents two major computational theories in crack analysis of concrete, namely, the discrete crack modelling approach and the smeared crack modelling approach.

Part I Fundamental Concepts of Linear Elastic Fracture Mechanics

4.1 Stress Intensity Factor and K-Controlled Crack-Tip Fields of LEFM

Consider a central crack of length $2a$ in an infinite plate subjected to tension, as shown in Fig. 4.1 where a combination of Cartesian coordinates and polar coordinates is used for denoting the near-tip fields. For a near-tip point $P(x, y)$, with $x = a + r\cos\theta$ and $y = r\sin\theta$, the well-known linear elastic solutions of its stress components are given by

$$\begin{aligned}\sigma_x &= \sigma \frac{\sqrt{a}}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right); \quad \sigma_y = \sigma \frac{\sqrt{a}}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \tau_{xy} &= \sigma \frac{\sqrt{a}}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}$$

(4.1a-c)

Figure: 4.1 An infinite plate with a central crack subjected to tension

Similarly, when the crack is subjected to shear as shown in Fig. 4.2, the stress components are obtained as

$$\begin{aligned}\sigma_x &= -\tau \frac{\sqrt{a}}{\sqrt{2r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right); \quad \sigma_y = \tau \frac{\sqrt{a}}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\ \tau_{xy} &= \tau \frac{\sqrt{a}}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)\end{aligned}\quad (4.2a-c)$$

As seen, under the assumption of elasticity, there exists an inverse-square-root singularity at the tip of the crack in the near-tip stress fields. It is known that this singularity-contained structure of stress distribution is invariant among the near-tip solutions of all the elastic crack problems, which differ from each other only by a constant. This single constant defines the amplitude of the stress in the vicinity of the crack tip, with which the crack-tip stress fields are uniquely determined. The significance of this constant in characterising the crack-tip fields in an elastic body was first recognised by Irwin (1957), who named it the stress intensity factor, K , which has become one of the most important fracture mechanics concepts in the theory of LEFM.

Figure 4.2: An infinite plate with a central crack subjected to shear

A crack may be subjected to three different types of loading that cause displacements of the crack surfaces, as shown in Fig. 4.3. Mode I loading, where the load is applied normal to the crack plane, tends to open the crack. Mode II loading refers to in-plane shear and causes the two crack surfaces to slide against each other. Mode III loading, where out-of-plane shear is applied, tends to tear the two crack surfaces apart. This last mode of deformation, called out-of-plane shear mode, does not occur in the plane elastic problem. The stress intensity factors for the three types of loading are denoted by K_I , K_{II} , and K_{III} , respectively. The crack-tip stresses for each mode of loading can now be rewritten in terms of the corresponding stress intensity factor as

Mode I:

$$\begin{aligned}\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right); \\ \sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right); \\ \tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}\tag{4.3a-c}$$

Mode II:

$$\begin{aligned}\sigma_x &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right); \\ \sigma_y &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}; \\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)\end{aligned}\quad (4.4a-c)$$

where $\tau_{xz} = \tau_{yz} = 0$ for plane stress and plane strain; and $\sigma_z = 0$ for plane stress and $\sigma_z = \nu(\sigma_x + \sigma_y)$ for plane strain.

Mode III:

$$\begin{aligned}\tau_{xz} &= -\frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}; \\ \tau_{yz} &= \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}\end{aligned}\quad (4.5a, b)$$

where $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$.

Similarly, the crack-tip displacement fields are obtained as

Mode I:

$$\begin{aligned}u_x &= \frac{K_I}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right); \\ u_y &= \frac{K_I}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right)\end{aligned}\quad (4.6a, b)$$

Mode II:

$$\begin{aligned}u_x &= \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right); \\ u_y &= -\frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 - 2 \sin^2 \frac{\theta}{2} \right)\end{aligned}\quad (4.7a, b)$$

Mode III:

$$u_z = \frac{2K_{III}}{G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2}\quad (4.8)$$

where G is the shear modulus; $k = 3 - 4\nu$ for plane strain, and $k = (3 - \nu)/(1 + \nu)$ for plane stress; and ν is Poisson's ratio. Note that the square-root of r characteristics of the near-tip displacement fields is the source of the inverse-square-root singularity of the near-tip stress fields.

Figure 4.3: Three independent modes of deformation at the crack tip

By comparing Eqs. (4.3) and (4.4) with Eqs. (4.1) and (4.2), which are the exact solutions for an infinite plate with a central crack as $r \rightarrow 0$, the stress intensity factors for this particular problem are found to be

$$K_I = \sigma\sqrt{\pi a}; \quad K_{II} = \tau\sqrt{\pi a} \quad (4.9a, b)$$

As seen, the stress intensity factor is a function of the applied load and the size of the crack; when a finite body is involved, it is also a function of the geometric configuration of the problem. Obviously, the proportionality of K to the applied load reflects the linear nature of the theory of elasticity. In general, the stress intensity factors can be defined as

$$\begin{aligned} K_I &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_y(r, \theta = 0); \\ K_{II} &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \tau_{xy}(r, \theta = 0); \\ K_{III} &= \lim_{r \rightarrow 0} \sqrt{2\pi r} \tau_{yz}(r, \theta = 0) \end{aligned} \quad (4.10a-c)$$

According to Eqs. (4.10), the stress intensity factor governs the magnitude of the stress singularity at the crack tip. The larger the applied load and the size of the crack are (resulting in a larger K), the greater the rate of stress increase in the singularity-dominated zone becomes, provided that the crack remains stationary.

As seen in Eqs. (4.3)-(4.8), the stress intensity factor completely defines the crack-tip fields (or the stress redistribution due to the occurrence of the crack); if K is known, it is possible to obtain all the components of stress, strain, and displacement as functions of r and θ . As a function of the applied load and the size of the crack as well as the geometry of the problem, the stress intensity factor represents both the strength of the crack-tip singularity and the mode of deformation, providing a link between the crack-contained local material behaviour and the global structural response. Based on the principle of linear superposition, the crack-tip stress fields of a mixed-mode crack can be written as

$$\sigma_{ij}(r, \theta) = \frac{1}{\sqrt{2\pi r}} \{K_I f_{ij}^I(\theta) + K_{II} f_{ij}^{II}(\theta) + K_{III} f_{ij}^{III}(\theta)\} \quad (4.11)$$

where the functions of $f_{ij}^I(\theta)$, $f_{ij}^{II}(\theta)$ and $f_{ij}^{III}(\theta)$ are defined by Eqs. (4.3)-(4.5).

Although the K -controlled crack-tip fields ($r \ll a$) have been obtained using the theory of elasticity, these solutions do not reflect the physical reality of the material deformation at the tip of a crack in engineering materials, because no real material can sustain infinite stresses at the inverse-square-root singularity predicted by the stress solutions. It is known that the singularity in the elastic solutions is the result of two mathematical simplifications: the assumption that the configuration of a real crack (whose crack-tip radius is finite) is a sharp crack (whose crack-tip radius is zero), and the assumption that the inelastic material behaviour at the crack tip is elastic. To remove the crack-tip singularity, it is necessary to take into account inelastic material behaviour, such as plasticity in metals and fracture process zone in concrete, by introducing the inelastic-zone concept to the crack tip.

Despite the inaccuracy in describing the actual material behaviour in the close vicinity of the crack tip, the elastic solutions of Eqs. (4.3)-(4.8) are of great importance in LEFM. The singularity-containing solutions of the stress fields at the tip of a sharp crack in a linear elastic material reveal the nature of fracture in its ultimate form, i.e., causing unbounded stresses at the crack tip. With the help of these elastic solutions, Irwin envisioned the physical significance of the constants contained in these solutions and coined the term “stress intensity factor” to characterise the singularity at the crack tip, which has turned out to be one of the most important concepts in fracture mechanics. Furthermore, these elastic solutions may still be valid outside the inelastic zone at the crack tip, if the size of the zone is sufficiently small compared to the length dimensions of the cracked structure.

Outside the K -controlled crack-tip fields, the general solutions for stress or displacement components include higher order terms of r . For example, the solution of stress for any type of loading (K) is given by

$$\sigma_{ij}(r, \theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \sum_{m=0}^{\infty} A_m r^{\frac{m}{2}} h_{ij}^{(m)}(\theta) \quad (4.12)$$

where A_m is the amplitude for the m -th term and $h_{ij}^{(m)}$ is a dimensionless function of θ . Note that the higher order terms depend on the geometry of the problem. As $r \rightarrow 0$, the leading term approaches infinity while the other terms remain finite or approach zero.

Analytical and numerical methods for calculating stress intensity factors have been extensively discussed in many textbooks of fracture mechanics and there are also handbooks on K for different geometries and loading conditions. The present book focuses on the physical interpretation of the stress intensity factors and the K -controlled crack-tip fields because of their theoretical importance for the subsequent development of nonlinear fracture mechanics of concrete; for detailed coverage on stress intensity factors, refer to the textbooks (Sanford, 2003; Anderson, 2005).

4.2 Energy Principles

With the crack-tip fields now being completely defined in terms of the stress intensity factor, an energy description of the fracture process needs to be explained using Griffith's fracture theory (Griffith, 1921), whose innovative concept of introducing fracture energy into the study of cracked materials laid a solid foundation for the later development of fracture mechanics. It will be shown that the fracture energy required for creating a unit surface of an open crack, i.e., the energy release rate, has a close relationship with the stress intensity factor.

4.2.1 The Griffith fracture theory

Griffith derived his fracture criterion by considering the problem shown in Fig. 4.1, where an infinite plate of unit thickness that contains a crack of length $2a$ is subjected to uniform tensile stress σ . He first obtained the decrease of strain energy in the plate due to the introduction of the crack based on the stress analysis of Inglis (1913), as

$$U_a = -\frac{1}{E} \pi a^2 \sigma^2 \quad (\text{plane stress}) \quad (4.13a)$$

$$U_a = -\frac{1-\nu^2}{E} \pi a^2 \sigma^2 \quad (\text{plane strain}) \quad (4.13b)$$

where E = the modulus of elasticity. Griffith then introduced "the fundamental conception of the new theory", i.e., surface energy, which is the fracture energy required to form the open crack, as

$$U_s = 4a\gamma_s \quad (4.14)$$

where γ_s = the surface energy per unit area. Note that in Eq. (4.14) the crack surface is $4a$ because a crack has two surfaces. According to the Griffith energy criterion, crack growth is a natural course of energy transfer between the strain energy of an elastic

body and the fracture energy required for creating a new crack surface. This requires the following condition to be satisfied:

$$\frac{d}{dA}(U_a + U_s) = \frac{1}{2} \frac{d}{da}(U_a + U_s) = 0 \quad (4.15)$$

Therefore, the critical load for crack propagation is obtained as

$$\sigma_c = \sqrt{\frac{2E\gamma_s}{\pi a}} \quad (4.16a)$$

in the case of plane stress, and

$$\sigma_c = \sqrt{\frac{2E\gamma_s}{(1-\nu^2)\pi a}} \quad (4.16b)$$

in the case of plane strain. Equation (4.16) is referred to as the Griffith fracture criterion, which is valid for ideally brittle materials.

4.2.2 The energy release rate G

The Griffith energy concept is of a global character: for a crack to propagate, the energy stored in the system must be sufficient to overcome the fracture energy of the material. Based on the Griffith fracture theory, Irwin generalised the concept by defining an energy release rate G , which is a measure of the energy available for a unit extension of the crack and represents “the force tending to cause crack extension” (Irwin, 1957). Figure 4.4 shows a cracked plate under a given load P , which causes the crack to extend by an amount dA . Since the load is fixed at P , the system is under load control. Obviously, the fracture energy dissipated in propagating the crack originates from the work done by the external force $Pd\Delta$ and the released elastic strain energy $-dU$ in the system, i.e.,

$$G = P \frac{d\Delta}{dA} - \frac{dU}{dA} \quad (4.17)$$

The strain energy of the elastic system is given by

$$U = \int_0^\Delta P d\Delta = \frac{P\Delta}{2} = \frac{CP^2}{2} \quad (4.18)$$

where the relation $\Delta = CP$ is employed; C is the compliance of the system, which is the inverse of the plate stiffness. Note that

$$\frac{d}{dA} = \frac{\partial}{\partial A} + \frac{dP}{dA} \frac{\partial}{\partial P} \quad (4.19)$$

Figure 4.4: Cracked plate under a load P

After substituting Eq. (4.18) and $\Delta = CP$ into Eq. (4.17), the above derivative operations are carried out and the energy release rate is derived as

$$G = \frac{P^2}{2} \frac{\partial C}{\partial A} \quad (4.20)$$

It can be proved that Eq. (4.20) can be obtained under the condition of displacement control as well. Theoretically, if the derivative of the compliance of a test specimen with respect to crack length can be derived experimentally or numerically by a FE solution, then the energy release rate can be calculated from Eq. (4.20).

4.2.3 Relationship between K and G

Two fracture parameters have been introduced so far: the stress intensity factor K and the energy release rate G . While the first parameter uniquely defines the near-tip stress and displacement fields, the second one represents the crack driving force to open that crack. Obviously, a relationship exists between the two. The following deduction is based on Irwin's crack closure analysis (Irwin, 1957).

Figure 4.5 shows an infinite plate of unit thickness with fixed ends that contains a crack of size a . Since the problem is a mode-I fracture, the terms K and G are denoted by K_I and G_I , respectively. Suppose that compressive forces are applied to the crack edge over an infinitesimal distance Δa to close the crack. Since the work required to

close the crack is equal to the fracture energy needed to extend the crack, the energy release rate can be calculated from

$$G_I = \lim_{\Delta a \rightarrow 0} \frac{\Delta W}{\Delta a} = \lim_{\Delta a \rightarrow 0} \frac{2}{\Delta a} \int_0^{\Delta a} \frac{1}{2} \sigma_y u_y dr \quad (4.21)$$

where ΔW is the required work, and u_y is the displacement in the y direction. Note that the factor 2 arises because the crack has two surfaces, and that the incremental work at r is calculated based on the fact that the stresses increase from zero.

Figure 4.5: Crack closure analysis

For the infinite plate under tension, the value of u_y is given by

$$u_y = \frac{2\sigma}{E} \sqrt{a^2 - x^2} = \frac{2K_I}{E} \sqrt{\frac{a^2 - x^2}{\pi a}} \quad (4.22)$$

Substituting $x = r - \Delta a + a$ into Eq. (4.22) and neglecting second-order terms lead to

$$u_y = \frac{2K_I}{E\sqrt{\pi}} \sqrt{2\Delta a - 2r + \frac{2r\Delta a}{a} - \frac{r^2}{a}} \cong \frac{2K_I}{E\sqrt{\pi}} \sqrt{2(\Delta a - r)} \quad (4.23)$$

The closing force is given by

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \quad (4.24)$$

By using Eqs. (4.23) and (4.24), Eq. (4.21) becomes

$$G_I = \lim_{\Delta a \rightarrow 0} \frac{2K_I^2}{E\pi\Delta a} \int_0^{\Delta a} \sqrt{\frac{\Delta a - r}{r}} dr \quad (4.25)$$

which leads to

$$G_I = \frac{K_I^2}{E} \quad (\text{plane stress}) \quad (4.26a)$$

and to

$$G_I = (1 - \nu^2) \frac{K_I^2}{E} \quad (\text{plane strain}) \quad (4.26b)$$

The obtained relation between G and K clearly reveals that the stress intensity factor has an energy basis. Similarly, it can be shown that for mode II:

$$G_{II} = \frac{K_{II}^2}{E} \quad (\text{plane stress}) \quad (4.27a)$$

$$G_{II} = (1 - \nu^2) \frac{K_{II}^2}{E} \quad (\text{plane strain}) \quad (4.27b)$$

and for mode III:

$$G_{III} = \frac{1}{1 - \nu} \frac{K_{III}^2}{E} \quad (\text{plane stress}) \quad (4.28a)$$

$$G_{III} = (1 + \nu) \frac{K_{III}^2}{E} \quad (\text{plane strain}) \quad (4.28b)$$

The total energy release rate in combined crack modes can be obtained by superposition:

$$G = G_I + G_{II} + G_{III} = \frac{1}{E} (K_I^2 + K_{II}^2 + \frac{1}{1 - \nu} K_{III}^2) \quad (\text{plane stress}) \quad (4.29a)$$

$$G = G_I + G_{II} + G_{III} = \frac{1 - \nu^2}{E} (K_I^2 + K_{II}^2 + \frac{1}{1 - \nu} K_{III}^2) \quad (\text{plane strain}) \quad (4.29b)$$

Since the K -controlled near-tip fields are uniquely defined for linear elastic materials, it can be proved that Eqs. (4.26)-(4.28) are valid for all geometries of specimens and types of cracks. In fact, G may even be considered as a physical parameter defined at the tip of a crack using the near-tip stress and displacement fields, as shown in Eq. (4.21). Hence, its relations with K must be invariant as the near-tip fields are uniquely established, regardless of any specific changes that may occur to the problem. For plates of finite size the influence of the geometric shape of the problem is reflected in the stress intensity factor K . Therefore, it is possible to obtain G for a problem of finite size by using the relationships between G and K .

4.2.4 The criterion for crack propagation

Since the stress at the tip of a crack in a K -controlled stress field is unbounded, a criterion for crack propagation cannot be stipulated in terms of the tip stress, but has to be described by using either of the two fracture parameters introduced so far, i.e., the energy release rate or the stress intensity factor, which are equivalent based on Eqs. (4.26)-(4.28). For ideally brittle materials, the Griffith criterion in Eq. (4.16a) can be rewritten in terms of the critical energy release rate, G_{Ic} , as

$$\sigma_c = \sqrt{\frac{EG_{Ic}}{\pi a}} \quad (4.30)$$

where

$$G_{Ic} = \frac{dU_s}{dA} = 2\gamma_s \quad (4.31)$$

Substituting Eq. (4.26a) into Eq. (4.30), the critical load is then given in terms of the critical stress intensity factor, K_{Ic} , as

$$\sigma_c = \frac{K_{Ic}}{\sqrt{\pi a}} \quad (4.32)$$

or

$$K_I = K_{Ic} \quad (4.33)$$

Equation (4.33) states that a crack propagates whenever K_I is equal to the threshold value, K_{Ic} , which is a measure of fracture toughness and is regarded as a material fracture property based on LEFM. In principle, the criterion for mode-II or mode-III shear-type fracture can also be expressed as

$$K_{II} = K_{IIc} \quad (4.34)$$

$$K_{III} = K_{IIIc} \quad (4.35)$$

For real engineering materials, stresses cannot increase unboundedly and yielding takes place in metallic materials to form a plastic zone ahead of an open crack. Therefore, the K -controlled stress states in Eqs. (4.3)-(4.8) do not describe the stress distribution inside the plastic zone. This raises a serious question: Is the K criterion still a valid criterion for crack growth?

It turns out that, if the plastic zone is sufficiently small compared to all the length dimensions in the structure and test specimen, the stress intensity factor is still a valid fracture parameter that can uniquely characterise the condition for crack propagation, even if a fracture actually nucleates inside the plastic zone. This requirement for the smallness of the plastic zone restricts its existence well within the singularity-controlled region where Eqs. (4.3)-(4.8) are valid except for the plastic zone, as shown in Fig. 4.6. Obviously, in a K -controlled stress field it is always possible

to associate the condition for crack propagation with a threshold value of K , whether the crack originates from the plastic zone or from the singularity-dominated region, implying the validity of Eqs. (4.33)-(4.35).

If the plastic zone is not small compared with the characteristic length of the problem (such as the crack size), there is no longer a singularity-dominated region where the stress varies as $1/\sqrt{r}$. Consequently, the stress intensity factor loses its physical significance as a fracture parameter, and the stresses outside the plastic zone are described by Eq. (4.12) in which higher-order terms become significant. This implies that LEFM is no longer valid and the problem has to be analysed based on NLFM.

Figure 4.6: Schematic illustration of K -controlled near-tip stresses with the presence of a plastic zone

Part II Fundamental Concepts of Nonlinear Fracture Mechanics of Concrete

4.3 Fracture Process Zone and Tension-Softening Phenomenon

Concrete is a heterogeneous material consisting of aggregates and cement pastes bonded together at the interface, and the material is inherently weak in tension due to the limited bonding strength and various pre-existing microcracks and flaws formed during hardening of the matrix. The tensile strength of concrete approximately ranges from 8 to 15% of its compressive strength. Under external loading a tension zone forms near the crack tip, in which complicated micro-failure mechanisms take place. These fracture processes include microcracking, crack deflection, crack branching, crack coalescence and debonding of the aggregate from the matrix, which are examples of inelastic toughening mechanisms that coexist with a crack when it propagates. In concrete the inelastic zone at the crack tip is extensively developed and therefore, in principle, LEFM cannot be used to study the fracture of concrete.

Figure 4.7 schematically illustrates the formation of an inelastic zone in concrete, which is known as a fracture process zone (FPZ) that can be roughly divided into a bridging zone and a microcracking zone, along with two idealizations of the FPZ. It is known that bridging is a result of the weak interface between the aggregates and the cement pastes, and is an important toughening mechanism in concrete. Within the damage zone the effective modulus of elasticity is reduced from that of the undamaged material E to E^* , if the process zone is modelled as a region of strain softening, as shown in Fig. 4.7(b). Inspired by the concept of the Dugdale and Barenblatt cohesive zone models for studying small-scale plasticity at the crack tip (Dugdale, 1960; Barenblatt, 1962), Hillerborg and his colleagues envisioned a fictitious crack in place of the physical FPZ and subjected it to closure tractions, as shown in Fig. 4.7(c) (Hillerborg, Modeer and Peterson, 1976). As is illustrated in the figure, the closure stress associated with the bridging grains and microcracks is a maximum f_t at the tip of the FPZ and decreases to zero at the continuous crack tip where the crack opening displacement (COD) reaches its critical value W_c , beyond which an open crack forms. Known as the tension-softening phenomenon, the relation between the closure stress and the COD, which completely defines the fracture energy of concrete, describes the local material behaviour inside the FPZ when fracture takes place in concrete.

Figure 4.8 presents a numerically obtained load-deformation relation of a laboratory size, notched plain concrete beam under bending, and the growth of a FPZ at the notch tip based on Hillerborg's fictitious crack model (FCM). The correspondence between the various points of the figure is meant to convey a clear picture of the FPZ and how it develops during beam failure. As seen, in the pre-peak region the tip stress at the notch reaches the tensile strength of concrete at point A, signalling the initiation of a FPZ. Upon reaching the peak load at point B, the FPZ has grown to a length of 50 mm and

the tensile stress at the notch tip decreases to $0.55f_t$. The tip stress of the notch drops to zero at point C in the post-peak region, as the FPZ stretches to its full length of 150 mm, reaching approximately three fourths of the beam height. Beyond point C , a fully-open crack appears ahead of the notch and continues to extend until the beam breaks into two at point D . The dashed line drawn across the load-deformation curve divides the structural response into two regions. As seen, the pre-peak nonlinearity and the tension softening in the pre- and the early post-peak regions (somewhere above point C) are mainly the work of microcracking. The remaining structural response before final failure at point D is the result of the fracture processes that involve mainly aggregate interlocking and other frictional effects.

Figure 4.7: Concept of the FPZ and tension softening in concrete: (a) the FPZ in front of an open crack, (b) a region of strain softening with reduced effective modulus of elasticity inside the FPZ, and (c) Hillerborg and colleagues' fictitious crack model with tension softening inside the FPZ

Figure 4.8: Typical load–deformation relation of a notched concrete beam under bending and development of the FPZ in front of the notch: (a) a notched beam; (b) load–displacement relation; and (c) development of the FPZ at points A, B, C and D

As clearly shown by this example, with the presence of such a large FPZ at the notch tip the fracture of concrete cannot be described by a single fracture parameter, such as K_{Ic} or G_{Ic} , as in the case of brittle materials. The fracture behaviour and structural response of the beam can only be described when the FPZ and the tension-softening characteristics are taken into account based on the principles of NLFM of concrete, which will be discussed below.

4.4 Fracture Energy G_F and Tension-Softening Law

As discussed above, a concrete fracture initiates in the FPZ ahead of an open crack through complicated micro-failure processes, and the fracture energy is consumed in overcoming the resistance of various toughening mechanisms to form an open crack at the end of the FPZ. The amount of fracture energy required to break a unit area of concrete is generally regarded as a material property (although it varies slightly with size) that determines the fracture behaviour of the material through the fundamental relationship between the cohesive stress and the COD in the FPZ, which is known as the tension-softening law of concrete. Just like the constitutive relationship of a continuous material that governs the fundamental material behaviour (whether it is elastic or inelastic), the tension-softening law with the fracture energy as its defining characteristic is the constitutive relationship for the material in the FPZ that describes

the transitional material behaviour from the continuous state to the discontinuous state. In the following, the fracture energy is discussed with reference to the numerical example of Fig. 4.8 and the standard RILEM test method for its measurement (RILEM, 1985). Tension-softening relations are then introduced using experimental and numerical methods.

4.4.1 Fracture energy G_F

In the load-displacement relations as shown in Fig. 4.8, the area enclosed by the response curve and the horizontal axis represents the work done by the external load to fracture the beam. Suppose that the crack growth is stable and the work done by the external load is spent solely in crack propagation. Based on the Griffith energy criterion, crack growth in an elastic body in the equilibrium state is a natural process of energy transfer between the strain energy of the body and the fracture energy required for creating a new crack surface so that a state of minimum potential energy is achieved for the system at a given load level. In the present case, the work is consumed in breaking the unnotched part of the beam cross-section, i.e., the ligament ahead of the notch. Denoting the work of the external load by W_F and the ligament area by A_{lig} , the energy needed to create a crack of unit area, G_F , is obtained as

$$G_F = \frac{W_F}{A_{lig}} \quad (4.36)$$

In fracture mechanics of concrete, G_F represents the mode-I fracture energy, which is often simplified as fracture energy.

As mentioned before, at point C of the response curve in Fig. 4.8 the FPZ has grown to its full length with the COD of the fictitious crack at the end of the FPZ reaching its critical value W_c , with which the tensile stress decreases to zero and an open crack forms. Along this FPZ a full relationship between the cohesive stress and the COD is obtained and plotted in Fig. 4.9, which is known as the tension-softening relation of concrete. Obviously, this relationship can also be obtained from the stress-COD relations at the notch tip, where an open crack has just been created by fracturing the intact material there. Note that the area enclosed by this tension-softening curve with the horizontal axis in Fig. 4.9 is exactly the fracture energy defined by Eq. (4.36), i.e.,

$$G_F = \int_0^{W_c} \sigma_n(W) dW \quad (4.37)$$

Figure 4.9: Relationships between the cohesive stress and the COD along the FPZ at point C of the response curve in Fig. 4.8

In 1985, the RILEM Technical Committee 50-FMC (Fracture Mechanics of Concrete) proposed a test method for determining the fracture energy G_F of mortar and concrete by means of stable three-point bend tests on notched beams (RILEM, 1985), as shown in Fig. 4.10. The size of the beam depends on the maximum size of aggregate, d_a , according to Table 4.1. As seen from the table, the depth of the beam is at least six times the aggregate size, and the span to depth ratio can vary between 4 and 8. To ensure that the work performed by the external load will be spent entirely in fracturing the beam at the notch, a notch size as large as half the beam depth is used. With such a large notch at the midspan, the maximum bending moment is expected to be kept reasonably low to ensure that all deformations outside the fracture zone are purely elastic due to the low stresses there.

To ensure stable crack propagation during the test, the testing machine has to be stiff enough or furnished with a closed-loop servo control. Recommendations with regard to the stiffness of the machine are given in RILEM (1985). Note that a test is regarded as stable if the load and the deformation change slowly during the whole test, i.e., without any sudden jump. To ensure pure bending the supports and loading arrangements should be arranged in such a manner that the forces acting on the beam are statically determined, as shown in Fig. 4.10(a). During the test the vertical displacement at midspan is measured and continuously plotted along with the gradually applied load P until complete fracture of the specimen. As shown in Fig. 4.10(b), the area under the load-deformation curve, W_o , represents the work done by the external load to fracture the beam. Besides this major work of fracture, there are other types of loads which also perform work of fracture that is secondary but

may not be negligible. These include the self-weight of the specimen and the loading equipment carried by it. Therefore, the fracture energy should be calculated as

$$G_F = \frac{W_F}{A_{lig}} = \frac{W_0 + mg\delta_0}{(H - a_0)B} \quad (4.38)$$

where W_F is the total work done by all the loads involved, mg is the weighted sum of the loads that perform the secondary work, δ_0 is the displacement at failure, and H , B and a_0 are respectively the specimen height, width and notch size.

From a fracture mechanics point of view the RILEM method is conceptually clear, simple and accurate as a test method for measuring the fracture energy of mortar and concrete. From an experimental point of view, the fracture test on a three-point-bend beam is both practical and reliable. For these reasons the principles embodied in the RILEM method still serve as the guidelines for designing test methods for the measurement of this fracture-related material constant for various types of structural concrete.

Figure 4.10: Determination of fracture energy G_F based on the RILEM method: (a) notched beam under three-point bending, and (b) load–deformation relation

4.4.2 Tension-softening law

Through the great efforts of many researchers the concept of the tension-softening law of concrete has been firmly established as the basis for the fracture mechanics of concrete, and the general shape of this constitutive relation has been obtained by both experimental and numerical methods.

Table 4.1: Recommended sizes of beams for measuring G_f based on the RILEM method (RILEM, 1985)

d_a^* (mm)	Depth, H (mm)	Width, B (mm)	Length, L (mm)	Span, L_s (mm)
1.0-1.6	100 ± 5	100 ± 5	840 ± 10	800 ± 5
16.1-32	200 ± 5	100 ± 5	1190 ± 10	1130 ± 5
32.1-48	300 ± 5	150 ± 5	1450 ± 10	1385 ± 5
48.1-64	400 ± 5	200 ± 5	1640 ± 10	1600 ± 5

* d_a : Maximum size of aggregate

Figure 4.11 presents the experimental results obtained from a series of direct tension tests on normal-weight concrete ($\gamma = 2370 \text{ kg/m}^3$, $f_c = 47 \text{ MPa}$, $E = 39 \text{ GPa}$, and $f_t = 3.2 \text{ MPa}$) by a group of scientists from the Delft University of Technology in the 1980s, where the data points of individual experiments represent the relation between the normalised stress, σ/f_t , and the COD, W (Hordijk, 1991). A regression analysis on these test data was carried out, and the optimum relation (standard deviation = 0.052) was given by

$$\frac{\sigma}{f_t} = \left[1 + \left(c_1 \frac{W}{W_c} \right)^3 \right] \exp \left(-c_2 \frac{W}{W_c} \right) - \frac{W}{W_c} (1 + c_1^3) \exp(-c_2) \quad (4.39)$$

The best-fit curve was obtained for $c_1 = 3$, $c_2 = 6.93$ and $W_c = 160 \text{ mm}$, and the corresponding fracture energy was found to be $G_F = 0.1 \text{ N/mm}$ based on Eq. (4.37), resulting in a relation of $W_c = 5.14 G_F / f_t$.

Because of the difficulty of carrying out a stable uniaxial tension test to obtain the tension-softening relation, Roelfstra and Wittmann (1986) proposed an inverse modelling method for determining this relationship using the fictitious crack approach and data fitting. In this method a conventional fracture test (such as a bending test) is simulated numerically by assuming and modifying a bilinear tension-softening relation that contains four parameters as shown in Fig. 4.12 through iterative computations to reduce the differences between the experimental and numerical results. They found that the best-fitting response curve (such as a load-deflection curve) could be obtained when a good approximation to the actual tension-softening characteristics was reached. The original formulation has been modified by other researchers to improve the convergence of solutions, and the revised formulation by Nomura et al. is outlined below (Nomura, Mihashi, Suzuki, et al., 1990).

Figure 4.11: Regression analysis for the σ - w relation; data points and fitted relation (Hordijk, 1991)

Figure 4.12: Definition of f_t , s_p , W_p , and W_2 in the bilinear softening diagram

Let $F_E(i)$ represent the experimentally obtained load-deformation curve recorded in a time sequence ($i = 1, 2, \dots, r$) at a given location. The numerically-calculated load-deformation curve at the same spot is denoted by $F_M(i; \{x\})$, which is obtained by assuming a bilinear softening relation with the four parameters $\{x\} = \{f_t, s_p, W_p, W_2\}$. By minimising the following sum of the square of differences of the two curves, the parameters for an optimum softening relation can be determined:

$$S(\{x\}) = \sum_{i=1}^r [F_E(i) - F_M(i; \{x\})]^2 \rightarrow \min \quad (4.40)$$

Let $\{x^{(k)}\}$ and $\{\Delta x^{(k)}\}$ represent, respectively, the k th estimation of the softening parameters and the corresponding incremental modification. The next estimation is then given by $\{x^{(k+1)}\} = \{x^{(k)}\} + \{\Delta x^{(k)}\}$. The incremental modification $\{\Delta x^{(k)}\}$ is estimated from the following iterative equation:

$$\{\Delta y^{(k)}\} \cong [J^{(k)}] \{\Delta x^{(k)}\} \quad (4.41a)$$

where

$$\Delta y_i^{(k)} = F_E(i) - F_M(i; \{x^{(k)}\}) \quad (4.41b)$$

$$J_{ij}^{(k)} = \left[\frac{\partial F_M(i; \{x\})}{\partial x_j} \right]_{x_j = x_j^{(k)}} \quad (4.41c)$$

The flow-chart for calculating the parameters of an optimum softening relation is illustrated in Fig. 4.13. For details of the solution procedure, refer to Nomura et al. (1990).

Figure 4.13: Flowchart of the numerical procedure for inverse modelling

Hordijk compared the experimental results presented in Fig. 4.11 with the optimum bilinear relations obtained by Wittmann et al. for a number of concrete mixes and loading rates, in which the stress at the break-point was fixed at $f_t/4$, and found agreement between them (Wittmann, Rokugo, Bruhwiler, et al. 1988; Hordijk, 1991). He observed that these optimum relations with only limited differences from each other appear to be a good approximation of the directly determined softening curve given by Eq. (4.39), as shown in Fig. 4.14 where the averaged bilinear relation is presented. As seen, both the initial slope and the slope of the second part of the descending branch coincide very well, while the parameter W_c is found to be coincidentally the same.

Figure 4.14: Bilinear softening relation by inverse modelling as compared with the relation obtained from uniaxial tensile tests (Hordijk, 1991)

The success of the inverse modelling method discussed above highlights the relevance of adopting a bilinear relation to represent the real tension-softening characteristics of concrete, which, according to Cotterell and Mai (1996), “has the advantage of simplicity and has enough parameters to enable the load-deflection curve to be accurately predicted by numerical simulation.” Figure 4.15 presents two types of bilinear relation that have been proposed for a wide range of concrete. Realistic predictions of fracture processes and structural responses in various concrete structures have been obtained by using these models. When these proposed relations fail to represent the real tension-softening characteristics for a particular type of concrete, the inverse modelling method can be employed to obtain an optimum softening relation for that particular material. With a bilinear softening relation the mode-I fracture energy is given by

$$G_F = \frac{1}{2}(f_t W_1 + s_1 W_c) \quad (4.42)$$

Figure 4.15: Proposed bilinear tension-softening models (a) by Petersson (1981) and (b) by Rokugo et al. (1989)

Part III Two Numerical Modelling Theories for Crack Analysis of Concrete

4.5 The Discrete Crack Modelling Approach

In this approach, a crack is treated as a geometric entity and the FPZ forms a part of the boundary condition, including the geometric shapes and cohesive forces acting on the crack surface. From an engineering point of view, this is the most accurate physical model to study cracks because the approach reflects most closely the physical reality of the cracked concrete and its numerical formulation represents an accurate mathematical description of the problem. Crack propagation results in a continuous change of the FE topology, which affects the element connectivity and the stiffness matrix. Applications of the discrete modelling approach for crack analysis of concrete have greatly increased in recent years, and some discrete crack models have been incorporated into some commercial FE codes that have crack analysis functions.

4.5.1 Fictitious crack model by Hillerborg and colleagues

As shown in Fig. 4.7(c), the fictitious crack model (FCM) proposed by Hillerborg, Modeer and Petersson (1976) for studying crack formation and growth in concrete brought two new concepts to the study of concrete fracture. The first concept is the physical presence of a FPZ (“microcracked zone” in their original words) in front of an open crack as a result of strain localization. The second concept is the existence of a tension-softening law inside the FPZ, stipulating the relationship between the cohesive force and the crack-opening displacement (COD). These concepts have provided the theoretical basis of the fracture mechanics of concrete.

Hillerborg and his co-workers developed these concepts based on experimental evidence of uniaxial tension tests on concrete bars that were performed by various researchers in the late 1960s. These experimental studies showed that crack formation in concrete is an evolutionary process of strain build-up in the pre-peak region, and strain localization and formation of a visible crack in the post-peak region, as illustrated in Fig. 4.16. Therefore, in their modelling of concrete fracture they imagined the zone of strain localization as a fictitious crack that “corresponds to a microcracked zone with some remaining ligaments for stress transfer.” They further suggested that this fictitious crack “may be looked upon as a reality. Stresses may be present in a microcracked zone as long as the corresponding displacement is small.” As shown in Fig. 4.16, for a bar of length L the total deformation in the post-peak region is given by

$$\Delta L = \delta_e + \delta_{ine} + W \quad (4.43)$$

Figure 4.16: The evolution of crack formation

where d_e is the elastic deformation, d_{in} is the inelastic deformation, and W is the COD of the fictitious crack. For the crack to propagate, the stress at the crack tip was assumed to reach the tensile strength of the material. They further assumed that the stress transferred through the fictitious crack was a function of the COD:

$$\sigma = f(W) \quad (4.44)$$

where $f(W)$ is a tension-softening curve unique to the material that must be determined from experiments, as discussed in the preceding section. As W reaches a critical value, W_c , the transferred stress drops to zero and the fictitious crack becomes a stress-free open crack.

For simplicity, Hillerborg and his colleagues further assumed that the inelastic deformation in the test specimen was negligible, i.e., the behaviour of the bulk material was linear elastic. Based on this assumption, Eq. (4.43) can be rewritten as

$$\Delta L = \delta_e + W = L \frac{\sigma}{E} + W = L \frac{f(W)}{E} + W \quad (4.45)$$

where E is the elastic modulus. Hence, if E and $f(W)$ are given, the load-deformation curve of the tensile bar is completely determined. As simple as Eq. (4.45) is, it underlines the general principles for crack analysis in concrete. First, it illustrates the fundamental importance of the tension-softening law: without it, crack analysis in its

strict sense cannot be carried out. Second, it shows that after cracking, the general deformation of the cracked body is composed of two parts, i.e., the deformation of the elastic body and the crack-opening width of the crack. Third, the elastic body outside the cracking zone is subjected to the stress transferred through the fictitious crack, and therefore, its deformation is also a function of the COD.

As such, the essence of crack analysis is to determine the general deformation of the body after the occurrence of cracks. If the total deformation of the cracked body is determined, strain and stress analyses can then be carried out without any particular difficulty. It should be noted that, though the assumption of linear elasticity for the bulk material makes the problem easier to solve, it is not an essential requirement and the FCM can be applied to more general situations including nonlinear behaviour of the bulk material. In general, however, the following hypotheses are used when applying the FCM in crack analysis of concrete:

1. The bulk material behaviour is isotropic linear elastic, and hence is defined by elastic modulus E and Poisson's ratio ν .
2. A crack initiates at a point when the maximum principal stress at that point reaches the tensile strength f_t , and the crack forms normal to the direction of the principal tensile stress.
3. As a crack forms and its crack opening increases, stress transfer takes place across the crack surfaces until a critical COD is reached. For pure mode-I opening, the stress transferred between the surfaces of the crack is a function of the COD, as defined by Eq. (4.44).

It is obvious that Eqs. (4.43) and (4.45) are derived based on the known cracking behaviour of a tensile bar, which is idealized as the crack forms perpendicular to the bar axis and its crack width is uniform. In most crack analysis problems, however, this is not the case because the growth of a fictitious crack is a highly nonlinear process which usually requires numerical procedures to find the solution.

4.5.2 Numerical formulation of a single-crack problem

The analytical concept of the crack-tip-controlled modelling of a single crack, formulated and referred to as the crack equations by Ohtsu (1990), is discussed below. The method offers an easy way of regulating cracking behaviour, which will be very useful when modelling multiple cracks. Figure 4.17 shows a single crack of the mode-I type, propagating in the direction normal to the maximum principal stress at the tip of the crack. The basic equations governing the next-step crack growth (by moving the present tip of the crack to the next node along the presumed future crack path) are obtained using the principle of superposition in elasticity. Assuming the solid body to be linear elastic, the overall equations are derived by linear combinations of individual solutions that make use of the influence coefficients due to unit external loads (Fig.

4.17(a)) and unit cohesive forces at the fictitious crack (Fig. 4.17(b)), respectively. Note that the external loads and the cohesive forces are unknowns and will be determined by solving the following crack equations.

Figure 4.17: Concept of the crack-tip-controlled modelling of a single crack: (a) forces and displacements at the crack due to unit external loads, (b) forces and displacements at the crack due to a pair of unit cohesive forces, and (c) load condition for crack propagation

As shown in Fig. 4.17, the crack and its fictitious prolongation are represented by separating pairs of dual nodes, a special kind of node consisting of two nodal points of the same coordinates that are rigidly connected to each other before separation: refer to Fig. 4.18 for crack path modelling using dual nodes. In formulating crack equations, subscript l represents the limit value of a nodal force, and superscripts i and j denote the i -th and j -th nodes. Let CR represent the reaction at the tip of the crack under a unit external load, and CI^i denote the tip force due to a pair of unit cohesive forces at the i -th node of the fictitious crack. For the crack to propagate, the tip tensile force must reach the tensile strength of concrete, which is given by

$$Q_i = CR \cdot P + \sum_{i=1}^M CI^i F^i \quad (4.46)$$

where Q_i is the limit nodal force (the tensile strength of concrete times the surface area apporportioned to a nodal point); P is the external load; F^i represents the pair of cohesive forces at the i -th node; and M is the number of nodes inside the fictitious crack. Equation (4.46) is the condition for stable crack propagation.

The COD at the i -th node is given by

$$W^i = BK^i \cdot P + \sum_{j=1}^M AK^{ij} F^j \quad (4.47)$$

where $i = 1, \dots, M$. Here, BK^i is the compliance at the i -th node due to the external load P . The influence coefficient AK^{ij} is the displacement at the i -th node due to a pair of unit cohesive forces at the j -th node. By virtue of the reciprocity theorem, $AK^{ij} = AK^{ji}$. The geometric shape of the fictitious crack is determined by Eq. (4.47). Note that CR , CI^i , BK^i and AK^{ij} are all determined by linear elastic FE computations based on the FE models shown in Fig. 4.17(a) and (b).

Figure 4.18: Crack-path modelling with dummy elements and dual nodes: Dummy element (i) is composed of nodes 1 to 3, and node 1 and node 3 share the same coordinates.

Along the fictitious crack, the cohesive forces F^i and the crack-opening displacements W^i must obey the tension-softening law of concrete

$$F^i = f(W^i) \quad (4.48)$$

where $i = 1, \dots, M$. Equations (4.46)-(4.48) form the so-called crack equations, which are composed of the crack growth condition, the crack shape function, and the cohesive force-COD relation of the fictitious crack. These equations represent a mathematical formulation of the crack propagation in a single-crack problem. With the number of equations $(2M + 1)$ matching the number of unknowns $(2M + 1)$, the problem can be uniquely solved to obtain the external load P and the cohesive forces F , since these equations are linearly independent. Then, the stress and displacement fields due to the next-step crack propagation can be calculated under the newly-obtained boundary condition, as shown in Fig. 4.17(c). This computational process is continued until structural failure. As such, the crack equations are the constitutive equations for crack growth in concrete, and the crack analysis is a continuing process of finding the loading history and the equilibrium states of a propagating crack.

As cracks in concrete structures are generally curvilinear due to the existence of shear force on the crack surface and their exact locations are unknown prior to numerical analysis, remeshing is necessary for modelling discrete crack propagation. A simple mesh modification method is introduced, which employs triangular elements that are connected in a regular manner (Shi, 2009). As shown in Fig. 4.19(a)-(c), after obtaining a new stress field the next-step crack propagation is set in the direction normal to the tip tensile force Q , and the presumed future crack path marked by a row of dual nodes ahead of the crack is then shifted parallel by moving the nearest nodes of corresponding elements to the new positions indicated in each case. Except a few interchanges between the normal nodes and the dual nodes in the two cases shown in Fig. 4.19(b) and (c), no additional nodes and meshes need to be generated in this process. Though Fig. 4.19(a)-(c) only illustrate the cases when the crack curves to the left, the same rules apply when the crack curves to the right. As shown in Fig. 4.19(d), in the equivalent cases of Fig. 4.19(b) and (c) reconstructions of the two meshes on the right side of the crack are required prior to constructing the new crack path.

Note that this simple scheme should be used with caution because it may result in ill-shaped elements. This problem can be avoided by readjusting the shapes of the modified elements with poor aspect ratio after remeshing, while keeping the newly-revised future crack path unchanged. Apparently, some additional computations are required to recalculate the stress and displacement fields before moving on to the next step of the crack analysis.

Figure 4.19: Remeshing scheme: (a) for left-curving without interchanges of normal and dual nodes ($r \leq \theta/2$); (b) for left-curving with interchanges of normal and dual nodes ($\theta/2 < r \leq \theta$); (c) for left-curving with interchanges of normal and dual nodes ($r > \theta$); and (d) for right-curving: changing element compositions and forming new meshes

4.5.3 Numerical formulation of a multiple-crack problem

The fundamental difference between single-crack and multiple-crack problems is whether the next-step cracking behaviour is predictable or not. The mathematical formulation for single-crack problems discussed in the previous section is based on the mode of crack propagation, which is the only valid mode for a single crack under tension. When multiple cracks are involved, the next-step cracking behaviour cannot be uniquely determined because there are multiple potential cracking modes for the given problem, including crack propagation, crack arrest and crack closure. Therefore, crack equations for a multiple-crack problem must be established by taking into account these potential cracking modes.

The following formulations are based on the single-active-crack modes in which only one crack is assumed to be propagating. Figure 4.20 illustrates two cracks of the mode-I type, crack A and crack B, where crack propagation is set in the direction normal to the tensile force at the tip of each fictitious crack. In formulating crack equations, subscripts a and b represent, respectively, crack A and crack B, and l stands for the limit value of a nodal force. Superscripts i , j and k denote the corresponding nodes at designated cracks. For clarity, the cohesive forces and the CODs of the inactive crack are marked by asterisks. To begin with, crack A is assumed to be the sole propagating crack. Hence, the tensile force at its tip must reach the nodal force limit Q_{la} , given by

$$Q_{la} = CR_a \cdot P_a + \sum_{i=1}^N CI_{aa}^i F_a^i + \sum_{j=1}^M CI_{ab}^j F_b^{*j} \quad (4.49)$$

where N and M are the number of nodes inside each fictitious crack, respectively. Note that the tensile forces at the tip of crack A, CR_a , CI_{aa}^i and CI_{ab}^j , are due to a unit external load, a pair of unit cohesive forces at the i -th node of crack A, and a pair of unit cohesive forces at the j -th node of crack B, respectively. The external load P_a is the required load for propagating crack A, while crack B remains inactive. It should be noted that the tip force components due to the cohesive forces of crack B in Eq. (4.49) represent crack interaction.

The CODs along the two fictitious cracks are given by

$$W_a^i = BK_a^i \cdot P_a + \sum_{k=1}^N AK_{aa}^{ik} F_a^k + \sum_{j=1}^M AK_{ab}^{ij} F_b^{*j} \quad (4.50)$$

$$W_b^{*j} = BK_b^j \cdot P_a + \sum_{i=1}^N AK_{ba}^{ji} F_a^i + \sum_{k=1}^M AK_{bb}^{jk} F_b^{*k} \quad (4.51)$$

where $i = 1, \dots, N$; $j = 1, \dots, M$. Here, the compliances BK_a^i at crack A and BK_b^j at crack B are due to the external load. The influence coefficients AK_{aa}^{ik} and AK_{ab}^{ij} are the CODs at the i -th node of crack A due to a pair of unit cohesive forces at the k -th node of crack A, and a pair of unit cohesive forces at the j -th node of crack B, respectively. Similarly, the influence coefficients AK_{ba}^{ji} and AK_{bb}^{jk} represent the CODs at the j -th node of crack B due to a pair of unit cohesive forces at the i -th node of crack A, and a pair of unit cohesive forces at the k -th node of crack B, respectively. According to the reciprocity theorem, $AK_{aa}^{ik} = AK_{aa}^{ki}$, $AK_{bb}^{jk} = AK_{bb}^{kj}$, and $AK_{ab}^{ij} = AK_{ba}^{ji}$.

Finally, imposing the tension-softening law of concrete along each fictitious crack leads to

$$F_a^i = f(W_a^i) \quad (4.52)$$

$$F_b^{*j} = f(W_b^{*j}) \quad (4.53)$$

where $i = 1, \dots, N$; $j = 1, \dots, M$. Equations (4.49)-(4.53) form the crack equations, stipulating the conditions for crack A to propagate. With the number of equations

$(2N + 2M + 1)$ matching the number of unknowns $(2N + 2M + 1)$, the problem is solved uniquely to obtain the external load P_a , the cohesive forces and CODs at the two cracks.

Figure 4.20: Crack-tip-controlled modelling of multiple cracks: (a) forces and displacements at the cracks due to unit external loads, (b) forces and displacements at the cracks due to a pair of unit cohesive forces at crack A, (c) forces and displacements at the cracks due to a pair of unit cohesive forces at crack B, (d) load condition for the growth of crack A, and (e) load condition for the growth of crack B

Alternatively, when crack B is assumed to be the only active crack, the crack equations are derived as

$$Q_{lb} = CR_b \cdot P_b + \sum_{i=1}^N CI_{ba}^i F_a^{*i} + \sum_{j=1}^M CI_{bb}^j F_b^j \quad (4.54)$$

$$W_a^{*i} = BK_a^i \cdot P_b + \sum_{k=1}^N AK_{aa}^{ik} F_a^{*k} + \sum_{j=1}^M AK_{ab}^{ij} F_b^j \quad (4.55)$$

$$W_b^j = BK_b^j \cdot P_b + \sum_{i=1}^N AK_{ba}^{ji} F_a^{*i} + \sum_{k=1}^M AK_{bb}^{jk} F_b^k \quad (4.56)$$

$$F_a^{*i} = f(W_a^{*i}) \quad (4.57)$$

$$F_b^j = f(W_b^j) \quad (4.58)$$

where the external load P_b is the required load for propagating crack B, while crack A is assumed to be inactive. Equations (4.54)-(4.58) are the crack equations that set the conditions for crack B to propagate.

Equations (4.49)-(4.58) are the two sets of crack equations required for modelling two discrete cracks. The influence coefficients employed in the crack equations are determined by linear elastic FE computations based on the FE models shown in Fig. 4.20(a-c). Upon solving the two sets of crack equations, the true cracking mode can be identified based on the minimum load criterion, which predicts the onset of crack propagation at the minimum load, i.e.,

$$P = \min(P_a, P_b) \quad (4.59)$$

After setting up the true crack paths for the next-step crack propagation, the stress and displacement fields are calculated under the condition of the obtained load and cohesive forces, as shown in Fig. 4.20(d, e). This process can be repeated until structural failure.

The theoretical basis for the minimum load criterion can be found in the Griffith energy principle, which states that crack extension occurs when the energy available for crack growth is sufficient to overcome the resistance of the material. Obviously, the Griffith energy principle is satisfied at the minimum load for crack propagation. Hence, the minimum load criterion is equivalent to an energy criterion for the growths of multiple cracks.

Obviously, the above solution procedure can be readily extended to problems with an arbitrary number of cracks. The flowchart of the computational procedure is shown in Fig. 4.21. As seen, numerical results are checked to eliminate invalid solutions upon solving the crack equations and obtaining the stress fields. These invalid solutions are encountered when an assumed cracking mode is irrelevant to the problem, and are manifested either by the tip tensile force exceeding the tensile strength at an inactive crack, or by the overlapping of the crack surfaces with negative CODs obtained at an inactive crack. In a situation like this, the crack tip is readjusted

by releasing or closing the tip nodes and the problem is recalculated, as illustrated in Fig. 4.21. To close the tip of a crack, the two disconnected nodes next to the tip of the crack are reconnected, while the previous normal traction acting at these nodes is referred to as the transient material strength of the cracked material (based on the fact that the material damage due to cracking is irreversible). By readjusting the crack-tip position, other cracking modes with geometrically admissible strain fields will emerge, which include simultaneous propagations of several cracks, and crack growth accompanied by crack closure. For further details of the solution procedures and how to extend them to solve a mixed-mode fracture problem, refer to Shi (2009).

Figure 4.21: Solution procedure for the crack-tip-controlled modelling of multiple cracks based on the single-active-crack modes

4.6 The Smeared Crack Modelling Approach

In this approach, after cracking of concrete the stiffness of a cracked element and its material strength are reduced to simulate the damaging effect of cracking on the element, which is assumed to be caused by numerous microcracks continuously distributed or smeared over the region. Obviously, this smeared-crack concept is a computational convenience and the method is best suited to studying the effect of fracture on general structural behaviours, such as load-carrying capacity and structural stability. The advantage of the method lies in its simplicity and cost-effectiveness since the topology of the FE mesh remains unchanged due to the continuity assumption for the cracked concrete. This modelling approach does have some weaknesses, such as its tendency to cause diffused crack patterns and the directional bias. Smeared crack models are widely used in engineering practice and have been incorporated in commercial FE codes that have crack analysis functions.

4.6.1 Crack band model

Based on the continuity assumption for cracked concrete in the smeared crack approach, Bazant and Oh (1983) modelled the fracture process zone (FPZ) by a band of uniformly distributed imaginary cracks with a fixed width of w_c , as shown in Fig. 4.22. Stable crack propagation is then simulated by progressively reducing the stiffness and strength of the cracked material within this band, based on a strain-softening relationship. For concrete, the band width is normally assumed to be three times the aggregate size. In numerical simulation the width of the FPZ is assumed to be constant to avoid spurious mesh sensitivity. This ensures that the energy dissipation due to fracture per unit length of crack is a constant, which is equal to the fracture energy of the material, G_F .

For simplicity, the cracking in Fig. 4.22 is assumed to be the mode-I type of fracture, i.e., no shear stress on the crack surface. If concrete is idealised as a homogeneous material, then the effect of cracking is simulated by changing the isotropic elastic moduli matrix to an orthotropic one, thus reducing the stiffness in the direction normal to the crack plane. The softening behaviour of concrete is modelled by superimposing the fracturing strain ε_f on the elastic strain as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \varepsilon_f \end{Bmatrix} \quad (4.60)$$

where $\sigma_x, \sigma_y, \sigma_z$ = the principal stresses; $\varepsilon_x, \varepsilon_y, \varepsilon_z$ = the principal strains; ε_f = the fracture strain by the opening of microcracks; E = the modulus of elasticity; and ν = Poisson's ratio. Note that $\varepsilon_f = \delta_f/w_c$, where $\delta_f = \sum \delta_f^i$ is the sum of all the openings of individual microcracks intersecting the z-axis over the crack band width w_c .

Figure 4.22: Crack band model for simulating crack propagation in a tension plate with a crack of length a

The fracture starts when the maximum principal stress σ_z at the crack tip reaches the tensile strength of concrete, f_t . As the microcracks in the crack band start to open, a strain-softened FPZ gradually develops with increasing ε_f and decreasing σ_z . Although a realistic strain-softening relationship between σ_z and ε_f is nonlinear, the following linear approximation is often adopted in practice:

$$\varepsilon_f = f(\sigma_z) = \frac{1}{C_f} (f_t - \sigma_z) \quad (4.61)$$

where $C_f = f_t/\varepsilon_0$; ε_0 = fracture strain at the end of the strain-softening relation; refer to Fig. 4.23(a). Theoretically, the microcracks coalesce into a continuous open crack when σ_z vanishes with the total strain reaching ε_z^f . Substituting Eq. (4.61) into Eq. (4.60) leads to

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & \frac{E}{E_t} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \varepsilon_0 \end{Bmatrix} \quad (4.62)$$

where E_t is the tangent softening modulus of the postpeak σ_z - ε_z relation, as shown in Fig. 4.23(b). Note that E_t is defined as

$$\frac{1}{E_t} = \frac{1}{E} - \frac{1}{C_f} \leq 0 \quad (4.63)$$

By definition, the fracture energy G_f is obtained by the following integral:

$$G_F = w_c \int_0^{\varepsilon_0} \sigma_z(\varepsilon) d\varepsilon \quad (4.64)$$

Based on the linear strain-softening relationship of Fig. 4.23, G_f is expressed as

$$G_F = \frac{1}{2} f_t \varepsilon_0 w_c \quad \text{or} \quad G_F = \frac{f_t^2}{2C_f} w_c \quad \text{or} \quad G_F = \frac{1}{2} \left(\frac{1}{E} - \frac{1}{E_t} \right) f_t^2 w_c \quad (4.65a-c)$$

Figure 4.23: The stress–strain relationship in the crack band model: (a) linear strain-softening relation; and (b) stress–total strain relation

With the known material properties f_t , G_f and w_c , the basic parameters of the strain-softening relationship in Fig. 4.23 are obtained as

$$C_f = \frac{f_t^2}{2G_F} w_c; \quad \varepsilon_0 = \frac{f_t}{C_f} = \frac{2G_F}{f_t w_c}; \quad E_t = \frac{f_t^2 E}{f_t^2 - 2EG_F / w_c} \quad (4.66a-c)$$

As shown from the above formulations, application of the crack band model requires the size of the FE mesh to be limited to w_c . In later development of the smeared modelling approach, this requirement has been eased to allow the effect of cracking to be smeared over a zone of the finite element with the average strain-softening relationship being adjusted to conserve the fracture energy G_f , as shown in Fig. 4.24 (a). The length of the smeared crack zone is referred to as the crack characteristic length, h_c . For parallel cracks in a rectangular element, the characteristic length is usually taken as the height of the element; for inclined cracks it is often defined as the

square root of the element area or as the maximum projected length of the element in the direction parallel to the crack; referred to Fig. 4.24(b) and (c). The empirical nature of these assumptions stems from the fact that the smeared crack approach is an approximate modelling method, and therefore, the adequacy of model parameters such as the crack characteristic length should be carefully verified before solving practical problems. Note that in the case of $h_c = w_c$, a smeared crack model becomes the crack band model.

Figure 4.24: The average stress–strain relation in a smeared crack model and the crack characteristic length: (a) stress–strain relation in a cracked element; (b) crack characteristic length for parallel cracks, $h_c = h_p h_2$ etc.; (c) crack characteristic length for inclined cracks, $h_c = \sqrt{h' h''}$, h''' , etc.

4.6.2 Non-orthogonal crack model

Due to aggregate interlocking, shear tractions develop on the crack surfaces, causing the principal stress direction to rotate. Hence, it is possible that under continued loading the current principal stress at a stress integration point or Gauss point reaches the tensile strength of concrete again in a direction deviating from the normal of the originally formed crack plane. In the non-orthogonal smeared crack model proposed by De Borst and Nauta (1985), a second crack is allowed to form after the change in

principal stress directions has exceeded a threshold value, such as 30 or 45 degrees. If on subsequent loading a further rotation in principal stress directions would occur, even a third crack is allowed to develop. This modelling concept is different from the crack band model and most of the other computational models in the smeared crack approach in which a subsequent crack is allowed to occur only in the direction perpendicular to the original crack plane.

The non-orthogonal crack model is formulated below to obtain an incremental stress-strain relationship. As with the crack band model, the total crack strain increment is decomposed into a concrete strain increment $\Delta\epsilon^{co}$ and a crack strain increment $\Delta\epsilon^{cr}$:

$$\Delta\epsilon = \Delta\epsilon^{cr} + \Delta\epsilon^{co} \quad (4.67)$$

The crack strain increment $\Delta\epsilon^{cr}$ is contributed by all the individual cracks at a particular integration point as

$$\Delta\epsilon^{cr} = \Delta\epsilon_1^{cr} + \Delta\epsilon_2^{cr} + \dots \quad (4.68)$$

where $\Delta\epsilon_1^{cr}$ is the strain increment of the first crack; $\Delta\epsilon_2^{cr}$ is the strain increment of the second crack, and so on, with each strain increment being determined in the global Cartesian coordinates.

Following algebraic manipulation, the overall relation between global stress and strain is obtained as

$$\Delta\sigma = \left\{ D^{co} - D^{co} N \left[D^{cr} + N^T D^{co} N \right]^{-1} N^T D^{co} \right\} \Delta\epsilon \quad (4.69)$$

where D^{co} is the constitutive matrix of uncracked concrete given by

$$D^{co} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \quad (4.70)$$

and D^{cr} is the constitutive matrix of local cracks given by

$$D^{cr} = \begin{bmatrix} D_1^{cr} & 0 & \dots \\ 0 & D_2^{cr} & \dots \\ \dots & \dots & \dots \end{bmatrix} \quad (4.71)$$

where the number of columns and rows of D^{cr} depends on the number of cracks at the Gauss point. As shown by the zero off-diagonal terms, coupling effects between different cracks are ignored. For the i -th crack, D_i^{cr} is given by

$$D_i^{cr} = \begin{bmatrix} D_i^I & 0 \\ 0 & D_i^{II} \end{bmatrix} \quad (4.72)$$

where D_i^I is a mode-I stiffness modulus, and D_i^{II} is a mode-II shear stiffness modulus. Again, no coupling is considered between shear and normal strain on the crack plane. Details of this stiffness will be explained later.

The transformation matrix $N = [N_1 \ N_2 \ \dots]$ combines all the individual crack transformation matrices, in which

$$N_i = \begin{bmatrix} \cos^2 \theta_i & -\sin \theta_i \cos \theta_i \\ \sin^2 \theta_i & \sin \theta_i \cos \theta_i \\ 2 \sin \theta_i \cos \theta_i & \cos^2 \theta_i - \sin^2 \theta_i \end{bmatrix} \quad (4.73)$$

where θ_i is the inclination angle of the normal of the i -th crack with the x -axis, as shown in Fig. 4.25. Again, the size of N depends on the number of cracks at the Gauss point.

Figure 4.25: Local and global coordinates at the i -th crack

For the linear strain-softening relationship shown in Fig. 4.26, the mode-I stiffness modulus $D_{i,l}^I$ is obtained as

$$D_{i,l}^I = \frac{EE_t}{E - E_t} = -\frac{f_t^2 h_c}{2G_F} \quad (4.74)$$

Note that $D_{i,l}^I$ is negative in the present formulation, while its counterpart C_f in the crack band model is defined as positive. For the bilinear strain-softening relationship of Fig. 4.26, which was proposed by Cai et al. (2006), the fracture energy G_F is obtained as

Figure 4.26: Linear and bilinear strain-softening relationships in local coordinates of the i -th crack

$$G_F = -\frac{h_c}{2} \left[\frac{(\alpha_1 f_t + f_t)(1 - \alpha_1) f_t}{D_{i,bl}^I} + \frac{\alpha_1^2 f_t^2}{\alpha_2 D_{i,bl}^I} \right] = -\frac{f_t^2 h_c}{2} \left[\frac{\alpha_2 + (1 - \alpha_2) \alpha_1^2}{\alpha_2 D_{i,bl}^I} \right] \quad (4.75)$$

from which the mode-I stiffness modulus $D_{i,bl}^I$ is obtained as

$$D_{i,bl}^I = \frac{\alpha_2 + (1 - \alpha_2) \alpha_1^2}{\alpha_2} \left(-\frac{f_t^2 h_c}{2G_F} \right) = \frac{\alpha_2 + (1 - \alpha_2) \alpha_1^2}{\alpha_2} D_{i,l}^I \quad (4.76)$$

where α_1 and α_2 are bilinear softening shape parameters defined in Fig. 4.26: α_1 = ratio of the tensile strength at the concave point, and α_2 = ratio of the mode-I stiffness modulus of the first slope to that of the second slope. Note that for $\alpha_1 = 0$ or $\alpha_2 = 1$, $D_{i,bl}^I = D_{i,l}^I$, the strain softening becomes linear. Figure 4.27 shows how the shapes of the bilinear diagram change and their relationship with the linear mode-I softening modulus $D_{i,l}^I$ as α_1 is fixed at 1/3 while α_2 is assigned values of 0.1, 0.2 and 0.3, respectively. Although in engineering practice linear softening relations are frequently employed, bilinear softening relations are sometimes desirable because they lead to better results, especially when conducting laboratory tests.

A simple shear retention factor β for reducing the tangential shear modulus of the fracture plane is often used in modelling concrete cracking, with which the mode-II shear stiffness modulus can be expressed as

$$D_i^{II} = \beta G = \beta \frac{E}{2(1 + \nu)} \quad (4.77)$$

where G is the elastic shear modulus. For more realistic modelling, Cai proposed defining β as a decreasing function of the crack normal strain as

Figure 4.27: Bilinear mode-I strain-softening relations for $\alpha_1 = 1/3$; $\alpha_2 = 0.1, 0.2$ and in local coordinates of the i -th crack (Cai, et al., 2006)

$$\beta = \beta_{\max} \left(1 - \frac{\epsilon_n^{cr}}{\epsilon_n^f} \right)^p \quad (4.78)$$

where p is a constant defining the shear softening shape (Cai, Robberts and van Rensburg, 2006). As shown in Fig. 4.28, if $p = 0$, $\beta = \beta_{\max}$ (constant); if $p = 1$, shear softening is linear; if $p = 2$, shear softening is nonlinear. Normally, $\beta_{\max}$ of less than 0.5 is applied to concrete. Figure 4.29 shows the stress-strain relationship of a cracked element with unloading, reloading, closing and reopening strategies. Note that there are two approaches for a partially opened crack to close, i.e., the secant approach returning back to the origin and the elastic unloading approach with immediate closing of the crack. A computational flowchart for crack analysis using a smeared crack model is illustrated in Fig. 4.30 for reference.

It should be pointed out that a close interrelationship exists between the non-orthogonal crack model and the other types of smeared crack model (De Borst and Nauta1985). For instance, by restricting the number of cracks allowed to occur at each Gauss point to one crack in any principal stress direction with the condition of $h_c = w_c$, a crack band model is obtained. By restricting the second and the third crack to occur normal to the existing crack plane, the orthogonal smeared crack models can be derived. Obviously, a rotational crack model can be obtained by constantly renewing the principal stress directions at Gauss points and by performing crack analysis accordingly.

Figure 4.28: Relationship between shear retention factor and crack normal strain (Cai, et al., 2006)

Figure 4.29: Stress–strain relationship of a cracked element with unloading, reloading, closing and reopening crack response

Figure 4.30: A general computational flowchart for crack analysis using the smeared crack modelling approach

4.6.3 Localised smeared crack model using the secant modulus of elasticity for strain softening

In the crack band model and the non-orthogonal crack model, the stress-strain relations of cracked elements are formulated in Eqs. (4.62) and (4.69) using the tangent softening modulus E_t of the postpeak stress-strain relation. Although theoretically sound, this type of formulation is known to have a potential problem of non-convergence in its numerical solutions, especially when a crack problem becomes highly nonlinear with extensive propagation of cracks. To overcome this problem, Bhattacharjee and Leger (1994) proposed formulating the cracked stress-strain relation using a secant modulus of elasticity E_n for the postpeak region, as shown in Fig. 4.31. By continuously reducing E_n and removing the excessive stress during loading, the condition of strain softening can be strictly imposed with less likelihood of stress fluctuation along the softening slope of E_t near a convergence point; refer to Fig. 4.31(b).

Figure 4.31: Stress–strain relationship in local coordinates using the secant stiffness E_n for strain softening: (a) stress–strain relationship with E , E_t and E_n ; (b) loading with E_n , and (c) unloading/reloading with E_n

For a two-dimensional plane stress condition, the constitutive matrix relating local stresses and strains is defined as

$$D_{ns} = \frac{E}{1-\eta\nu^2} \begin{bmatrix} \eta & \eta\nu & 0 \\ \eta\nu & 1 & 0 \\ 0 & 0 & \mu \frac{1-\eta\nu^2}{2(1+\nu)} \end{bmatrix} \quad (4.79)$$

where

$$\eta = \frac{E_n}{E}; \mu = \frac{1+\nu}{1-\eta\nu^2} \left(\frac{\eta\varepsilon_n - \varepsilon_s}{\varepsilon_n - \varepsilon_s} - \eta\nu \right); (0 \leq \mu \leq 1) \quad (4.80a, b)$$

where $\varepsilon_n, \varepsilon_s$ = normal strain components in the local axis normal to and parallel with the crack plane, respectively. Note that when $\eta = 1$ and $\mu = 0$, Eq. (4.79) maintains the pre-softening isotropic elastic stress-strain relationship. The local constitutive relationship matrix, D_{ns} , can be transformed into the global coordinates as

$$[D]_{xy} = [T]^T [D]_{ns} [T] \quad (4.81)$$

where

$$[T] = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\ -2 \sin \theta \cos \theta & 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \quad (4.82)$$

With increasing strain softening, the damaged modulus of elasticity, E_n , and hence the parameters η and μ decrease gradually, and may eventually reach zero after complete fracture ($\varepsilon_n > \varepsilon_n^f$). The constitutive matrix in Eq. (4.79) is updated as the values of parameters η and μ change. During unloading and reloading, the secant modulus E_n remains unchanged, as shown in Fig. 4.31(c); the parameter μ , however, may vary during that process based on the actual strain values.

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