Pattern recognition, infeasible systems of linear inequalities, and graphs

Full-function complexes of pattern recognition allow a human or technological user to mine relevant feature data in two main directions that can be considered interconnected, depending on the goals that must be achieved by the complexes.

One direction of data mining in pattern recognition is most often referred to as *unsupervised learning*. The complex deals with a massive collection of vectors whose components represent qualitative or quantitative descriptions of various parameters that are specific for the problem domain of the user. Although some categorical labels could have preliminarily been assigned to those vectors, in order to reflect a knowledge of the domain, the complex treats the data without using any earlier classifying information. Instead, the task consists in an exploratory analysis of the massive amount of high-dimensional vectors from the feature space, which aims at the elucidation of the inner structure of the data cloud. Typically, one is interested in how many relatively dense and isolated subclouds, called *clusters*, can be discovered in the whole data cloud, and how each of them can be given a concise characterization in working terms of the problem domain.

Various strong mathematical mechanisms, as well as heuristics, are involved for preprocessing the input sample of vectors and obtaining a resulting hierarchical picture of the data cloud. Let us mention just two questions that must be answered by the designers of a complex of pattern recognition. How incomplete or missing information on the components of vectors from the feature space should be dealt with? Is there any possibility to artificially decrease the complexity of the data sample by means of an information-preserving map of the sample into a derived feature space of much lower dimension? It is clear that for obtaining concise descriptions of relatively isolated data subclouds, outermost vectors, say the vectors lying on the boundaries of the convex hulls of the subclouds, are most relevant; for this reason certain methods of thinning irrelevant vectors may be provided.

The essential topics in unsupervised learning are the choice of *metrics* that allow the recognition complex to measure the *similarity* or *distance* between vectors and between *clusters* of vectors, and the choice of the presentation format for the cluster hierarchy revealed to the user. It is convenient to visualize the hierarchy with the help of interactively scaled tree-like graphical structures that make it possible to easily reveal information on the cluster membership and on metric intercluster dissimilarities.

Although the exploratory cluster analysis surely plays an important role in data mining, the result of unsupervised learning of the recognition complex should consist in the generation of decision rules, which would allow the complex to refer any new vector of the feature space to a large isolated cluster, thus recognizing the new vector as a representative of a certain category. Such a recognition rule is based on the

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procedure of comparison of the similarities or distances between the new unclassified vector and the large isolated clusters.

The aim of this book is to present a mathematical toolset finding an application to the construction of pattern recognition complexes that solve the recognition problem, in its *geometric setting*, in the *supervised learning* mode.

Such complexes of pattern recognition begin their work with preprocessing of a *training sample*, that is, a massive collection of vectors from a high-dimensional feature space that are preliminarily divided into groups that partially represent logically uniform *classes* or *categories*. These groups reflect a certain knowledge domain in the boundaries of which every new unclassified vector entering into the complex must be referred to one of the classes.

The variety of approaches to supervised recognition learning includes such universally accepted methodologies as *nearest-neighbor classifiers*, *neural networks*, and *support vector machines*.

At consecutive stages of preprocessing, the groups from the training sample are aggregated, with the use of hierarchical tree-like structures, into two extended groups that partially represent the corresponding generalized classes.

Given an odd integer k, a k*-nearest-neighbor classifier* finds, for a new unclassified vector from the feature space, its k distinct nearest neighbors from the training sample; a majority of these neighbors belongs to one of the extended groups and, as a consequence, that group votes for the referring of the vector to the generalized class represented by the group. A hierarchically organized procedure of making similar k-nearest-neighbor decisions, that is applied to each of the extended subgroups of the training sample, allows the complex to recognize the new vector as a representative of the class partially described by a group from the training sample.

Dealing with an extended subgroup of vectors from the training sample, which is, in turn, divided into two subgroups at some stage of a hierarchical learning process, a *neural network* represents a collection of interconnected layers of neurons. *Neurons* are elementary computational operators that reflect vectors of the feature space to weighted values of a *sigmoid function* taken at certain weighted sums of the components of those vectors. As the result of supervised training, the neural network combines the responses of individual neurons into a decision, based on a mechanism of *thresholds*, which refers a new unclassified vector to some generalized subclass.

A *support vector machine* tries to find, at a step of a hierarchically organized procedure, three parallel hyperplanes of the feature space, namely the *maximal-margin hyperplane* which separates the vectors of two subgroups from the training sample and, at the same time, maximizes the distance between two *margin hyperplanes* containing the nearest vectors of the training sample that belong to different subgroups. The quadratic optimization technique allows the recognition complex to find the maximal-margin hyperplanes (when training subgroups are affinely separable) or to motivate the search for nonlinear separating surfaces (when the subgroups cannot be separated by hyperplanes). The hierarchical collection of the separating hyperplanes and surfaces makes it possible to refer new unclassified vectors from the feature space to some classes partially represented by the vectors of the training sample.

Thus, the task of the recognition complex that implements a supervised learning methodology often consists in the search for a geometric object that has a relatively simple formal description and, at the same time, strictly separates the vectors from distinct extended groups of the training sample.

In the context of the book, the above-mentioned task can be seen as the search for a separating hyperplane in an Euclidean feature space. In practice, information contained in almost any training sample leads to a situation where a unique separating hyperplane cannot be found, because the linear inequality system underlying the problem of the discrimination of the two extended groups turns out to be infeasible. Indeed, let $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ be the two extended groups of vectors from the training sample, processed at some step of the hierarchical supervised learning procedure. These are just two finite sets of vectors of the feature space $\mathbb{R}^{n-1}$. Let us augment every vector from the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ by a new nth component which is equal to 1. We thus obtain two sets $\boldsymbol{B}, \boldsymbol{C} \subset \mathbb{R}^n$, for which the recognition complex tries to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x} \rangle > 0, & \boldsymbol{b} \in \boldsymbol{B}\,, \\ \langle \boldsymbol{c}, \mathbf{x} \rangle < 0, & \boldsymbol{c} \in \boldsymbol{C}\,, \end{cases} \tag{1}$$

where $\langle \boldsymbol{b}, \mathbf{x} \rangle$ denotes the standard scalar product $\sum_{k \in [n]} b_{ik} \mathrm{x}_k$, and $[n] := \{1, 2, \ldots, n\}$. The *strictness* of these *homogeneous* inequalities is motivated by the stability demands that must be satisfied by the decision rules generated by the pattern recognition complex.

If $\boldsymbol{x}$ is a solution to system (1), then classification of a new vector $\boldsymbol{g} \in \mathbb{R}^n$ (i.e., the referring of $\boldsymbol{g}$ to one of the extended classes partially represented by the sets $\boldsymbol{B}$ and $\boldsymbol{C}$) is performed on the basis of the sign of the scalar product $\langle \boldsymbol{x}, \boldsymbol{g} \rangle$. However, the system under consideration can turn out to be infeasible, and this most frequent case requires the development of special methods of problem-solving.

Even if system (1) as a whole has no solution, any of its feasible subsystems can be solved by the software of the recognition complex that implements techniques of linear optimization.

By means of the passage from system (1) to the infeasible system

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x} \rangle > 0, & \boldsymbol{b} \in \boldsymbol{B}\,, \\ \langle -\boldsymbol{c}, \mathbf{x} \rangle > 0, & \boldsymbol{c} \in \boldsymbol{C}\,, \end{cases}$$

which we will briefly describe here as the system

$$\{\langle \boldsymbol{a}, \mathbf{x} \rangle > 0 \colon\ \boldsymbol{a} \in \boldsymbol{A}\}\,, \tag{2}$$

the recognition complex deals with the mathematical construction that has the principal feature: if any subsystem, with two inequalities, of system (2) is feasible, then this simple condition guarantees that the recognition complex can involve in its computational arsenal a powerful technique for constructing certain *collective solutions* to infeasible system (2), and further use them as the components of hierarchical decision rules for recognition.

Recall that a *committee* of infeasible system (2) is defined as a finite subset of vectors $\mathcal{K} \subset \mathbb{R}^n$ satisfying the relation

$$|\{\boldsymbol{x} \in \mathcal{K}\colon\ \langle \boldsymbol{a}, \boldsymbol{x} \rangle > 0\}| > \tfrac{1}{2}|\mathcal{K}|\ ,$$

for each vector $\boldsymbol{a} \in \boldsymbol{A}$.

Suppose that a committee $\mathcal{K}$ of system (2) is found by the recognition complex. Then an unclassified vector of the feature space $\mathbb{R}^{n-1}$, lifted to the working $(n-1)$-dimensional affine subspace of the space $\mathbb{R}^n$ with the help of the additional nth component 1, can be recognized as an element of the classes, partially represented by the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$, according to the result of the majority voting procedure performed by the members of the committee $\mathcal{K}$.

The smart committee strategy of the recognition complex consists in the finding of solutions to a few *maximal feasible subsystems* (*MFSs*) of system (2), and in their combining into the committee decision rule, which operates with arrangements of separating hyperplanes.

A feasible subsystem of infeasible system (2) is called *maximal* if any additional inequality from the system turns the resulting collection of inequalities into an infeasible subsystem.

If $[m]$ is the set of indices with which the inequalities from infeasible system (2) are marked, then a *multi-index* $T \subseteq [m]$ corresponds to the subsystem composed of the inequalities with the indices from the set T.

If we let $\mathbf{J}$ denote the family of the multi-indices of all maximal feasible subsystems of system (2), then the *graph of MFSs* of system (2) is defined as the graph with the vertex set $\mathbf{J}$; an unordered pair $\{J, J'\} \subset \mathbf{J}$ is an edge of this graph if and only if the multi-indices J and J' cover the index set of system (2), that is, $J \cup J' = [m]$.

The high efficiency of supervised learning algorithms implemented by the recognition complex, which uses the graph of MFSs, is explained by the following three basic facts:

- The graph of MFSs is *connected*.
- The graph of MFSs is *not bipartite*.
- The complement $[m] - J$ of the multi-index $J \in \mathbf{J}$ of any *MFS* of system (2) is the multi-index of a *feasible subsystem*.

Since the graph of MFSs is not bipartite, it contains at least one *cycle of odd length*.

A fundamental result in the committee theory is formulated as follows: if the multi-indices of some MFSs represent the vertex set of a cycle of odd length in the graph of MFSs, then in order to construct a committee, it suffices to take one vector from the open cone of solutions to each MFS from the vertex set of the cycle.

Thus, the problem of constructing a committee with a small number of members can be reduced to the problem of finding a cycle of short odd length in the graph of MFSs. This derived problem is solved by the software of the recognition complex with the help of various strong and heuristic methods.

On the one hand, the obtained committee decision rule always allows the recognition complex to correctly discriminate the vectors from the two extended groups of the training sample and, on the other hand, it makes it possible to apply the procedure of committee voting to a new vector entering into the complex; the majority decision rule, governed by the committee, refers the new vector to a generalized class.

The complex repeatedly solves the two-class pattern recognition problem for each higher level extended group of vectors from the training sample, adding at every step some committee decision rule to a resulting hierarchical tree-like structure.

This structure represents the machine for recognition of new vectors, and it correctly recognizes any vector of the training sample.

1 Infeasible monotone systems of constraints

In discrete mathematics, the following research subjects are of prime importance: Let $\mathfrak{S} := \{\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_m\}$ be a finite nonempty *system of constraints* and $[m] := \{1, 2, \ldots, m\}$ the set of the *indices* of constraints with which the elements of the set $\mathfrak{S}$ are marked. Assigning to the set $[m]$ the *Boolean lattice* $\mathbb{B}(m)$ of all its subsets partially ordered by set-theoretical inclusion, we call an arbitrary element $B \in \mathbb{B}(m)$ the *multi-index of the subsystem* $\{\mathfrak{s}_i : i \in B\}$ of the system $\mathfrak{S}$; in many studies the shorter term *index of a subsystem* is used. To the relation $A \subseteq B$ of inclusion for the multi-indices $A, B \subseteq [m]$ corresponds the comparison relation $A \preceq B$ for the elements A and B in the lattice $\mathbb{B}(m)$. The set of *atoms* $\mathbb{B}(m)^{(1)} := \{\{1\}, \{2\}, \ldots, \{m\}\}$ of the lattice $\mathbb{B}(m)$ is in one-to-one correspondence with the set of constraints $\mathfrak{S}$. The *least element* $\hat{0}$ of the lattice $\mathbb{B}(m)$ is the multi-index of the *empty subsystem* $\emptyset$ of the system $\mathfrak{S}$, and the *greatest element* $\hat{1}$ of the lattice $\mathbb{B}(m)$ is the multi-index $[m]$ of the entire system $\mathfrak{S}$.

Let a map $\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\Gamma}$ into the family of subsets of some nonempty set Γ be given, with the following properties:

- The empty subsystem of the system $\mathfrak{S}$ is *feasible*, that is,

$$\pi(\hat{0}) \neq \emptyset\ ; \tag{1.1}$$

  one usually supposes $\pi(\hat{0}) := \Gamma$.
- Each constraint taken independently is realizable or, in other words, each subsystem consisting of one constraint is *feasible*:

$$B \in \mathbb{B}(m)^{(1)} \Longrightarrow \pi(B) \neq \emptyset\ . \tag{1.2}$$

- Further,

$$A,\ B \in \mathbb{B}(m), \quad A \preceq B \Longrightarrow \pi(A) \supseteq \pi(B)\ , \tag{1.3}$$

  and thus

$$A,\ B \in \mathbb{B}(m) \Longrightarrow \pi(A) \cap \pi(B) \supseteq \pi(A \vee B)\ ,$$

  where $A \vee B$ denotes the least upper bound (i.e., the set-union $A \cup B$) of the elements A and B in the lattice $\mathbb{B}(m)$.
- One often considers *infeasible systems* $\mathfrak{S}$ such that

$$\pi(\hat{1}) = \emptyset\ . \tag{1.4}$$

We will call any system of constraints $\mathfrak{S}$, for which the map π and the range family of this map associated with $\mathfrak{S}$ satisfy conditions (1.1)–(1.4), a *finite infeasible monotone system of constraints*.

1.1 Structural and combinatorial properties of infeasible monotone systems of constraints

In this chapter, we particularly describe some properties of constraint systems, which are essentially associated with the *representativity of sets*.

Speaking briefly, the mutual representativity of sets A and B of any kind is related to the answer to the question on the nonemptiness of their intersection $A \cap B$.

The subject of this chapter goes back to the standard problem of *combinatorial optimization*: for a nonempty family $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$ of nonempty and pairwise distinct subsets of a finite *ground set* $\mathrm{V}(\mathcal{A}) := \bigcup_{i=1}^{\alpha} A_i$, to describe, from the structural and combinatorial points of views, the properties of the pair $(\mathcal{A}, \mathfrak{B}(\mathcal{A}))$, where $\mathfrak{B}(\mathcal{A})$ is the family of all *minimal* (by inclusion) *systems of representatives* for $\mathcal{A}$ – several equivalent terms which will be mentioned later are used for naming these constructions; by definition, *any set from* $\mathfrak{B}(\mathcal{A})$ *has a nonempty intersection with each set from the family* $\mathcal{A}$ *and, under the removal of its arbitrary element, lacks this property*.

A subset $B \subseteq \mathrm{V}(\mathcal{A})$ satisfying the condition

$$B \cap A_i \neq \emptyset\,, \quad \forall\, i \in [\alpha]\,, \tag{1.5}$$

is called a *system of representatives*, *blocking set*, *transversal*, or *transversal set* of the family $\mathcal{A}$. From the graph-theoretic point of view, the family $\mathcal{A}$ is the family of *hyperedges of a hypergraph* on the set of *vertices* $\mathrm{V}(\mathcal{A})$ and, in this context, the set B with property (1.5) is called a *vertex cover* of the hypergraph.

Note that the system of representatives B of $\mathcal{A}$ contains as a subset at least one minimal system of representatives.

If a family $\mathcal{A}$ has the property

$$A_i \not\subseteq A_j \quad \forall\, i, j \in [\alpha],\;\; i \neq j\,,$$

(in other words, if the sets from $\mathcal{A}$ are pairwise *incomparable by inclusion*) then the following terms are used for such a family: a *Sperner family*, *clutter*, or *antichain*. Note that for an arbitrary family $\mathcal{A}$, the corresponding family $\mathfrak{B}(\mathcal{A})$ is by definition a Sperner family.

In the theory of combinatorial optimization, the family $\mathfrak{B}(\mathcal{A})$ is called the *blocker* of the family $\mathcal{A}$; its sets $B \in \mathfrak{B}(\mathcal{A})$ are called the *minimal* (by inclusion) *blocking sets*, *minimal transversals*, *minimal transversal sets*, or *minimal systems of representatives* (the last three terms should not be confused with related terms used in an analysis of systems of *distinct* representatives, which are irrelevant to the subject of our research).

The question on the systems of representatives of the family $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$, with the ground set $\mathrm{V}(\mathcal{A}) = [m]$, of nonempty pairwise distinct sets can be posed from several points of views. Recall some of them:

(a) Functions of logical algebra

The problem on systems of representatives is instructive for elucidation of the basic principles of the application of Boolean algebra and of the very effective trick of "the turning elements and sets into propositions."

The *proposition* stating that some set B is a system of representatives of a family $\mathcal{A}$ looks like

$$\Big(\bigvee_{\mathrm{x}\in A_1} \text{“}\mathrm{x} \in B\text{”}\Big) \wedge \Big(\bigvee_{\mathrm{x}\in A_2} \text{“}\mathrm{x} \in B\text{”}\Big) \wedge \cdots \wedge \Big(\bigvee_{\mathrm{x}\in A_\alpha} \text{“}\mathrm{x} \in B\text{”}\Big) = 1\,. \tag{1.6}$$

It is customary to write, for brevity, the character x instead of the proposition "$\mathrm{x} \in B$"; it is thought of as a *Boolean variable* (whose value is 1 when x belongs to the set B, and 0 otherwise.) It is yet convenient to write disjunction as sum, and conjunction as product. Then the proposition "B is a system of representatives of $\mathcal{A}$" of the form (1.6) can be reformulated as follows:

$$\Big(\sum_{\mathrm{x}\in A_1} \mathrm{x}\Big) \cdot \Big(\sum_{\mathrm{x}\in A_2} \mathrm{x}\Big) \cdots \Big(\sum_{\mathrm{x}\in A_\alpha} \mathrm{x}\Big) = 1\,, \tag{1.7}$$

and this equation is satisfied by those tuples of the values of Boolean variables $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_m$ only for which the proposition is true.

In order to find the minimal systems of representatives of $\mathcal{A}$, one transforms the left-hand side of equation (1.7) into the *minimal disjunctive normal form*

$$\prod_{\mathrm{x}\in B_1} \mathrm{x} + \prod_{\mathrm{x}\in B_2} \mathrm{x} + \cdots + \prod_{\mathrm{x}\in B_\beta} \mathrm{x} = 1\,,$$

by removing parentheses and using the absorption law. Such a form is unique. The sets $B_1, B_2, \ldots, B_\beta \subseteq \mathrm{V}(\mathcal{A})$ are the minimal systems of representatives of the family $\mathcal{A}$, and they compose its blocker $\mathfrak{B}(\mathcal{A})$.

(b) $(0, 1)$-matrices

One can put in correspondence with a cover $\mathcal{A}$ of the set $[m]$ the binary *incidence matrix* of size $\alpha \times m$ whose (i, j)th entry by definition equals to 1 when $j \in A_i$, and to 0 otherwise.

Incidence matrices serve as the main connecting link between combinatorial problems on the existence and choice, on the one hand, and matrix theory, on the other.

The *width* of an incidence matrix is the minimal number of its columns such that the sum of elements in each row of the submatrix formed by the selected columns is positive. The width of this matrix coincides with the least quantity among the cardinalities of the representative systems of the family $\mathcal{A}$.

(c) Transversal sets (vertex covers) of hypergraphs

In graph theory, as was noticed earlier, the set $\mathrm{V}(\mathcal{A})$ and the family $\mathcal{A}$ are interpreted as a hypergraph on the vertex set $\mathrm{V}(\mathcal{A})$, with the hyperedge family $\mathcal{A}$. If any hyperedge of a hypergraph is of cardinality r then such a hypergraph is called *r-regular*. The 2-regular hypergraphs are called *simple graphs* without isolated vertices.

The *transversal sets* of the hypergraph are precisely the systems of representatives of the hyperedge family $\mathcal{A}$. The minimal cardinality of transversal sets is called the *transversality number* of the hypergraph.

The importance of the problems on the transversality number of hypergraphs comes from the fact that many combinatorial questions can be reformulated in terms of finding the transversality number of some hypergraph.

(d) Bipartite graphs and binary relations

A simple graph $\mathbf{G}(V_1, V_2; \mathcal{E})$ is called *bipartite* if its vertex set $V_1 \dot\cup V_2$ is a union of two nonempty disjoint sets (*parts* or *classes*) V_1 and V_2 such that each edge from the family $\mathcal{E}$ is incident to vertices from different classes. If one orients all the edges of $\mathbf{G}(V_1, V_2; \mathcal{E})$ in the direction from V_1 to V_2 then the graph $\mathbf{G}(V_1, V_2; \mathcal{E})$ can be identified with a *binary relation* (some subset of the *Cartesian product* $V_1 \times V_2$) on the classes V_1 and V_2.

Let us put in correspondence with the set family $\mathcal{A}$ under consideration the bipartite graph $\mathbf{G}(\mathrm{V}(\mathcal{A}), \mathcal{A}; \mathcal{E})$ in which, by definition, for all vertices $v \in \mathrm{V}(\mathcal{A})$ and sets $A \in \mathcal{A}$, the inclusions $(v, A) \in \mathcal{E}$ are fulfilled if and only if $v \in A$. The systems of representatives of the family $\mathcal{A}$ are in one-to-one correspondence with subfamilies of the edge family $\mathcal{E}$ such that the sets of elements from the class $\mathcal{A}$, that are incident with them, cover this class.

(e) Systems of distinct representatives

This research topic in combinatorial optimization is the last among those we mention, and it is completely beyond the scope of our consideration. Mathematical constructions related to systems of distinct representatives are *bipartite matchings*, *transversal matroids*, and the *permanents of incidence matrices*.

A subset $B \subseteq \mathrm{V}(\mathcal{A})$ is called a *system of distinct representatives* of the family $\mathcal{A}$ if there is a bijection $\phi\colon B \to [\alpha]$ such that for each element $b \in B$ the inclusion $b \in A_{\phi(b)} \in \mathcal{A}$ holds.

It is evident that the family (possibly, empty) of all systems of distinct representatives of $\mathcal{A}$ is a subfamily of the family of all systems of representatives of $\mathcal{A}$. Note also that any system of distinct representatives (if such systems exist) contains as a subset at least one set from the blocker $\mathfrak{B}(\mathcal{A})$.

Recall the relationship between the combinatorial properties of two partially ordered sets (posets) that arise naturally together with specific partitions of Boolean lattices. Let $\mathbb{B}(m)$ be the Boolean lattice of rank $m \geq 1$, which is again thought of as the lattice of all subsets of the set $[m]$. Recall that this lattice represents the power set $\mathbf{2}^{[m]}$ of the set $[m]$ whose elements (*faces*) are partially ordered by set-theoretic inclusion. As earlier, we denote the least element of the lattice $\mathbb{B}(m)$, which is the empty subset $\emptyset$ of the set $[m]$, by $\hat{0}$. Let $\mathcal{V} := \{V_1, V_2, \ldots, V_\gamma\} \subseteq \mathbb{B}(m) - \{\hat{0}\}$ be some nonempty family of sets equipped with the ordering induced by the partial order $\preceq$ on $\mathbb{B}(m)$. We denote the sets of minimal and maximal elements of the poset $\mathcal{V}$ by $\mathbf{min}\,\mathcal{V}$ and $\mathbf{max}\,\mathcal{V}$, respectively. Recall that the *order ideal* $\mathfrak{I}(\mathcal{V}) = \mathfrak{I}(\mathbf{max}\mathcal{V})$ of the lattice $\mathbb{B}(m)$ generated by a set $\mathcal{V}$ is defined as the subposet $\mathfrak{I}(\mathcal{V}) := \{E \in \mathbb{B}(m) \colon\ \exists V \in \mathcal{V},\ E \preceq V\}$.

Let $\rho\colon\ \mathbb{B}(m) \longrightarrow \{0\}\,\dot\cup\,[m]$ be the *rank function* on $\mathbb{B}(m)$ that reflects the subsets of $[m]$ to their cardinalities. As earlier, we use the notation $\mathbb{B}(m)^{(1)} := \{D \in \mathbb{B}(m) \colon \rho(D) = 1\} = \{\{1\}, \{2\}, \ldots, \{m\}\}$ to denote the layer of *atoms* of the lattice $\mathbb{B}(m)$, that is, the family of one-element subsets of the set $[m]$. If $\mathcal{V}$ is an *antichain* or, in other words, if any two elements of the set $\mathcal{V}$ are *incomparable* in $\mathbb{B}(m)$, that is, $\mathcal{V} = \mathbf{min}\,\mathcal{V} = \mathbf{max}\,\mathcal{V}$, then the *unordered* family of sets

$$\Delta := \{F \subseteq [m] \colon\ F \in \mathfrak{I}(\mathcal{V})\} \tag{1.8}$$

is called the *abstract simplicial complex* on the set of *vertices* $\bigcup_{V \in \mathcal{V}} V$, with the family of *facets* $\mathcal{V}$. The sets from the complex Δ are called its *faces*, and the ideal $\mathfrak{I}(\mathcal{V})$ is called the *face poset* of the complex Δ.

The abstract simplicial complex is a fundamental construction in algebraic and combinatorial topology, discrete mathematics, and mathematical cybernetics; in some cases it is called the *independence system*, and in this context one says on the *independent sets* and *bases* of independence systems instead of the faces and facets of complexes.

We built complex (1.8) on the basis of an antichain of the Boolean lattice of subsets of a finite set, and we interpreted the antichain as the facet family. The notion of abstract simplicial complex is often introduced without initial addressing to the Boolean lattices and without using any orderings: Let $\mathcal{A}$ be a Sperner family; the abstract simplicial complex on the *vertex set* $\mathrm{V}(\mathcal{A})$, with the *facet family* $\mathcal{A}$, is defined as the set family

$$\Delta := \{F \subseteq \mathrm{V}(\mathcal{A}) \colon\ \exists A \in \mathcal{A},\ F \subseteq A\}\ . \tag{1.9}$$

Let us note once again that an arbitrary abstract simplicial complex Δ is characterized by the property

$$G \in \Delta\,, \quad F \subseteq G \quad \Longrightarrow \quad F \in \Delta\,;$$

in particular, $\emptyset \in \Delta$.

We will often deal with complexes that are reconstructed from the families of their facets, and we will use the notation $\Delta(\mathcal{A})$ to denote the complex with the given facet family $\mathcal{A}$.

Let $\Delta(\mathcal{A}) \subsetneq \mathbf{2}^{[m]}$, that is, $\mathcal{A} \neq \{\hat{1}\}$. Consider the complement $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ of the family $\Delta(\mathcal{A})$ up to the power set $\mathbf{2}^{[m]}$. If $B \in \mathbf{2}^{[m]} - \Delta(\mathcal{A})$ then $B \nsubseteq A$ or, equivalently, $B \cap ([m] - A) \neq \emptyset$ for all facets $A \in \mathcal{A}$. In other words, such a set B is a system of representatives of the family $\mathcal{A}^{\perp} := \{[m] - A \colon\ A \in \mathcal{A}\}$. From this point of view, the blocker $\mathfrak{B}(\mathcal{A}^{\perp})$ is the set $\mathbf{min}\big(\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})\big)$ of minimal elements of the subposet $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})$. In addition, the subposet $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})$ corresponding to the family $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ carries the structure of an order filter of the lattice $\mathbb{B}(m)$.

Let $\mathcal{W} \subseteq \mathbb{B}(m)$. The *order filter* $\mathfrak{F}(\mathcal{W}) = \mathfrak{F}(\mathbf{min}\ \mathcal{W})$ generated by $\mathcal{W}$ is defined as the subposet $\mathfrak{F}(\mathcal{W}) := \{E \in \mathbb{B}(m) \colon\ \exists W \in \mathcal{W},\ E \succeq W\}$.

Thus, the family $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ is represented in the lattice $\mathbb{B}(m)$ by the order filter $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A}) = \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))$.

Any antichain $\mathcal{A} \subset \mathbb{B}(m) - \{\hat{1}\}$ then induces the partition of the lattice $\mathbb{B}(m)$ of the form

$$\mathbb{B}(m) = \mathfrak{I}(\mathcal{A}) \,\dot{\cup}\, \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))\ . \tag{1.10}$$

We will below recall some basic combinatorial properties of the pair $(\mathfrak{I}(\mathcal{A}), \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp})))$.

Let us return to a specific interpretation of the lattice $\mathbb{B}(m)$ mentioned at the beginning of the chapter. Let $[m]$ be the set of indices marking the constraints that form some infeasible monotone system of constraints $\mathfrak{S}$ described by means of (1.1)–(1.4). The monotonicity property prescribed to the system $\mathfrak{S}$ means that *each subsystem of a feasible subsystem from $\mathfrak{S}$ is feasible, and each subsystem containing an infeasible subsystem is also infeasible*.

Let $\mathbf{I}$ be the family of multi-indices of *minimal* (*by inclusion*) *infeasible or irreducible infeasible subsystems – IISs*, and let $\mathbf{J}$ be the family of multi-indices of *maximal* (*by inclusion*) *feasible subsystems – MFSs* of the system $\mathfrak{S}$. A key construction associated with this system is the partition

$$\mathbb{B}(m) = \mathfrak{I}(\mathbf{J})\ \dot{\cup}\, \mathfrak{F}(\mathbf{I})\ . \tag{1.11}$$

Since

$$\mathbf{I} = \mathfrak{B}(\mathbf{J}^{\perp})\ , \tag{1.12}$$

that is, *the family of the multi-indices of minimal infeasible subsystems is the blocker of the family of complements of the multi-indices of maximal feasible subsystems*, we have $\mathfrak{F}(\mathbf{I}) = \mathfrak{F}(\mathfrak{B}(\mathbf{J}^{\perp}))$.

We will call a system $\mathfrak{S}$ *irreducible* when $\bigcap_{J \in \mathbf{J}} J = \emptyset$, and *reducible* otherwise.

1.2 Abstract simplicial complexes and monotone Boolean functions

Let $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ again be a finite family of finite and pairwise distinct sets. Recall once again that a set B is called a *system of representatives* of the family $\mathcal{A}$ if for all $i \in [\alpha]$ it holds $B \cap A_i \neq \emptyset$; in the family of the systems of representatives of $\mathcal{A}$ one distinguishes the blocker $\mathfrak{B}(\mathcal{A})$ of this family, that is, the family of all the minimal (by inclusion) systems of representatives of $\mathcal{A}$. A set B by definition belongs to the blocker $\mathfrak{B}(\mathcal{A})$ if and only if the following conditions are satisfied: (1) B is a system of representatives of $\mathcal{A}$; (2) for any element $b \in B$ there exists an index $j \in [\alpha]$ such that $(B - \{b\}) \cap A_j = \emptyset$.

Recall also that a family $\mathcal{A}$ is called a Sperner family if for all indices $i, j \in [\alpha]$, $i \neq j$, the condition $A_i \not\subseteq A_j$ is satisfied.

The following result is a key research tool in discrete mathematics:

Proposition 1.1. *If $\mathcal{A}$ is a Sperner family then*

$$\mathfrak{B}\big(\mathfrak{B}(\mathcal{A})\big) = \mathcal{A} \ .$$

Note that the main property of blockers recalled in Proposition 1.1 complements observation (1.12) made with respect to the multi-index families of minimal infeasible and maximal feasible subsystems of infeasible monotone systems of constraints, by the dual result

$$\mathbf{J} = \mathfrak{B}(\mathbf{I})^{\perp} \ . \tag{1.13}$$

If Δ is an abstract simplicial complex and $F \in \Delta$, then the *dimension* $\dim(F)$ of a face F by definition is less than its cardinality by 1: $\dim(F) = |F| - 1$. The *dimension* $\dim \Delta$ of the complex Δ is the quantity $\max\{\dim F\colon \ F \in \Delta\}$. $f_j(\Delta)$ denotes the number of j-dimensional faces of Δ. One calls the ordered collection of the integers $f_j(\Delta)$ the *f-vector* of the complex Δ. By definition, $f_{-1}(\Delta) = 1$, and $f_0(\Delta)$ is the number of vertices of the complex Δ. $\#\Delta$ denotes the total number of faces of the complex Δ.

If A is some subset of the set $[m]$ then we will denote its complement $[m] - A$ by $A^{\perp}$ and, as earlier, for a cover $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ of the set $[m] = \bigcup_{i=1}^{\alpha} A_i$ we will denote the corresponding family of complements $\{{A_1}^{\perp}, {A_2}^{\perp}, \dots, {A_\alpha}^{\perp}\}$ by $\mathcal{A}^{\perp}$.

Considering an abstract simplicial complex $\Delta(\mathcal{A})$ with facet family $\mathcal{A}$, and the corresponding order ideal $\mathfrak{I}(\mathcal{A})$ of the Boolean lattice $\mathbb{B}(m)$, it often turns out to be convenient to study the structure of the construction $\mathfrak{F}\big(\mathfrak{B}(\mathcal{A}^{\perp})\big)^{\perp}$, closely related to the complement $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A}) = \mathfrak{F}\big(\mathfrak{B}(\mathcal{A}^{\perp})\big)$, instead of the complement itself; the above-mentioned construction represents the face poset of the simplicial complex with the facets that are the complements of the minimal systems of representatives of the family $\mathcal{A}^{\perp}$ up to the set $[m]$. Thus, a study of the pair $\big(\Delta(\mathcal{A})\mathbf{2}^{[m]} - \Delta(\mathcal{A})\big)$ is most commonly substituted by a study of the pair of simplicial complexes

$$\big(\ \Delta(\mathcal{A})\ ,\ \Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)\ \big)\ ; \tag{1.14}$$

here $\Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big) = \big(\mathbf{2}^{[m]} - \Delta(\mathcal{A})\big)^{\perp}$.

From the combinatorial topological point of view, the set family $\Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)$ is the complex called the Alexander dual of $\Delta(\mathcal{A})$.

When one digresses from the key aspect of the representativity of sets characterizing the relationship between families (1.14), Alexander duals are traditionally defined as follows:

If $\Delta \subsetneqq \mathbf{2}^{[m]}$ is an abstract simplicial complex on the vertex set $[m]$ then the complex $\Delta^{\vee}$, the *Alexander dual* of Δ, is the family

$$\Delta^{\vee} := \{G^{\perp}\colon\ G \subseteq [m],\ G \notin \Delta\} \ .$$

Thus, for a Sperner cover $\mathcal{A}$ of the set $[m]$ such that $\mathcal{A} \neq \{[m]\}$, we have $\Delta(\mathcal{A})^{\vee} = \Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)$.

The following simple observation is related to the basic fact that the number of rank j elements (or, in the language of combinatorial poset theory, the jth *Whitney number of the second kind*), which correspond in the Boolean lattice $\mathbb{B}(m)$ to the j-subsets of the set $[m]$, is the binomial coefficient $\binom{m}{j} := \frac{m!}{j!(m-j)!}$:

Proposition 1.2. *Let $\Delta \subsetneq \mathbf{2}^{[m]}$ be an abstract simplicial complex on the vertex set $[m]$. Then for all j, $-1 \le j \le m-1$, it holds $f_j(\Delta) + f_{m-j-2}(\Delta^\vee) = \binom{m}{j+1}$.*

This observation makes it possible to come to several conclusions. In the first statement we use the *Kronecker delta* $\delta(s,t)$ which is equal, by definition, to 1 when $s = t$, and to 0 when $s \neq t$. The point is that the number of the faces of all dimensions in the complex $\Delta^\vee$ determine the dimension of the complex Δ.

Corollary 1.3. *Let $\Delta \subsetneq \mathbf{2}^{[m]}$ be an abstract simplicial complex on the vertex set $[m]$. Then*

$$\begin{aligned}\dim(\Delta) &= m - \sum_{j=-1}^{m-1} \delta\left(f_j\left(\Delta^\vee\right), \tbinom{m}{j+1}\right) - 1 \\ &= m - \max\left\{i \in \{0\} \dot\cup [m] : \ f_{i-1}\left(\Delta^\vee\right) = \tbinom{m}{i}\right\} - 2 .\end{aligned}$$

The total number of faces in the partition under consideration is equal to the number of sets in the power set $\mathbf{2}^{[m]}$, namely $\#\Delta + \#\Delta^\vee = 2^m$.

Let us denote the number of all feasible and infeasible subsystems, of cardinality k, of an infeasible monotone system $\mathfrak{S}$ by ν_k and τ_k, respectively. We have $\nu_k + \tau_k = f_{k-1}(\Delta(\mathbf{J})) + f_{m-k-1}(\Delta(\mathbf{I}^\perp)) = \binom{m}{k}$, for all $k \in [m]$, and $\#\Delta(\mathbf{J}) + \#\Delta(\mathbf{I}^\perp) = 2^m$.

Speaking of combinatorial tools of the exact enumeration of the faces of complexes with the known structure of their facet families, it is worth recalling that the combinatorial *inclusion–exclusion principle* is formulated, in one of its various forms, as follows:

– The number $\mathrm{N}_k(\mathcal{A})$ of k-subsets of the set $[m]$ containing as a subset at least one set $A_i \in \mathcal{A}$ is
$$\mathrm{N}_k(\mathcal{A}) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{m - \left|\bigcup_{t\in T} A_t\right|}{m-k} ; \tag{1.15}$$

– the number $\mathrm{N}_k(\mathcal{A}^\perp)$ of k-subsets of the set $[m]$ containing as a subset at least one set ${A_i}^\perp \in \mathcal{A}^\perp$ is
$$\mathrm{N}_k(\mathcal{A}^\perp) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{\left|\bigcap_{t\in T} A_t\right|}{m-k} ; \tag{1.16}$$

– the number $\mathrm{R}_k(\mathcal{A})$ of k-subsets of the set $[m]$ contained in at least one set $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ is
$$\mathrm{R}_k(\mathcal{A}) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{\left|\bigcap_{t\in T} A_t\right|}{k} ; \tag{1.17}$$

– the number $\mathrm{R}_k(\mathcal{A}^\perp)$ of k-subsets of the set $[m]$ contained in at least one set $A_i{}^\perp \in \mathcal{A}^\perp$ is

$$\mathrm{R}_k(\mathcal{A}^\perp) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{m - \left|\bigcup_{t\in T} A_t\right|}{k} . \tag{1.18}$$

If the structure of the family $\mathbf{J}$ of the multi-indices of maximal feasible subsystems (or the structure of the family $\mathbf{I}$ of the multi-indices of minimal infeasible subsystems) of the system $\mathfrak{S}$ is known then the inclusion–exclusion principle allows us to find the number ν_k of all feasible subsystems of cardinality k and the number τ_k of all infeasible subsystems of cardinality k. Let us make use of relations (1.15) and (1.17), taking into account that any feasible subsystem is contained in at least one MFS, and any infeasible subsystem contains at least one IIS. We come to the conclusion:

Proposition 1.4. *Let $\mathfrak{S}$ be a finite infeasible monotone system of constraints, and let $\mathbf{I}$ and $\mathbf{J}$ be the families of the multi-indices of its IISs and MFSs, respectively. Let τ_k and ν_k be the numbers of infeasible subsystems and of feasible subsystems, of cardinality k, respectively. Then*

$$\tau_k = \binom{m}{k} - \nu_k = -\sum_{s\in[\#\mathbf{I}]} (-1)^s \cdot \sum_{\substack{T\subseteq[\#\mathbf{I}]:\\ |T|=s}} \binom{m - \left|\bigcup_{t\in T} I_t\right|}{m-k} ,$$

$$\nu_k = \binom{m}{k} - \tau_k = -\sum_{s\in[\#\mathbf{J}]} (-1)^s \cdot \sum_{\substack{T\subseteq[\#\mathbf{J}]:\\ |T|=s}} \binom{\left|\bigcap_{t\in T} J_t\right|}{k} . \tag{1.19}$$

If the quantities τ_k and ν_k are known then Corollary 1.3 allows us to determine for the system $\mathfrak{S}$ the extremal sizes of its IISs and MFSs.

Proposition 1.5. *Let $\mathfrak{S}$ be a finite infeasible monotone system of constraints.*

– *The cardinality of the smallest IIS is*

$$\begin{aligned}\min_{I\in\mathbf{I}} |I| &= \sum_{t=0}^{m} \delta\left(\nu_t, \tbinom{m}{t}\right)\\ &= \max\left\{k \in \{0\}\,\dot\cup\,[m] :\ \nu_k = \tbinom{m}{k}\right\} + 1 .\end{aligned}$$

– *The cardinality of the largest MFS is*

$$\begin{aligned}\max_{J\in\mathbf{J}} |J| &= m - \sum_{t=0}^{m} \delta\left(\tau_{m-t}, \tbinom{m}{t}\right)\\ &= m - \max\left\{k \in \{0\}\,\dot\cup\,[m] :\ \tau_{m-k} = \tbinom{m}{k}\right\} - 1 .\end{aligned}$$

Now we will briefly discuss some combinatorial characteristics of hypergraphs, which are put in correspondence with simple graphs by means of representative systems.

Let $\mathbf{H}([m], \mathcal{A})$ be a hypergraph with the vertex set $[m]$ and with the hyperedge family $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$. As usual, $\mathfrak{B}(\mathcal{A})$ denotes the blocker of the family $\mathcal{A}$. For what hypergraphs $\mathbf{H}([m], \mathcal{A})$ are the corresponding hypergraphs $\mathbf{H}([m], \mathfrak{B}(\mathcal{A}))$ finite simple graphs? In order to answer this question, we will use the following auxiliary statement:

Proposition 1.6. *If a family $\mathcal{A}$ is Sperner then $\bigcap_{E \in \mathfrak{B}(\mathcal{A})} E = \emptyset$ if and only if the family $\mathcal{A}$ contains no one-element sets.*

Proof. Let us denote the family $\mathfrak{B}(\mathcal{A})$ by $\mathcal{E}$. In view of Proposition 1.1, $\mathcal{A} = \mathfrak{B}(\mathcal{E})$.

The *sufficiency*: since $\bigcap_{E \in \mathcal{E}} E = \emptyset$, none of the elements x of $\bigcup_{A \in \mathcal{A}} A = \bigcup_{E \in \mathcal{E}} E$ is a system of representatives of $\mathcal{E}$, that is, $\mathcal{A}$ contains no one-element sets.

To prove the *necessity*, suppose to the contrary that $\bigcap_{E \in \mathcal{E}} E =: X \neq \emptyset$. Then any one-element subset $\{x\} \subseteq X$ is a minimal system of representatives of $\mathcal{E}$, that is, $\{x\} \in \mathcal{A}$, a contradiction. ☐

Let us determine conditions that must be satisfied by a set family $\mathcal{A}$ such that the dimension of the complex $\Delta(\mathfrak{B}(\mathcal{A})^{\perp})$ is less than or equal to a given value or, on the contrary, the dimension is greater than the value.

According to Corollary 1.3,

$$\dim \Delta(\mathfrak{B}(\mathcal{A})^{\perp}) = m - \sum_{j=-1}^{m-1} \delta\left(f_j\left(\Delta(\mathcal{A}^{\perp})\right), \tbinom{m}{j+1}\right) - 1 .$$

If this dimension should not be greater than k, then the relation

$$\sum_{j=-1}^{m-1} \delta\left(f_j\left(\Delta(\mathcal{A}^{\perp})\right), \tbinom{m}{j+1}\right) \geq m - k - 1$$

should hold.

On the other hand, if $\dim \Delta(\mathfrak{B}(\mathcal{A})^{\perp}) > k$, then the relation

$$\sum_{j=-1}^{m-1} \delta\left(f_j\left(\Delta(\mathcal{A}^{\perp})\right), \tbinom{m}{j+1}\right) < m - k - 1$$

should hold.

Let us sum up this information as a proposition:

Proposition 1.7. *If $\mathcal{A}$ is a family of nonempty finite and pairwise distinct sets covering the set $[m]$ then, for a fixed k, the following relations hold:*

(1) *If for all j, $-1 \leq j \leq m-k-3$, it holds $f_j\left(\Delta(\mathcal{A}^{\perp})\right) = \tbinom{m}{j+1}$ then* $\dim \Delta(\mathfrak{B}(\mathcal{A})^{\perp}) \leq k$. *In particular, if equality holds for all j, $-1 \leq j \leq m-4$, then the complex $\Delta(\mathfrak{B}(\mathcal{A})^{\perp})$ is a simple graph* (*possibly, with isolated vertices*).

(2) *If $f_{m-k-3}\left(\Delta(\mathcal{A}^{\perp})\right) < \tbinom{m}{k+2}$ then* $\dim \Delta(\mathfrak{B}(\mathcal{A})^{\perp}) > k$.

Let us return to consider finite infeasible monotone systems of constraints $\mathfrak{S}$ for which the maps π and the range families $\mathbf{2}^{\Gamma}$ of these maps, which are put in correspondence with the systems, satisfy conditions (1.1)–(1.4).

For combinatorial analysis of systems $\mathfrak{S}$, a unique relevant property is perhaps the emptiness or nonemptiness of the images $\pi(B)$ of various multi-indices $B \in \mathbb{B}(m)$ of subsystems under the map π. For this reason, the setting of the problem of analyzing the above-mentioned systems in the language of monotone Boolean functions is universally accepted; we will return to the question of optimal *inference* of these functions in Section 4.1.

Let $\mathbf{B}$ denote the two-element set $\{0, 1\}$. The *unit discrete m-dimensional cube* $\mathbf{B}^m$ is equipped with the following relation $\leq$ of partial ordering: for two binary tuples $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ one supposes $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if and only if $\alpha_i \leq \beta_i$, for all $i \in [m]$. The *Boolean function* $\mathfrak{f}\colon \mathbf{B}^m \to \mathbf{B}$ is called *monotone* if $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ implies $\mathfrak{f}(\boldsymbol{\alpha}) \leq \mathfrak{f}(\boldsymbol{\beta})$. Any monotone Boolean function induces the partition $\mathbf{B}^m = \mathfrak{f}^{-1}(0) \,\dot{\cup}\, \mathfrak{f}^{-1}(1)$. The set $\mathfrak{f}^{-1}(0)$ is composed of the so-called *zeros* of the function $\mathfrak{f}$, and the set $\mathfrak{f}^{-1}(1)$ is composed of the *units* of this function. The subset $\mathbf{max}\,\mathfrak{f}^{-1}(0)$ of the maximal elements of the subposet $\mathfrak{f}^{-1}(0)$ is called the set of *upper zeros* of the function $\mathfrak{f}$; the subset $\mathbf{min}\,\mathfrak{f}^{-1}(1)$ of the minimal elements of the subposet $\mathfrak{f}^{-1}(1)$ is called the set of *lower units* of the function $\mathfrak{f}$.

With the system $\mathfrak{S}$ can be naturally put in correspondence a monotone Boolean function $\mathfrak{f}$, which is defined as follows:

$$\mathfrak{f}(\boldsymbol{\alpha}) = 0 \Longleftrightarrow \pi\Big(\bigcup_{\{a_i\} \in \mathbb{B}(m)^{(1)}\colon\ \alpha_i = 1} \{a_i\}\Big) \neq \emptyset\,,$$

$$\mathfrak{f}(\boldsymbol{\alpha}) = 1 \Longleftrightarrow \pi\Big(\bigcup_{\{a_i\} \in \mathbb{B}(m)^{(1)}\colon\ \alpha_i = 1} \{a_i\}\Big) = \emptyset\,.$$

The elements of the set $\mathfrak{f}^{-1}(1)$ are in one-to-one correspondence with the multi-indices of *infeasible* subsystems of the system $\mathfrak{S}$: $\boldsymbol{\alpha}$ is a unit of $\mathfrak{f}$ if and only if the set $\bigcup_{\substack{a_i \in \mathbb{B}(m)^{(1)}\colon \\ \alpha_i = 1}} \{a_i\}$ is the multi-index of an infeasible subsystem of $\mathfrak{S}$. In a similar matter, the elements of the set $\mathfrak{f}^{-1}(0)$ are in one-to-one correspondence with the multi-indices of *feasible* subsystems of the system $\mathfrak{S}$: $\boldsymbol{\alpha}$ is a zero of $\mathfrak{f}$ if and only if the set $\bigcup_{\substack{a_i \in \mathbb{B}(m)^{(1)}\colon \\ \alpha_i = 1}} \{a_i\}$ is the multi-index of a feasible subsystem of $\mathfrak{S}$. Finally, $\boldsymbol{\alpha}$ is a lower unit of $\mathfrak{f}$ if and only if $\bigcup_{\substack{a_i \in \mathbb{B}(m)^{(1)}\colon \\ \alpha_i = 1}} \{a_i\}$ is the multi-index of a minimal infeasible subsystem of the system $\mathfrak{S}$, and $\boldsymbol{\alpha}$ is an upper zero of $\mathfrak{f}$ if and only if $\bigcup_{\substack{a_i \in \mathbb{B}(m)^{(1)}\colon \\ \alpha_i = 1}} \{a_i\}$ is the multi-index of a maximal feasible subsystem of the system $\mathfrak{S}$.

Various examples of collections $(\mathfrak{S}, \Gamma, \pi)$ satisfying conditions (1.1)–(1.4) are provided by finite infeasible systems of equations or inequalities, or by mixed systems of equations and inequalities, over vector spaces.

We will especially be interested in collections $(\mathfrak{S}, \mathbb{R}^n, \pi)$, where

$$\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n,\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\} \tag{1.20}$$

is a finite infeasible system, of rank n, of homogeneous strict linear inequalities over the finite dimensional space $\mathbb{R}^n$. $\langle \boldsymbol{a}_i, \mathbf{x}\rangle := \sum_{k \in [n]} a_{ik} \mathrm{x}_k$ denotes here the standard scalar product; $\|\boldsymbol{a}_i\| := \sqrt{\langle \boldsymbol{a}_i, \boldsymbol{a}_i \rangle}$ is the Euclidean norm of the vector $\boldsymbol{a}_i$. The map π by definition assigns to the multi-index $T \in \mathbb{B}(m)$ the open cone of solutions to the subsystem $\{\langle \boldsymbol{a}_t, \mathbf{x}\rangle > 0 : \; t \in T\}$ as follows:

$$\begin{aligned} \pi(T) : &= \bigcap_{t \in T} \{\boldsymbol{x} \in \mathbb{R}^n : \; \langle \boldsymbol{a}_t, \boldsymbol{x} \rangle > 0\} , \\ \pi(\hat{0}) : &= \mathbb{R}^n . \end{aligned} \tag{1.21}$$

The map π defined by (1.21) induces the partition

$$\mathbb{B}(m) = \{T \in \mathbb{B}(m) : \; \pi(T) = \emptyset\} \;\dot{\cup} \{T \in \mathbb{B}(m) : \; \pi(T) \neq \emptyset\}$$

or, in other words, a partition of the form (1.11) of the lattice $\mathbb{B}(m)$ into the ideal $\mathfrak{I}(\mathbf{J})$ and filter $\mathfrak{F}(\mathbf{I})$ generated by the families of the multi-indices of maximal feasible subsystems and of minimal infeasible subsystems, respectively.

Notes

Infeasible systems of constraints are an integral part of the studies in discrete mathematics and mathematical cybernetics [96]; among a wide variety of constructions with the monotonicity property, the most close to the material of the present research are infeasible systems of linear inequalities [39–43, 96].

One can become familiar with basic problems of combinatorial optimization in works [13, 33, 63, 80, 81, 115]. Partially ordered sets and, in particular, Boolean lattices are thoroughly studied in [3, 12, 62, 85, 105, 134].

Mentioning the problem on representative systems in the context of logical functions, we follow [166].

One can become familiar with the studies of (0, 1)-matrices in works [111, 112, 123, 138, 139]. We give a short description of the link of the problem on covers with (0, 1)-matrices, following [137].

We speak of transversal sets of hypergraphs, following [104]. See, for example, [3] on bipartite graphs and binary relations.

A thorough review of works devoted to the transversality number of hypergraphs is given in [46]. One can become familiar with the studies of systems of distinct representatives in works [102, 103]. Matching theory is presented in [87]. Information on transversal matroids can be found in [3, 13]. See also [5, 7, 13, 122, 135] on the mentioned and related questions. A survey of the studies of permanents is given in [106]; the connection between permanents and systems of distinct representatives is discussed in [122].

The fundamental statement from Proposition 1.1 was proved in [36, 82, 83].

Abstract simplicial complexes are described in detail in [25, 116, 121, 132, 133].

Books [86, 90, 110, 155] are devoted to Boolean functions. Minimization of Boolean functions in the class of disjunctive normal forms is discussed in many works, see [10, 125, 151, 154, 160–162]; a review of this research can be found in [113, 127].

See [3, 9, 72, 122, 134] on the combinatorial inclusion–exclusion principle.

The literature of (hyper)graph theory is enormous, we mention here just a few books: [13, 28, 32, 67, 104, 136, 147, 156, 166].

The setting of the problem of analyzing infeasible systems in the language of monotone Boolean functions is universally accepted after the appearance of seminal work [159].