2 Complexes, (hyper)graphs, and inequality systems

In many problems of combinatorial optimization, it is necessary to distinguish in an abstract simplicial complex its facet or a collection of facets with some properties, say of maximal dimension. Recall that in our study complexes often serve as combinatorial models of (in)feasible monotone systems of constraints – the multi-indices (or the constructions marked with them) of feasible subsystems represent the faces of complexes. Thus, a complex with a unique facet corresponds to a feasible system, and a complex with several facets emulates an infeasible system. In this context, it is interesting to investigate the structural properties of the facet family of a complex and, in particular, to investigate a specific graph associated with the facet family; this graph describes the *covers* of the vertex set by *facet pairs*. For this construction, one traditionally uses the term *graph of an independence system*.

The *hypergraph of an independence system* arises when considering the *covers* of the vertex set of a complex by *arbitrary subfamilies of facets*.

In the algorithmic context, those situations are of interest where the graphs of independence systems are connected because the connectedness of such graphs can be efficiently used for constructing algorithms of determining facet families.

In this chapter, sufficient conditions of the connectedness of graphs provided by specific classes of complexes are discussed. Special attention will be paid to the study of the *graph of maximal feasible subsystems* (*graph of MFSs*) of a finite infeasible system of linear inequalities. In Section 5.1, the connectedness of this graph will serve as the basis of algorithms of designing decision rules in applied problems of pattern recognition.

2.1 The graph of an independence system

In this section, we will use the notation (V, Δ) to denote the abstract simplicial complex Δ, with the facet family $\mathbf{max}\,\Delta$, on the vertex set $V := \bigcup_{H \in \mathbf{max}\,\Delta} H$.

The term *graph of an independence system* $\mathtt{ISG}(V, \Delta)$ is used to refer to a simple graph defined as follows:

– *the vertex set of the graph* $\mathtt{ISG}(V, \Delta)$ *is the facet family* $\mathbf{max}\,\Delta$*;*
– *the edge family of the graph* $\mathtt{ISG}(V, \Delta)$ *is the family of all the unordered pairs of facets* $\{H, H'\} \subseteq \mathbf{max}\,\Delta$ *that cover the vertex set:*

$$H \cup H' = V\ .$$

Let us denote by $\overrightarrow{\mathtt{ISG}}(V, \Delta)$ the *oriented graph of the independence system*, with the vertex set $\mathbf{max}\,\Delta$, and with the arc family $\vec{\mathcal{E}}$, which is obtained from the graph $(\mathbf{max}\,\Delta, \mathcal{E}) := \mathtt{ISG}(V, \Delta)$ by an assignment of a direction to each edge of the family $\mathcal{E}$.

DOI 10.1515/9783110481068-003,

Given two complexes (V, Δ) and (V', Δ'), a map $\varphi\colon V \to V'$ between their vertex sets is called a *homomorphism* (or a *simplicial map*) if the image $\varphi(F)$ of each face $F \in \Delta$ is a face of Δ', that is, $\varphi(F) \in \Delta'$. If there exists a bijective homomorphism $\mathfrak{i}\colon V \to V'$ such that the inverse map $\mathfrak{i}^{-1}\colon V' \to V$ is also a homomorphism then the complexes (V, Δ) and (V', Δ') are said to be *isomorphic*, and in this case the map $\mathfrak{i}$ is called an *isomorphism*. If the complexes (V, Δ) and (V', Δ') are isomorphic then we will write $(V, \Delta) \simeq (V', \Delta')$ for mentioning this property.

Since the simple graphs are abstract simplicial complexes, the same definitions are relevant to them: a map φ from the vertex set of a simple graph $\mathbf{G}$ into the vertex set of another simple graph $\mathbf{G}'$ is called a *homomorphism* if the images of the end vertices of any edge of the graph $\mathbf{G}$ under the map φ are either the end vertices of an edge of the graph $\mathbf{G}'$ or these images coincide. A homomorphism $\mathfrak{i}$ of $\mathbf{G}$ to $\mathbf{G}'$ is called an *isomorphism* if it is one-to-one and if the inverse map $\mathfrak{i}^{-1}$ is also a homomorphism; if these conditions are satisfied then the graphs $\mathbf{G}$ and $\mathbf{G}'$ are called *isomorphic*.

Isomorphic complexes (V, Δ) and (V', Δ') are evidently assigned isomorphic graphs $\mathtt{ISG}(V, \Delta)$ and $\mathtt{ISG}(V', \Delta')$.

Proposition 2.1. *If there exists a surjective homomorphism $\varphi\colon V \to V'$ of a complex (V, Δ) to a complex (V', Δ'), then there exists a homomorphism of the graph $\mathtt{ISG}(V, \Delta)$ to the graph $\mathtt{ISG}(V', \Delta')$.*

Proof. If the complex (V, Δ) is the power set $\mathbf{2}^V$ of the set V, then the corresponding graph of the independence system $\mathtt{ISG}(V, \Delta)$ is the isolated vertex $\{V\}$, and the complex (V', Δ') represents the power set $\mathbf{2}^{V'}$ because $\varphi(V) = V' \in \Delta'$, and thus the graph $\mathtt{ISG}(V', \Delta')$ is the isolated vertex $\{V'\}$; in this case the map $\{V\} \mapsto \{V'\}$ is the unique homomorphism $\mathtt{ISG}(V, \Delta) \to \mathtt{ISG}(V', \Delta')$.

Assume that (V, Δ) is not the power set $\mathbf{2}^V$, that is, the vertex set of the graph $\mathtt{ISG}(V, \Delta)$ is not a singleton. For each facet $H \in \mathbf{max}\,\Delta$, pick an arbitrary facet $\gamma(H) \in \mathbf{max}\,\Delta'$ of the complex Δ' such that $\gamma(H) \supseteq \varphi(H)$, having thus defined some map $\gamma\colon \mathbf{max}\,\Delta \to \mathbf{max}\,\Delta'$ between the facet families of the complexes under consideration.

- If the graph $\mathtt{ISG}(V, \Delta)$ is edgeless then the map γ is its homomorphism to the graph $\mathtt{ISG}(V', \Delta')$.
- If the graph $\mathtt{ISG}(V, \Delta)$ is not edgeless then for each of its edge $\{H_1, H_2\}$ we have $\gamma(H_1) \cup \gamma(H_2) \supseteq \varphi(H_1) \cup \varphi(H_2) = \varphi(H_1 \cup H_2) = \varphi(V) = V'$; this means

  $$\gamma(H_1) \cup \gamma(H_2) = V' ,$$

  that is,
  - either $\gamma(H_1) = \gamma(H_2)$, the complex Δ' is the power set $\mathbf{2}^{V'}$ of the set V', and the graph $\mathtt{ISG}(V', \Delta')$ represents the isolated vertex $\{V'\}$,
  - or $\{\gamma(H_1), \gamma(H_2)\}$ is an edge of the graph $\mathtt{ISG}(V', \Delta')$ corresponding to the complex (V', Δ') which is not the power set $\mathbf{2}^{V'}$;

  therefore, the map γ is a homomorphism from $\mathtt{ISG}(V, \Delta)$ to $\mathtt{ISG}(V', \Delta')$. □

Let us now address the question on the representation of an arbitrary graph as the graph of an independence system.

Proposition 2.2. *Any finite simple graph* $\mathbf{G}$ *is isomorphic to the graph of some independence system.*

Proof. If the graph $\mathbf{G}$ represents an isolated vertex then it is isomorphic to the graph of the independence system $(V, \mathbf{2}^V)$, for any nonempty set V.

Let $\overline{\mathbf{G}} := (V, \mathcal{E})$ be the graph with vertex set V, $|V| > 1$, and edge family $\mathcal{E}$, which is the *complement* of the graph $\mathbf{G}$. Let us consider the common collection $A := V \cup \mathcal{E}$ of vertices and edges, which is the family of nonempty faces of the complex $\overline{\mathbf{G}}$, and denote by A_v the set A after the removal of the vertex v and all the edges that are incident with v (the end vertices of the mentioned edges, different from v, are not removed). For any two distinct vertices v and u of the graph $\overline{\mathbf{G}}$, the sets A_v and A_u are incomparable by inclusion because $v \in A_u - A_v$ and $u \in A_v - A_u$; in other words, $\{A_v : v \in V\}$ is a Sperner family.

Let us define an independence system (A, Δ) on the vertex set A as follows: some collection $F \subseteq A$ of vertices and edges of the graph $\overline{\mathbf{G}}$ is a face of the complex (A, Δ), $F \in \Delta$, if and only if there is a vertex $w \in V$ for which $F \subseteq A_w$; that is, (A, Δ) is the complex with the facet family $\mathbf{max}\,\Delta = \{A_v : v \in V\}$, and $\#\mathbf{max}\,\Delta = |V|$. Two distinct facets A_u and A_v of the complex (A, Δ) cover the set A if and only if the pair $\{u, v\}$ is not an edge of the graph $\overline{\mathbf{G}}$, $\{u, v\} \notin \mathcal{E}$. Thus, the map $V \to \mathbf{max}\,\Delta$, $v \mapsto A_v$, is an isomorphism of the graph $\mathbf{G}$ onto the graph of the independence system $\mathtt{ISG}(A, \Delta)$. □

Corollary 2.3. *Any finite simple graph* $\mathbf{G} = (V, \mathcal{E})$ *with vertex set* V *and edge family* $\mathcal{E}$ *is the graph of an independence system* $\mathtt{ISG}(A, \Delta)$ *associated with a complex* (A, Δ) *and, besides,* $2|A| \leq |V|^2 + |V| - 2\#\mathcal{E}$.

Let us determine the following partial order relation on the family of ordered pairs of subsets of a finite nonempty set V: $(V_1, V_2) \preceq (V'_1, V'_2)$ when $V_1 \subseteq V'_1$ and $V_2 \subseteq V'_2$, or $V_1 \subseteq V'_2$ and $V_2 \subseteq V'_1$.

We will use the following auxiliary statement whose proof is given on page 25, it follows the proof of Proposition 2.5:

Proposition 2.4. *Let* (V, Δ) *be a complex such that for the corresponding graph of the independence system* $(\mathbf{max}\,\Delta, \mathcal{E}) := \mathtt{ISG}(V, \Delta)$ *one has* $\#\mathcal{E} > 1$. *If there exists a partition of the edge family* $\mathcal{E} = \mathcal{C} \dot\cup \mathcal{D}$ *into nonempty subfamilies* $\mathcal{C}$ *and* $\mathcal{D}$ *such that none of the edges from* $\mathcal{C}$ *is adjacent to an edge from* $\mathcal{D}$, *then there do not exist edges* $\vec{c} \in \vec{\mathcal{C}}$ *and* $\vec{d} \in \vec{\mathcal{D}}$ *of the oriented graph* $(\mathbf{max}\,\Delta, \vec{\mathcal{E}}) := \overrightarrow{\mathtt{ISG}}(V, \Delta)$ *and nonempty vertex subsets* $V_1, V_2 \subseteq V$ *for which* $V_1 \cup V_2 = V$ *and* $\vec{c} \succeq (V_1, V_2) \preceq \vec{d}$.

Let some finite nonempty multifamily $\mathcal{V} := \{v_i := (X_i, X'_i) : i \in [m]\}$ of the ordered pairs of subsets of a nonempty set X be given. Define for this multifamily the operation of intersection of subset pairs: $(X_{i_1}, X'_{i_1}) \cap (X_{i_2}, X'_{i_2}) := (X_{i_1} \cap X_{i_2}, X'_{i_1} \cap X'_{i_2})$.

Let us consider a complex $(V, \Delta_\cap)$ for which, by definition

$$F \in \Delta_\cap \iff \bigcap_{v \in F} v \neq (\emptyset, \emptyset) . \tag{2.1}$$

We now turn to the graphs of independence systems whose connectedness is induced by the connectedness of topological spaces.

Proposition 2.5. *Let V be some finite multifamily of ordered pairs $v_i := (\boldsymbol{Z}_i, \boldsymbol{Z}'_i)$, $i \in [m]$, of closed subsets $\boldsymbol{Z}_i \subset \boldsymbol{Z}$ and $\boldsymbol{Z}'_i \subset \boldsymbol{Z}$ of a connected topological space $\boldsymbol{Z}$, that cover the space: $\boldsymbol{Z}_i \cup \boldsymbol{Z}'_i = \boldsymbol{Z}$.*

If $\#\mathbf{max}\,\Delta_\cap > 1$ then the graph of the independence system $\mathtt{ISG}(V, \Delta_\cap)$ is connected.

Proof. Let us first show that the graph $(\mathbf{max}\,\Delta_\cap, \mathcal{E}) := \mathtt{ISG}(V, \Delta_\cap)$ has no isolated vertices. Let $H \in \mathbf{max}\,\Delta_\cap$ be an arbitrary facet of the complex under consideration. Since $H \in \Delta_\cap$, we have $(\boldsymbol{A}, \boldsymbol{A}') := \bigcap_{v \in H} v \neq (\emptyset, \emptyset)$, by convention (2.1). Specifically, let us suppose $\boldsymbol{A} \neq \emptyset$. Fix some element $\boldsymbol{a} \in \boldsymbol{A}$. It follows from the maximality of H that for any pair $(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V - H$, we have $\boldsymbol{a} \notin \boldsymbol{Z}_i$ and, because of $\boldsymbol{Z}_i \cup \boldsymbol{Z}'_i = \boldsymbol{Z}$, the inclusion $\boldsymbol{a} \in \boldsymbol{Z}'_i$ holds, that is, $V - H \in \Delta_\cap$. Thus, there exists a facet $H' \in \mathbf{max}\,\Delta_\cap$, $H' \supseteq V - H$, such that $H \cup H' = V$, that is, the vertex H is not an isolated vertex of the graph $\mathtt{ISG}(V, \Delta_\cap)$. If $\#\mathbf{max}\,\Delta_\cap = 2$, then the proposition is proved.

Suppose that $\#\mathbf{max}\,\Delta_\cap > 2$ and, as a consequence, $\#\mathcal{E} > 1$.

We will use the oriented graph $(\mathbf{max}\,\Delta_\cap, \vec{\mathcal{E}}) := \overrightarrow{\mathtt{ISG}}(V, \Delta_\cap)$ of the independence system $(V, \Delta_\cap)$, with the vertex set $\mathbf{max}\,\Delta_\cap$ and the edge family $\vec{\mathcal{E}}$ that represents the pairs, ordered in an arbitrary way, of end vertices of the edges of the graph $\mathtt{ISG}(V, \Delta_\cap)$.

Let us assign to an arbitrary element $\boldsymbol{z} \in \boldsymbol{Z}$ faces $F_{\boldsymbol{z}} \subset V$ and $F'_{\boldsymbol{z}} \subset V$ of the complex $\Delta_\cap$, defined as follows: $F_{\boldsymbol{z}} := \{(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V\colon\ \boldsymbol{z} \in \boldsymbol{Z}_i\}$ and $F'_{\boldsymbol{z}} := \{(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V\colon \boldsymbol{z} \in \boldsymbol{Z}'_i\}$; note that they form a cover $F_{\boldsymbol{z}} \cup F'_{\boldsymbol{z}}$ of the vertex set V of the complex $\Delta_\cap$. Thus, for an element $\boldsymbol{z} \in \boldsymbol{Z}$ there is an arc $\vec{e} := (H, H') \in \vec{\mathcal{E}}$ for which

$$F_{\boldsymbol{z}} \subseteq H \in \mathbf{max}\,\Delta_\cap \quad \text{and} \quad F'_{\boldsymbol{z}} \subseteq H' \in \mathbf{max}\,\Delta_\cap$$

or

$$F_{\boldsymbol{z}} \subseteq H' \in \mathbf{max}\,\Delta_\cap \quad \text{and} \quad F'_{\boldsymbol{z}} \subseteq H \in \mathbf{max}\,\Delta_\cap ,$$

that is,

$$(F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{e} . \tag{2.2}$$

Assume that the graph $\mathtt{ISG}(V, \Delta_\cap)$, with no isolated vertices, is disconnected, that is, there exists a partition $\mathcal{E} = \mathcal{C} \,\dot\cup\, \mathcal{D}$ of its edge family into nonempty subfamilies $\mathcal{C}$ and $\mathcal{D}$ such that none of the edges from $\mathcal{C}$ is adjacent to an edge from $\mathcal{D}$.

In the space $\boldsymbol{Z}$, distinguish its subsets

$$\boldsymbol{Y} := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ \exists \vec{c} \in \vec{\mathcal{C}},\ (F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{c}\} ,$$
$$\boldsymbol{Y}' := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ \exists \vec{d} \in \vec{\mathcal{D}},\ (F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{d}\} .$$

Since $\mathcal{E} = \mathcal{C} \dot\cup \mathcal{D}$, it follows from (2.2) that

$$\boldsymbol{Y} \cup \boldsymbol{Y}' = \boldsymbol{Z} \,. \tag{2.3}$$

Further, we have $\boldsymbol{Y} \cap \boldsymbol{Y}' = \emptyset$ because, in the contrary case, for any $\boldsymbol{x} \in \boldsymbol{Y} \cap \boldsymbol{Y}'$ there would exist arcs $\vec{c} \in \vec{\mathcal{C}}$ and $\vec{d} \in \vec{\mathcal{D}}$ such that $\vec{c} \succeq (F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d}$, a contradiction with Proposition 2.4.

For any subset $U \subseteq V$ of the vertex set of the complex $\Delta_\cap$, define subsets

$$\mathbf{T}(U) := \bigcap_{\substack{\boldsymbol{Z}_i \subseteq \boldsymbol{Z}:\\ (\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in U}} \boldsymbol{Z}_i \quad \text{and} \quad \mathbf{T}'(U) := \bigcap_{\substack{\boldsymbol{Z}'_i \subseteq \boldsymbol{Z}:\\ (\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in U}} \boldsymbol{Z}'_i$$

of the space $\boldsymbol{Z}$, with $\mathbf{T}(\emptyset) = \mathbf{T}'(\emptyset) := \boldsymbol{Z}$. Let us show that

$$\boldsymbol{Y} = \bigcup_{\boldsymbol{x} \in \boldsymbol{Y}} (\mathbf{T}(F_{\boldsymbol{x}}) \cap \mathbf{T}'(F'_{\boldsymbol{x}})) \,. \tag{2.4}$$

For this, it suffices to show that

$$\mathbf{T}(F_{\boldsymbol{x}}) \cap \mathbf{T}'(F'_{\boldsymbol{x}}) \subseteq \boldsymbol{Y} \,, \tag{2.5}$$

for any element $\boldsymbol{x} \in \boldsymbol{Y}$.

Assume that inclusion (2.5) does not hold for some element $\boldsymbol{a} \in \boldsymbol{Y}$. Then there exists an element $\boldsymbol{b} \in \boldsymbol{Y}' \cap \mathbf{T}(F_{\boldsymbol{a}}) \cap \mathbf{T}'(F'_{\boldsymbol{a}})$. As a consequence, there exist arcs $\vec{c} \in \vec{\mathcal{C}}$ and $\vec{d} \in \vec{\mathcal{D}}$ such that $\vec{d} \succeq (F_{\boldsymbol{b}}, F'_{\boldsymbol{b}}) \succeq (F_{\boldsymbol{a}}, F'_{\boldsymbol{a}}) \preceq \vec{c}$, a contradiction with Proposition 2.4. Thus, relations (2.4) and (2.5) hold. It follows from the closedness of $\mathbf{T}(U)$ and $\mathbf{T}'(U)$, for $U \subseteq V$, and from the finiteness of V, taking into account (2.4), that $\boldsymbol{Y}$ is also closed.

Let us show that $\boldsymbol{Y} \neq \emptyset$. Let $\vec{c} := (H_1, H_2) \in \vec{\mathcal{C}} \neq \emptyset$. We have $(\boldsymbol{A}, \boldsymbol{A}') := \bigcap_{v \in H_1} v \neq (\emptyset, \emptyset)$. Specifically, suppose $\boldsymbol{A} \neq \emptyset$ and pick some element $\boldsymbol{x} \in \boldsymbol{A}$. Then

$$F_{\boldsymbol{x}} = H_1 \,. \tag{2.6}$$

It follows from the maximality of H_1 that for $(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V - H_1$ we have $\boldsymbol{x} \in \boldsymbol{Z}'_i$, that is,

$$F'_{\boldsymbol{x}} \supseteq V - H_1 \,. \tag{2.7}$$

Relations (2.6) and (2.7) imply

$$(F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \succeq (H_1, V - H_1) \,. \tag{2.8}$$

Assume that $\boldsymbol{Y} = \emptyset$. Since $\boldsymbol{Y} \cup \boldsymbol{Y}' = \boldsymbol{Z}$, we have $\boldsymbol{x} \in \boldsymbol{Y}'$. It follows from the definition of the set $\boldsymbol{Y}'$ that there exists an arc $\vec{d} \in \vec{\mathcal{D}}$ such that $(F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d}$. Then, taking into account (2.8), we obtain

$$\vec{\mathcal{C}} \ni \vec{c} := (H_1, H_2) \succeq (H_1, V - H_1) \preceq (F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d} \in \vec{\mathcal{D}} \,,$$

a contradiction with Proposition 2.4. Thus, $\boldsymbol{Y} \neq \emptyset$.

It can be proved analogously that the set $\boldsymbol{Y}'$ is nonempty and closed. Thus, we have partitioned the connected space $\boldsymbol{Z}$ into two nonempty disjoint closed subsets $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ – such a contradiction proves the *weak connectedness* of the oriented graph of the independence system $\overrightarrow{\mathtt{ISG}}(V, \Delta_\cap)$ and, as a consequence, the *connectedness* of the underlying undirected graph $\mathtt{ISG}(V, \Delta_\cap)$, thus completing the proof. □

Proof of Proposition 2.4. Assume the converse: let there exist arcs $\vec{c} = (C_1, C_2) \in \vec{\mathcal{C}}$, $\vec{d} = (D_1, D_2) \in \vec{\mathcal{D}}$ of the graph $\overrightarrow{\mathtt{ISG}}(V, \Delta)$, and nonempty subsets $V_1, V_2 \subseteq V$ such that $V_1 \cup V_2 = V$ and $\vec{c} \succeq (V_1, V_2) \preceq \vec{d}$. Without loss of generality, we will suppose that $C_1 \supseteq V_1$, $C_2 \supseteq V_2$, and $V_1 \subseteq D_1$, $V_2 \subseteq D_2$. Since V_1 and V_2 cover the set V then moreover $C_1 \cup D_2 = V$ and thus $e := \{C_1, D_2\} \in \mathcal{E}$. Specifically, suppose $e \in \mathcal{C}$. Then $e \cap d = \{D_2\}$, that is, the edges $e \in \mathcal{C}$ and $d \in \mathcal{D}$ of the graph $\mathtt{ISG}(V, \Delta)$ are adjacent, a contradiction. □

Let $\mathbf{F}(\boldsymbol{Z}) := \{f_i\colon\ i \in [m]\}$, $m > 1$, be a finite system of real continuous functions $f_i\colon\ \boldsymbol{Z} \to \mathbb{R}$ over a *connected* topological space $\boldsymbol{Z}$. Let us consider three classes of complexes $\Delta_\geq$, $\bar{\Delta}$ and $\Delta_>$, on the vertex set $\mathbf{F}(\boldsymbol{Z})$, defined via their nonempty faces F as follows:

- $F \in (\mathbf{F}(\boldsymbol{Z}), \Delta_\geq)$ if and only if

$$\bigcap_{f \in F} \{(\alpha_f, \boldsymbol{z}) \in (\mathbb{R} - \{0\}) \times \boldsymbol{Z}\colon\ \alpha_f f(\boldsymbol{z}) \geq 0\} \neq \emptyset$$

  or, equivalently, the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0\colon f \in F\}$ is feasible for some factor $\alpha_F^\star \in \mathbb{R} - \{0\}$;
- $F \in (\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ if and only if $\bigcap_{f \in F}(\overline{\mathbf{C}_>(f)}, \overline{\mathbf{C}_<(f)}) \neq (\emptyset, \emptyset)$, where $\overline{\mathbf{C}_>(f)}$ and $\overline{\mathbf{C}_<(f)}$ denote the closures of the sets

$$\mathbf{C}_>(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) > 0\} \quad \text{and} \quad \mathbf{C}_<(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) < 0\}\,,$$

  respectively;
- $F \in (\mathbf{F}(\boldsymbol{Z}), \Delta_>)$ if and only if

$$\bigcap_{f \in F} \{(\alpha_f, \boldsymbol{z}) \in \mathbb{R} \times \boldsymbol{Z}\colon\ \alpha_f f(\boldsymbol{z}) > 0\} \neq \emptyset$$

  or, equivalently, the inequality system $\{\alpha_F f(\boldsymbol{z}) > 0\colon f \in F\}$ is feasible for some factor $\alpha_F^\star \in \mathbb{R}$.

Remark 2.6. *If F is a nonempty face of the complex $\Delta_\geq$ on the vertex set $\mathbf{F}(\boldsymbol{Z})$, and F' is a face of this complex such that $F' \supseteq F$ (if in particular F' is a facet containing F), that is, the subsystem $\{\alpha_{F'} f(\boldsymbol{z}) \geq 0\colon\ f \in F'\}$ of the (in)feasible system $\{\alpha_{\mathbf{F}(\boldsymbol{Z})} f(\boldsymbol{z}) \geq 0\colon\ f \in \mathbf{F}(\boldsymbol{Z})\}$ is feasible for some factor $\alpha_{F'}^\star$, then the system*

$$\{\alpha_F f(\boldsymbol{z}) \geq 0\colon\ f \in F\}$$

is also feasible when $\alpha_F = \alpha_{F'}^\star$; an analogous observation is true for the faces F of the complex $\Delta_>$. □

Let us comment on the definitions of the complexes $\Delta_{\geq}$ and $\Delta_{>}$ by the example of the complex $\Delta_{\geq}$ and an *infeasible* system of inequalities

$$\{f(\boldsymbol{z}) \geq 0\colon\ f \in \mathbf{F}(\boldsymbol{Z})\}\ . \tag{2.9}$$

Let Γ be a complex whose faces are the *feasible* subsystems of system (2.9); in other words, if the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0\colon\ f \in F\}$ is feasible for the factor $\alpha^\star_F := 1$ then $F \in \Gamma$. If $H \in \mathbf{2}^{\mathbf{F}(\boldsymbol{Z})} - \Gamma$ is an *infeasible* subsystem of system (2.9) then $H \in \Delta_{\geq} - \Gamma$ if and only if the inequality system $\{\alpha_H h(\boldsymbol{z}) \geq 0\colon\ h \in H\}$ is feasible for some factor $\alpha^\star_H \in \mathbb{R} - \{0, 1\}$. If $G \subseteq H$, $G \neq \emptyset$, then, according to Remark 2.6, the subset G is a face of the complex $\Delta_{\geq}$ because the system $\{\alpha^\star_H g(\boldsymbol{z}) \geq 0\colon\ g \in G\}$ is feasible. Thus,

$$\Delta_{\geq} = \Gamma \cup \Delta\big(\mathbf{max}\,(\Delta_{\geq} - \Gamma)\big)\ .$$

For a nonempty face F of the complex $\Delta_{\geq}$, denote by $\boldsymbol{A}^\star_F$ the set of all the nonzero real factors $\alpha^\star_F$ for which the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0\colon\ f \in F\}$ is feasible. If $\mathcal{H}(F) := \{H \in \mathbf{max}\,\Delta_{\geq}\colon\ H \supseteq F\}$ is the subfamily of facets of $\Delta_{\geq}$ that contain the face F, then the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0\colon\ f \in F\}$ turns out to be feasible for any factor

$$\alpha^\star_F \in \bigcap_{H \in \mathcal{H}(F)} \boldsymbol{A}^\star_H \subseteq \bigcup_{H \in \mathcal{H}(F)} \boldsymbol{A}^\star_H \subseteq \boldsymbol{A}^\star_F\ .$$

We proceed by investigating the connectedness of the graphs of independence systems that correspond to the complexes of the three above-defined classes.

Proposition 2.7. *If* $\#\mathbf{max}\,\Delta_{\geq} > 1$ *then the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$ *is connected.*

Proof. Let us consider a multifamily $V := \big\{(\mathbf{C}_{\geq}(f), \mathbf{C}_{\leq}(f))\colon\ f \in \mathbf{F}(\boldsymbol{Z})\big\}$ of ordered pairs whose sets $\mathbf{C}_{\geq}(f)$ and $\mathbf{C}_{\leq}(f)$ are defined as follows:

$$\mathbf{C}_{\geq}(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) \geq 0\} \quad \text{and} \quad \mathbf{C}_{\leq}(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) \leq 0\}\ .$$

The definition of the complex $(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$ and definition (2.1) of the complex $(V, \Delta_{\cap})$ imply that they are isomorphic, $(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq}) \simeq (V, \Delta_{\cap})$; an isomorphism is provided by the map $\mathbf{F}(\boldsymbol{Z}) \to V$, $f \mapsto (\mathbf{C}_{\geq}(f), \mathbf{C}_{\leq}(f))$. Since the set V satisfies the conditions of Proposition 2.5, the graph of the independence system $\mathtt{ISG}(V, \Delta_{\cap})$ is connected and, as a consequence, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$, isomorphic to the graph $\mathtt{ISG}(V, \Delta_{\cap})$, is also connected. ☐

If $\boldsymbol{X}$ is a subset of the space $\boldsymbol{Z}$ then we will use the notation $\mathrm{Fr}\,(\boldsymbol{X})$ to denote its boundary in $\boldsymbol{Z}$.

Proposition 2.8. *If* $\#\mathbf{max}\,\bar{\Delta} > 1$ *and the set* $f^{-1}(0)$ *is nowhere dense for each function* $f \in \mathbf{F}(\boldsymbol{Z})$*, then the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ *is connected.*

Proof. The multifamily $V := \{(\overline{\mathbf{C}_{>}(f)}, \overline{\mathbf{C}_{<}(f)}) : f \in \mathbf{F}(\boldsymbol{Z})\}$ satisfies the conditions of Proposition 2.5: indeed, $\overline{\mathbf{C}_{>}(f)} \cup \overline{\mathbf{C}_{<}(f)} = \boldsymbol{Z}$, for each continuous function $f \in \mathbf{F}(\boldsymbol{Z})$ because $f^{-1}(0) \subseteq \mathrm{Fr}(\mathbf{C}_{>}(f)) \cup \mathrm{Fr}(\mathbf{C}_{<}(f))$ and the preimage $f^{-1}(0)$ is by condition nowhere dense in $\boldsymbol{Z}$. The map $\mathbf{F}(\boldsymbol{Z}) \to V$, $f \mapsto (\overline{\mathbf{C}_{>}(f)}, \overline{\mathbf{C}_{<}(f)})$, is an isomorphism of the complexes $(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ and $(V, \Delta_{\cap})$; according to Proposition 2.5, the graph of the independence system $\mathtt{ISG}(V, \Delta_{\cap})$ is connected and thus the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is also connected. □

Proposition 2.9. *Let* $\mathbf{F}(\boldsymbol{Z})$ *be a system of continuous functions, together with the corresponding complex* $(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$*, such that* $\#\mathbf{max}\,\Delta_{>} > 1$*. If the sets* $f^{-1}(0)$ *are nowhere dense for each function* $f \in \mathbf{F}(\boldsymbol{Z})$*, then for each facet* $H \in \mathbf{max}\,\Delta_{>}$ *the inequality system*

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & \text{if } f \in H\,, \\ -\alpha f(\boldsymbol{z}) > 0\,, & \text{if } f \in \mathbf{F}(\boldsymbol{Z}) - H\,, \end{cases} \tag{2.10}$$

where $\alpha \in \mathbb{R} - \{0\}$*, is feasible.*

Proof. By the hypothesis of the proposition, the system

$$\{\alpha f(\boldsymbol{z}) > 0 \colon\ f \in H\} \tag{2.11}$$

is feasible; its nonempty set of solutions $\boldsymbol{S} \subset \boldsymbol{Z}$ is open. The maximality of H implies that for each function $f \in \mathbf{F}(\boldsymbol{Z}) - H$ and for any point $\boldsymbol{z} \in \boldsymbol{S}$ it holds

$$\alpha f(\boldsymbol{z}) \le 0\,. \tag{2.12}$$

Since the preimage $f^{-1}(0)$ is nowhere dense for any function $f \in \mathbf{F}(\boldsymbol{Z})$, it follows from (2.12) that the constraint system

$$\{-\alpha g(\boldsymbol{z}) > 0 \colon\ \boldsymbol{z} \in \boldsymbol{S}\} \tag{2.13}$$

is feasible for each function $g \in \mathbf{F}(\boldsymbol{Z}) - H$ and, besides, its solutions represent an open subset of the solution set $\boldsymbol{S}$ to system (2.11) – denote it by $\boldsymbol{S}_g$; thus, $\boldsymbol{S}_g$ is the solution set to the feasible system

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & f \in H\,, \\ \alpha(-g(\boldsymbol{z})) > 0\,. \end{cases}$$

Without loss of generality we will suppose that $\mathbf{F}(\boldsymbol{Z}) - H = \{f_1, f_2, \ldots, f_k\}$. If $k = 1$ then $\mathbf{F}(\boldsymbol{Z}) - H = \{g\}$, and the proposition is proved. If $k > 1$ then let us consider each function $g \in \{f_2, f_3, \ldots, f_k\}$ one by one. Repeating the above argument, we see that for each index i, $2 \le i \le k$, the subsystem

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & f \in H \dot{\cup} \{f_1, \ldots, f_{i-1}\}\,, \\ \alpha(-f_i(\boldsymbol{z})) > 0 \end{cases}$$

is feasible. As a consequence, the initial system (2.10) is also feasible. □

Corollary 2.10. *Under the hypothesis of Proposition* 2.9*,*

(i) $H \in \mathbf{max}\,\Delta_{>} \implies \mathbf{F}(\boldsymbol{Z}) - H \in \Delta_{>}$,

(ii) *the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *has no isolated vertices.*

Proposition 2.11. *Suppose that* $\#\mathbf{max}\,\Delta_{>} > 1$*. If for the system* $\mathbf{F}(\boldsymbol{Z})$ *the sets* $h^{-1}(0)$ *are nowhere dense for all functions* $h \in \mathbf{F}(\boldsymbol{Z})$*, and the condition*

$$f, g \in \mathbf{F}(\boldsymbol{Z})\,,\;\; f \neq g \quad\implies\quad f^{-1}(0) \cap g^{-1}(0) = \emptyset \tag{2.14}$$

is satisfied, then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *is connected.*

Proof. Recall that, according to Proposition 2.8, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is connected. Let us show that $\Delta_{>} = \bar{\Delta}$.

The inclusion $\Delta_{>} \subseteq \bar{\Delta}$ holds: indeed, let F be a nonempty face of the complex $\Delta_{>}$, that is, for some real factor $\alpha^{\star}_{F}$ the inequality system $\{\alpha^{\star}_{F} f(\boldsymbol{z}) > 0\colon\; f \in F\}$ is feasible. Then $\bigcap_{f\in F} \overline{\mathbf{C}_{>}(f)} \neq \emptyset$ and thus $F \in \bar{\Delta}$.

Let us prove the reverse inclusion $\Delta_{>} \supseteq \bar{\Delta}$. Let $F \in \bar{\Delta}$, that is, $\bigcap_{f\in F}\big(\overline{\mathbf{C}_{>}(f)}, \overline{\mathbf{C}_{<}(f)}\big) \neq (\emptyset, \emptyset)$. Specifically, suppose that $\bigcap_{f\in F} \overline{\mathbf{C}_{>}(f)} \neq \emptyset$.

Since $\overline{\mathbf{C}_{>}(f)} = \mathbf{C}_{>}(f) \dot{\cup} \mathrm{Fr}(\mathbf{C}_{>}(f))$ and each function f is by condition continuous, the inclusion $\mathrm{Fr}(\mathbf{C}_{>}(f)) \subseteq f^{-1}(0)$ holds.

Taking into account condition (2.14), we have

$$\begin{aligned}\bigcap_{f\in F} \overline{\mathbf{C}_{>}(f)} &= \bigcap_{f\in F}\Big(\mathbf{C}_{>}(f) \dot{\cup} \mathrm{Fr}(\mathbf{C}_{>}(f))\Big)\\ &= \Big(\bigcap_{f\in F} \mathbf{C}_{>}(f)\Big) \cup \bigcup_{f\in F}\Big(\mathrm{Fr}(\mathbf{C}_{>}(f)) \;\cap \bigcap_{g\in F\colon\; g\neq f} \mathbf{C}_{>}(g)\Big) \neq \emptyset\,.\end{aligned} \tag{2.15}$$

If the intersection $\bigcap_{f\in F} \mathbf{C}_{>}(f)$ in expression (2.15) is nonempty then $F \in \Delta_{>}$.

If there exists a function $f \in F$ such that

$$\mathrm{Fr}(\mathbf{C}_{>}(f)) \;\cap \bigcap_{g\in F,\; g\neq f} \mathbf{C}_{>}(g) \neq \emptyset \tag{2.16}$$

in expression (2.15), then $\bigcap_{g\in F} \mathbf{C}_{>}(g) \neq \emptyset$. Thus, $F \in \Delta_{>}$ and, taking into account the above argument, $\bar{\Delta} = \Delta_{>}$. Since the graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is connected, the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$is also connected. ☐

Proposition 2.12. *Suppose that* $\#\mathbf{max}\,\Delta_{>} > 1$*, and for the system* $\mathbf{F}(\boldsymbol{Z})$ *the sets* $h^{-1}(0)$ *are nowhere dense for each function* $h \in \mathbf{F}(\boldsymbol{Z})$*. If the set*

$$\boldsymbol{Z}' := \boldsymbol{Z} - \bigcup_{\substack{f,g\in\mathbf{F}(\boldsymbol{Z})\colon\\ f\neq g}} (f^{-1}(0) \cap g^{-1}(0))$$

is connected then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *is connected.*

Proof. Let us consider the collection $\mathbf{F}(\boldsymbol{Z}') := \{f\,|_{\boldsymbol{Z}'} : f \in \mathbf{F}(\boldsymbol{Z})\}$ of the maps from $\mathbf{F}(\boldsymbol{Z})$ restricted to $\boldsymbol{Z}'$.

Let $\Delta'_>$ be a simplicial complex on the vertex set $\mathbf{F}(\boldsymbol{Z}')$ whose nonempty faces are by definition those subsets

$$F' := \{(f' := f\,|_{\boldsymbol{Z}'}) : f \in F\} \subseteq \mathbf{F}(\boldsymbol{Z}') \tag{2.17}$$

corresponding to the sets $F \subseteq \mathbf{F}(\boldsymbol{Z})$ for which the inequality systems

$$\{\alpha_{F'} f'(\boldsymbol{z}) > 0 : f' \in F'\} \tag{2.18}$$

are feasible for some factors $\alpha^\star_{F'} \in \mathbb{R}$.

For the nonempty subsets $F' \subseteq \mathbf{F}(\boldsymbol{Z}')$ defined in (2.17), we have

$$F' \in \Delta'_> \iff F \in \Delta_> . \tag{2.19}$$

The *sufficiency* is evident: indeed, if system (2.18) is feasible for some factor $\alpha^\star_{F'}$ then the system $\{\alpha^\star_{F'} f(\boldsymbol{z}) > 0 : f \in F\}$ is also feasible.

The *necessity*. Let $F \in \Delta_>$, that is, the system $\{\alpha_F f(\boldsymbol{z}) > 0 : f \in F\}$ is by definition feasible for some factor $\alpha^\star_F \in \mathbb{R}$; specifically, suppose that $\alpha^\star_F > 0$. If $\boldsymbol{z}^\star$ is a solution to the system $\{\alpha^\star_F f(\boldsymbol{z}) > 0 : f \in F\}$ then, because of the continuity of the functions $f \in \mathbf{F}(\boldsymbol{Z})$, elements of some neighborhood $\boldsymbol{O}_{\boldsymbol{z}^\star}$ of the solution $\boldsymbol{z}^\star$ are also solutions to the system. Since the sets $f^{-1}(0)$ are nowhere dense for each function $f \in \mathbf{F}(\boldsymbol{Z})$, we have $\boldsymbol{O}_{\boldsymbol{z}^\star} - \bigcup_{f \in F} f^{-1}(0) \neq \emptyset$. As a consequence, there exists an element $\boldsymbol{z}' \in \boldsymbol{Z}'$ such that $f'(\boldsymbol{z}') > 0$, for each function $f' := f\,|_{\boldsymbol{Z}'}$, that is, $F' \in \Delta'_>$. Thus, $\Delta(\mathbf{F}(\boldsymbol{Z}'), \Delta'_>) \simeq \Delta(\mathbf{F}(\boldsymbol{Z}), \Delta_>)$.

According to Proposition 2.11, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}'), \Delta'_>)$ is connected; therefore, the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_>)$ is also connected. □

We proceed by considering the graphs of independence systems that are associated with finite collections $\mathbf{F}(\mathbb{R}^n) := \{f_i : i \in [m]\}$, $m > 1$, of polynomial functions $f_i : \mathbb{R}^n \to \mathbb{R}$ on the real Euclidean space $\mathbb{R}^n$, $n > 1$.

For a nonempty tuple $F \subseteq \mathbf{F}(\mathbb{R}^n)$, we will use the notation $\boldsymbol{V}(F) := \bigcap_{f \in F}\{\boldsymbol{z} \in \mathbb{R}^n : f(\boldsymbol{z}) = 0\}$.

Remark 2.13. *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are two relatively prime polynomials then the set $\boldsymbol{V}(f, g)$ has the topological dimension at most $n - 2$.*

In order to see that it suffices to consider the situation where the polynomials f and g are irreducible – in this case the sets $\boldsymbol{V}(f)$ and $\boldsymbol{V}(g)$ are algebraic varieties, that is, these sets are irreducible. Since $g \neq \lambda f$, for any factor $\lambda \in \mathbb{R}$, the strict inclusion $\boldsymbol{V}(f, g) \subsetneq \boldsymbol{V}(f)$ holds. Thus, the algebraic dimension of the set $\boldsymbol{V}(f, g)$ does not exceed $n - 2$; recall that the topological dimension $\boldsymbol{V}(f, g)$ coincides with its algebraic dimension. □

Proposition 2.14. *If $\mathbf{F}(\mathbb{R}^n)$ is a collection of pairwise relatively prime polynomials then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected.*

Proof. According to Remark 2.13, the sets $f^{-1}(0) \cap g^{-1}(0)$ have the topological dimension at most $n-2$, for any distinct polynomials $f, g \in \mathbf{F}(\mathbb{R}^n)$. As a consequence, the complement $\mathbb{R}^n - \bigcup_{\substack{f,g \in \mathbf{F}(\mathbb{R}^n):\\ f \neq g}} (f^{-1}(0) \cap g^{-1}(0))$ is connected; therefore, Proposition 2.12 allows us to come to the conclusion that the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_{>})$ is connected. □

Let us consider some corollaries of the above argument; two of them follow from Proposition 2.7:

Corollary 2.15. *Let $(\boldsymbol{V}, \Delta)$ be the complex whose vertex set $\boldsymbol{V}$ is a finite subset of points on the sphere $\mathbb{S}^{n-1}$, and a nonempty subset $F \subseteq \boldsymbol{V}$ is a face if and only if the set F is contained in a closed hemisphere.*

The graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected.

Proof. Let us assign to each point $\boldsymbol{v} \in \boldsymbol{V}$ the linear functional $f_{\boldsymbol{v}} \colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{z} \mapsto \langle \boldsymbol{v}, \boldsymbol{z} \rangle$, and consider the collection $\mathbf{F}(\mathbb{R}^n) := \{f_{\boldsymbol{v}} \colon \boldsymbol{v} \in \boldsymbol{V}\}$ of all such functionals. Since the space $\mathbb{R}^n$ is connected, the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_{\geq})$ is also connected in accordance with Proposition 2.7. Recall that a nonempty subset F of the set $\boldsymbol{V}$ is contained in a closed hemisphere if and only if the inequality system $\{\langle \boldsymbol{v}, \mathbf{z} \rangle \geq 0 \colon \boldsymbol{v} \in F\}$ is feasible, that is, $\{f_{\boldsymbol{v}} \colon \boldsymbol{v} \in F\} \in \Delta_{\geq}$. As a consequence, $(\boldsymbol{V}, \Delta) \simeq (\mathbf{F}(\mathbb{R}^n), \Delta_{\geq})$ and the graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected. □

Corollary 2.16. *Let $(\boldsymbol{V}, \Delta)$ be the complex whose vertex set $\boldsymbol{V} := \{\boldsymbol{P}_i \colon i \in [m]\}$, $m > 1$, is a finite family of closed half-spaces of the space $\mathbb{R}^n$ with a nonempty intersection, and a nonempty subfamily $F \subseteq \boldsymbol{V}$ is a face if and only if the polyhedron $\bigcap_{\boldsymbol{P} \in F} \boldsymbol{P}$ is unbounded.*

The graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected.

Proof. Let us represent each half-space $\boldsymbol{P}_i$ in the form $\boldsymbol{P}_i := \{\boldsymbol{z} \in \mathbb{R}^n \colon \langle \boldsymbol{v}_i, \boldsymbol{z} \rangle \geq b_i\}$, where $\boldsymbol{v}_i \in \mathbb{S}^{n-1}$ and $b_i \in \mathbb{R}$. Since for a nonempty index subset $L \subseteq [m]$ the polyhedron $\bigcap_{i \in L} \boldsymbol{P}_i$ is unbounded if and only if the point set $\{\boldsymbol{v}_i \colon i \in L\}$ is contained in a closed hemisphere, the statement follows from Corollary 2.15. □

We conclude this section by pointing out at two corollaries of Proposition 2.11:

Corollary 2.17. *If $\mathbf{F}(\mathbb{R}^n) := \{f_i \colon i \in [m]\}$ is a collection of linear functionals $f_i \colon \mathbb{R}^n \to \mathbb{R}$ and $f_i \neq -\lambda f_j$, for any distinct indices $i, j \in [m]$ and for positive factors $\lambda \in \mathbb{R}$, then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_{>})$ is connected.*

Proof. Suppose that $f_i = \lambda f_j$ for some distinct indices $i, j \in [m]$ and for a real factor $\lambda > 0$. In this case, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_{>})$ is isomorphic to the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n) - \{f_j\}, \Delta'_{>})$ that corresponds to the complex $\Delta'_{>}$ on the vertex set $\mathbf{F}(\mathbb{R}^n) - \{f_j\}$. In this context, we can content ourselves with a discussion of the case when the functionals from $\mathbf{F}(\mathbb{R}^n)$ satisfy the condition $f_i \neq \lambda f_j$, for $i, j \in [m]$ and $\lambda > 0$. Taking into account that, analogously, $f_i \neq -\lambda f_j$, for $i, j \in [m]$ and $\lambda > 0$, we come to the conclusion that the hypothesis of Proposition 2.12 holds and, as a consequence, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_{>})$ is connected. □

Corollary 2.18. *Let* $(\boldsymbol{V}, \Delta)$ *be the complex whose vertex set* $\boldsymbol{V}$ *is a finite subset of points on the sphere* $\mathbb{S}^{n-1}$*, and a nonempty subset* $F \subseteq \boldsymbol{V}$ *is a face of* Δ *if and only if the set* F *is contained in an open hemisphere.*

If the set $\boldsymbol{V}$ *does not contain antipodes then the graph of the independence system* $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ *is connected.*

Proof. Let us assign to each point $\boldsymbol{v} \in \boldsymbol{V}$ the linear functional $f_{\boldsymbol{v}}\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{z} \mapsto \langle \boldsymbol{v}, \boldsymbol{z} \rangle$, and consider the collection $\mathbf{F}(\mathbb{R}^n) := \{f_{\boldsymbol{v}}\colon\ \boldsymbol{v} \in \boldsymbol{V}\}$ of all such functionals. The complex $(\boldsymbol{V}, \Delta)$ is isomorphic to the complex $\big(\mathbf{F}(\mathbb{R}^n), \Delta_>\big)$, and the lack of antipodes in $\boldsymbol{V}$ implies the fulfillment of conditions $f_{\boldsymbol{v}} \neq -\lambda f_{\boldsymbol{w}}$, for any points $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ and real factors $\lambda > 0$. According to Corollary 2.17, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected; as a consequence, the isomorphic graph $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is also connected. ☐

2.2 The hypergraph of an independence system

The so-called hypergraph of an independence system is a natural generalization of the notion of graph of this system:

The *hypergraph of the independence system* $\mathtt{ISH}(V, \Delta)$ corresponding to an abstract simplicial complex (V, Δ), with the facet family $\mathbf{max}\,\Delta$ and with the vertex set $V := \bigcup_{H \in \mathbf{max}\,\Delta} H$, is the hypergraph defined as follows:

- *the vertex set of the hypergraph* $\mathtt{ISH}(V, \Delta)$ *is the facet family* $\mathbf{max}\,\Delta$ *of the complex* Δ*;*
- *the hyperedge family of the hypergraph* $\mathtt{ISH}(V, \Delta)$ *is the family of all the unordered collections of facets* $\mathcal{H} \subseteq \mathbf{max}\,\Delta$ *of the complex* Δ *that cover the vertex set of the complex:*

$$\bigcup_{H \in \mathcal{H}} H = V\,.$$

Recall that any finite simple graph is isomorphic to the graph of some independence system – see Proposition 2.2. Similarly, the family of hypergraphs that are isomorphic to the hypergraphs $\mathtt{ISH}(V, \Delta)$ is quite large.

In this section, we will consider finite infeasible monotone systems of constraints $\mathfrak{S} := \{\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_m\}$ and the corresponding maps $\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\Gamma}$, where Γ is some nonempty set, such that

$$\pi(\{\emptyset\}) = \Gamma\,, \quad \pi([m]) = \emptyset\,;$$
$$\{i\} \in \mathbb{B}(m)^{(1)} \quad \Longrightarrow \quad \boldsymbol{D}_i := \pi(i) \neq \emptyset\,;$$
$$A,\ B \in \mathbb{B}(m) \quad \Longrightarrow \quad \pi(A) \cap \pi(B) = \pi(A \cup B)\,;$$

we studied similar systems in Chapter 1.

In other words, any such system $\mathfrak{S}$ represents an infeasible system of constraints

$$\mathbf{x} \in \boldsymbol{D}_i \subset \Gamma\,, \quad i \in [m]\,. \tag{2.20}$$

Let us denote the family of the multi-indices of MFSs of system (2.20) as usual by $\mathbf{J}$. The family of the multi-indices of all feasible subsystems of this system represents the abstract simplicial complex $\Delta(\mathbf{J})$ on the vertex set $[m]$, with the facet family $\mathbf{max}\,(\Delta(\mathbf{J})) := \mathbf{J}$.

Proposition 2.19. *A hypergraph $\Lambda := (\{v_1, \ldots, v_p\}, \mathcal{E})$, with the vertex set $\{v_1, \ldots, v_p\}$, $p > 1$, and with hyperedge family $\mathcal{E}$, is isomorphic to the hypergraph of the independence system $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ that corresponds to system* (2.20) *for some relevant quantity m, if and only if the hyperedge family $\mathcal{E}$, partially ordered by inclusion, represents an order filter in the Boolean lattice $\mathbb{B}(p)$ of subsets of the set $\{v_1, \ldots, v_p\}$ and, besides,*

$$\mathcal{E} \subseteq \mathbb{B}(p) - \{\mathbb{B}(p)^{(1)}\}\,. \tag{2.21}$$

Proof. The *necessity*. Suppose that the hypergraph Λ with pairwise distinct hyperedges is isomorphic to the hypergraph of the independence system $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ that corresponds to system (2.20); we denote an isomorphism providing this correspondence by $\varphi\colon \{v_1, \ldots, v_p\} \to \mathbf{J}$.

Let us show that inclusion (2.21) holds. Indeed, since by convention $p > 1$, system (2.20) is infeasible and thus the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ has no *loops*; as a consequence, the isomorphic hypergraph Λ also has no loops – in other words, $\mathcal{E} \cap \mathbb{B}(p)^{(1)} = \emptyset$.

Let us verify that the hyperedge family $\mathcal{E}$ is an order filter. Consider an arbitrary hyperedge, say $U := \{v_1, \ldots, v_k\} \in \mathcal{E}$, such that $U \neq \{v_1, \ldots, v_p\}$, and consider some hyperedge $W := \{v_1, \ldots, v_k, \ldots, v_s\} \supsetneqq U$ containing it as a subset; we have $\varphi(U) = \{J_{i_1}, \ldots, J_{i_k}\}$ and $\varphi(W) = \{J_{i_1}, \ldots, J_{i_k}, \ldots, J_{i_s}\}$, where $J_{i_1}, \ldots, J_{i_s}$ are the multi-indices of some maximal feasible subsystems of system (2.20). Since φ is an isomorphism, the family $\varphi(U)$ of the multi-indices of MFSs is a hyperedge of the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$, that is, according to the definition of this hypergraph, $\bigcup_{e=1}^{k} J_{i_e} = [m]$; moreover, $\bigcup_{e=1}^{s} J_{i_e} = [m]$ and thus the family $\varphi(W)$ is a hyperedge of the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$. Since the inverse map φ^{-1} is also an isomorphism, the family $W = \varphi^{-1}(\varphi(W))$ is a hyperedge of the hypergraph Λ.

The *sufficiency*. Let us show that if the hyperedge family $\mathcal{E}$ of the hypergraph Λ, partially ordered by inclusion, represents an order filter in the Boolean lattice $\mathbb{B}(p)$, and condition (2.21) is satisfied, then there exist an integer m and sets $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_m \subset \mathbb{N}$ such that the hypergraph of the independence system $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ corresponding to system (2.20) is isomorphic to the hypergraph Λ.

Recall that, by the hypothesis, $p > 1$, and consider the family

$$\mathcal{F} := \mathbb{B}(p) - \{\mathcal{E} \cup \{\hat{0}\}\}\,,$$

that is the order ideal $\mathbb{B}(p) - \mathcal{E}$ with the minimal element removed. Since the hypergraph Λ has no loops, the finite nonempty family $\mathcal{F} := \{F_1, \ldots, F_m\}$ contains all one-element subsets of the set $\mathbb{B}(p)^{(1)} := \{v_1, \ldots, v_p\}$. Suppose

$$J_k := \{i \in [m]\colon\ v_k \notin F_i\}$$

for all $k \in [p]$. The sets, constructed in such a way, satisfy the following conditions:

$$\emptyset \neq J_k \subset [m] \,, \tag{2.22}$$

$$k_1, k_2 \in [p], \; k_1 \neq k_2 \implies J_{k_1} - J_{k_2} \neq \emptyset \,, \tag{2.23}$$

$$\bigcup_{k \in L} J_k = [m] \iff \{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \in \mathcal{E} \,, \; \emptyset \neq L \subset [p] \,. \tag{2.24}$$

Let us show that relation (2.24) holds. Consider an arbitrary proper subset L of the set $[p]$.

Let a set $\{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\}$ be a hyperedge of the hypergraph Λ. Assume that $\bigcup_{k \in L} J_k \neq [m]$. Then there exists an index $i_0 \in [m]$ such that for all elements $k \in L$ we have $v_k \in F_{i_0}$. Thus, $F_{i_0} \supseteq \{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \in \mathcal{E}$, a contradiction with the fact that $\mathcal{E}$ is an order filter in $\mathbb{B}(p)$. We have verified that $\{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \in \mathcal{E} \implies \bigcup_{k \in L} J_k = [m]$.

Now suppose that $\bigcup_{k \in L} J_k = [m]$. Since for each index $i \in [m]$ we have $\{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \cap J_i \neq \emptyset$, then $\hat{0} \neq \{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \notin \mathcal{F}$; thus, $\bigcup_{k \in L} J_k = [m] \implies \{v_k \in \mathbb{B}(p)^{(1)} : \; k \in L\} \in \mathcal{E}$.

Conditions (2.22)–(2.24) guarantee that the sets $J_1, \ldots, J_p$ are the multi-indices of MFSs of some constraint system. Indeed, for each $j \in [p]$, suppose

$$\boldsymbol{D}_j := \{i \in [p] : \; j \in J_i\} \,. \tag{2.25}$$

The sets $J_1, \ldots, J_p$ only are the multi-indices of MFSs of system (2.20), with the sets $\boldsymbol{D}_j$ determined by relation (2.25). Indeed, for each J_i and for any $j \in J_i$, by construction, the inclusion $i \in \boldsymbol{D}_j$ holds; as a consequence, $\bigcap_{j \in J_i} \boldsymbol{D}_j \neq \emptyset$. On the other hand, let $\bigcap_{j \in L} \boldsymbol{D}_j \neq \emptyset$, where $L \neq \emptyset$. By construction, there exists a number $i \in [p]$ such that $i \in \boldsymbol{D}_j$ for each $j \in L$; thus, $L \subseteq J_i$.

By using the bijection $\varphi : \{v_1, \ldots, v_p\} \to \mathbf{J}$, $v_k \mapsto J_k$, we verify that, because of condition (2.24), the map φ is an isomorphism of the hypergraphs Λ and $\mathtt{ISH}([m], \Delta(\mathbf{J}))$. □

2.3 The graph of maximal feasible subsystems of an infeasible system of linear inequalities

In this section, we study the graphs of independence systems associated with the complexes (of the multi-indices) of the feasible subsystems of infeasible systems of linear inequalities.

We will investigate the properties of the finite infeasible system

$$\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0 : \; \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; \; \|\boldsymbol{a}_i\| = 1, \; i \in [m]; \; i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2}\} \tag{2.26}$$

of homogeneous strict linear inequalities, of rank n, over the real Euclidean space $\mathbb{R}^n$, whose set of determining vectors $\boldsymbol{A}(\mathfrak{S}) := \{\boldsymbol{a}_i : \; i \in [m]\}$ contains no pairs of antipodes.

Following the argument presented on page 18, we assign to system (2.26) the map

$$\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\mathbb{R}^n}\ ,\quad \hat{0} \neq T \mapsto \bigcap_{t\in T}\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \langle \boldsymbol{a}_t, \boldsymbol{x}\rangle > 0\}\ ,$$
$$\hat{0} \mapsto \mathbb{R}^n\ , \tag{2.27}$$

putting in correspondence with each nonempty multi-index $T \in \mathbb{B}(m)$ of the subsystem $\{\langle \boldsymbol{a}_t, \mathbf{x}\rangle > 0\colon\ t \in T\}$ of the system $\mathfrak{S}$ the open cone of its solutions. Since we will consider ordered pairs of subsets of the space $\mathbb{R}^n$ associated with these cones, we also use the synonymous notation $\mathbf{C}_>(T) := \pi(T)$ resembling the notation used in Section 2.1. The linear subspaces $\bigcap_{t\in T}\{\boldsymbol{x}\in\mathbb{R}^n\colon\ \langle \boldsymbol{a}_t, \boldsymbol{x}\rangle = 0\}$ will be denoted by $\mathbf{H}(T)$. For brevity, we use the notation $\mathbf{C}_>(i)$, instead of $\mathbf{C}_>(\{i\})$, for open half-spaces, and $\mathbf{H}(i)$, instead of $\mathbf{H}(\{i\})$, for hyperplanes; $\mathbf{C}_<(T) := -\mathbf{C}_>(T)$.

Let T be the multi-index of a feasible subsystem of the system $\mathfrak{S}$, and $L \subseteq T$ a multi-index such that for any index $i \in [m] - L$ the strict inclusion $\mathbf{H}(L \dot\cup \{i\}) \subsetneq \mathbf{H}(L)$ holds; in this case the *face*, open with respect to the subspace $\mathbf{H}(L)$, $\mathbf{H}(L) \cap \mathbf{C}_>(T-L)$ of the closed cone $\overline{\mathbf{C}_>(T)}$ will be denoted by $\mathcal{F}(L, T)$. If $\dim \mathbf{H}(L) = r$ then the face $\mathcal{F}(L, T)$ of the cone $\overline{\mathbf{C}_>(T)}$ is called *r-dimensional*.

Let us denote by $[\boldsymbol{x}, \boldsymbol{y}] := \operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ the closed segment connecting the points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$; $(\boldsymbol{x}, \boldsymbol{y}) := [\boldsymbol{x}, \boldsymbol{y}] - \{\boldsymbol{x}, \boldsymbol{y}\}$ is the corresponding open segment.

Let us denote by $\mathbf{J}$ the family of the multi-indices of MFSs of system (2.26). The abstract simplicial complex $\Delta(\mathbf{J})$, with the facet family $\mathbf{J}$, on the vertex set $[m]$, is the family of the multi-indices of all feasible subsystems of the system $\mathfrak{S}$.

With system (2.26) is put in correspondence a specific graph-theoretic construction from the common family of the graphs of independence systems defined in Section 2.1:

The *graph* $\mathtt{MFSG}(\mathfrak{S})$ *of maximal feasible subsystems* (*the graph of MFSs*) of the system $\mathfrak{S}$ is defined as the graph

$$\mathtt{MFSG}(\mathfrak{S}) := \mathtt{ISG}\big([m], \Delta(\mathbf{J})\big)$$

of the independence system that corresponds to the complex $\big([m], \Delta(\mathbf{J})\big)$. Thus, by definition,

- *the vertex set of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is the family* $\mathbf{J}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$*;*
- *the edge family of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is the family of all the unordered pairs* $\{J, J'\}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$ *that cover the index set of the inequalities of the system:*
$$J \cup J' = [m]\ .$$

Theorem 2.20. *The graph* $\mathtt{MFSG}(\mathfrak{S})$ *of maximal feasible subsystems of system* (2.26) *is connected.*

Proof. Let $(\boldsymbol{V}, \Delta)$ be the abstract simplicial complex on the vertex set $\boldsymbol{V} := \boldsymbol{A}(\mathfrak{S})$ representing the vectors that define system (2.26); a nonempty subset $F \subseteq \boldsymbol{V}$ is by definition a face of Δ if and only if the set F is contained in an open hemisphere of the unit sphere $\mathbb{S}^{n-1}$.

Using Corollary 2.18 and the isomorphism $[m] \to \boldsymbol{V}$, $i \mapsto \boldsymbol{a}_i$, of the complex $([m], \Delta(\mathbf{J}))$ onto the complex $(\boldsymbol{V}, \Delta)$, considered in Corollary 2.18, we verify that the graph $\mathtt{MFSG}(\mathfrak{S})$ is connected. □

It will be shown below that the problem of extracting MFSs of the system $\mathfrak{S}$, as well as the properties of the graph of MFSs of the system $\mathfrak{S}$, play a significant role in the solving of pattern recognition problems in their geometric setting.

When constructing algorithms of extracting MFSs of the system $\mathfrak{S}$, it is important to know those properties of its graph of MFSs that characterize the neighborhoods of vertices, say some estimates for the degrees of vertices. We will denote the neighborhood of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ by $\mathcal{N}(J_s)$.

It is useful to keep in mind the following properties of convex polyhedral cones:

Lemma 2.21. (i) *If $M \subset [m]$ and $\mathbf{C}_>(M) \neq \emptyset$, then $\mathbf{H}(M) \subset \overline{\mathbf{C}_>(M)}$.*

(ii) *If $L, M \subset [m]$, $\mathbf{C}_>(L) \neq \emptyset$ and $\mathbf{C}_>(M) \neq \emptyset$, then*

(1) $\overline{\mathbf{C}_>(L)} \cap \mathbf{C}_>(M) \neq \emptyset \Longleftrightarrow \mathbf{C}_>(L) \cap \mathbf{C}_>(M) \neq \emptyset$ *and*

(2) $\mathbf{H}(L) \cap \mathbf{C}_>(M) \neq \emptyset \Longrightarrow \mathbf{C}_>(L) \cap \mathbf{C}_>(M) \neq \emptyset$.

Lemma 2.22. *For the multi-index $J_s \in \mathbf{J}$ of an arbitrary maximal feasible subsystem of the system $\mathfrak{S}$ the inclusion $-\mathbf{C}_>(J_s) \subsetneq \mathbf{C}_>([m] - J_s)$ holds.*

Proof. Let us fix some vector $\boldsymbol{x}^* \in \mathbf{C}_>(J_s)$ and show that the inclusion $-\boldsymbol{x}^* \in \mathbf{C}_>([m] - J_s)$ holds. Assume the converse: let $j^* \in [m] - J_s$ be an index such that $-\boldsymbol{x}^* \notin \mathbf{C}_>(j^*)$. In this case, the relations $\boldsymbol{x}^* \in \overline{\mathbf{C}_>(j^*)}$ and $\overline{\mathbf{C}_>(j^*)} \cap \mathbf{C}_>(J_s) \neq \emptyset$ imply $\mathbf{C}_>(j^*) \cap \mathbf{C}_>(J_s) \neq \emptyset$, a contradiction with the maximality of the subsystem with the multi-index J_s. Thus, $-\mathbf{C}_>(J_s) \subseteq \mathbf{C}_>([m] - J_s)$. The equality $-\mathbf{C}_>(J_s) = \mathbf{C}_>([m] - J_s)$ is impossible because the set $\boldsymbol{A}(\mathfrak{S})$ of vectors determining the system (2.26) by convention contains no pairs of antipodes. □

Lemma 2.23. *Suppose that each subsystem of cardinality $k + 1$, where $2 \leq k \leq n - 1$, of the system $\mathfrak{S}$ is feasible. Given the multi-index $J_s \in \mathbf{J}$ of some of its maximal feasible subsystem, let us consider an arbitrary $(n - k)$-dimensional face $\mathfrak{F}(L, J_s)$ of the closed cone $\overline{\mathbf{C}_>(J_s)}$.*

(i) *The inclusion $-\mathfrak{F}(L, J_s) \subset \mathbf{C}_>\big([m] - J_s\big)$ holds, and*

(ii) $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$, *that is, the subsystem $\big\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; i \in L \cup ([m] - J_s)\big\}$ of the system $\mathfrak{S}$ is feasible.*

Proof. (i) Suppose to the contrary that there exists an index $j^* \in [m] - J_s$ such that $-\mathfrak{F}(L, J_s) \not\subset \mathbf{C}_>(j^*)$. Using Lemma 2.22, we have $-\mathfrak{F}(L, J_s) \subset -\overline{\mathbf{C}_>(J_s)} \subseteq \overline{\mathbf{C}_>\big([m] - J_s\big)} \subseteq \overline{\mathbf{C}_>(j^*)}$. Thus, the two cases are only possible:

(1) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j^*) \neq \emptyset$ and $-\mathfrak{F}(L, J_s) \cap \mathbf{H}(j^*) \neq \emptyset$;

(2) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j^*) = \emptyset$ and, as a consequence, $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(j^*)$.

In the first case, let us pick some points $\boldsymbol{x} \in -\mathcal{F}(L, J_s) \cap \mathbf{C}_>(j^*)$ and $\boldsymbol{y} \in -\mathcal{F}(L, J_s) \cap \mathbf{H}(j^*)$. Under $\lambda > 0$ the inclusion $\boldsymbol{z} := -\lambda\boldsymbol{x} + (1+\lambda)\boldsymbol{y} \in -\mathbf{C}_>(j^*)$ holds. Since $\boldsymbol{x}, \boldsymbol{y} \in -\mathcal{F}(L, J_s)$ and the set $-\mathcal{F}(L, J_s)$ is convex and open with respect to $\mathbf{H}(L)$, then moreover $\boldsymbol{z} \in -\mathcal{F}(L, J_s) \subseteq -\overline{\mathbf{C}_>(J_s)}$ for a sufficiently small $\lambda > 0$. Thus, $\boldsymbol{z} \in -\mathbf{C}_>(j^*) \cap -\overline{\mathbf{C}_>(J_s)}$ for a sufficiently small $\lambda > 0$; as a consequence, $-\mathbf{C}_>(j^*) \cap -\overline{\mathbf{C}_>(J_s)} \neq \emptyset$, a contradiction with the maximality of the feasible subsystem with the multi-index J_s.

Consider the second case: $-\mathcal{F}(L, J_s) \subset \mathbf{H}(j^*)$. Since the set $-\mathcal{F}(L, J_s)$ is open with respect to the subspace $\mathbf{H}(L)$, then $\mathbf{H}(L) \subseteq \mathbf{H}(j^*)$, that is, the rank of the subsystem with the multi-index $L \cup \{j^*\}$ equals the rank of the subsystem with the multi-index L, namely k. Since any subsystem, with $k+1$ inequalities, of the system $\mathfrak{S}$ is feasible, the rank k subsystem with the multi-index $L \cup \{j^*\}$ is also feasible, that is, $\mathbf{C}_>(L \cup \{j^*\}) \neq \emptyset$.

By definition, $\mathcal{F}(L, J_s) \subset \mathbf{C}_>(J_s - L)$; on the other hand, using Lemma 2.21 (i) for the multi-index $L \cup \{j^*\}$, we have $\mathcal{F}(L, J_s) \subset \mathbf{H}(L) = \mathbf{H}(L \cup \{j^*\}) \subseteq \overline{\mathbf{C}_>(L \cup \{j^*\})}$. Thus, $\mathbf{C}_>(J_s - L) \cap \overline{\mathbf{C}_>(L \cup \{j^*\})} \supseteq \mathcal{F}(L, J_s)$ or $\mathbf{C}(J_s \cup \{j^*\}) \neq \emptyset$, a contradiction with the maximality of the feasible subsystem with the multi-index J_s. This proves the inclusion $-\mathcal{F}(L, J_s) \subset \mathbf{C}_>([m] - J_s)$.

(ii) Since $-\mathcal{F}(L, J_s) \subset \mathbf{H}(L) \subseteq \overline{\mathbf{C}_>(L)}$, then $\overline{\mathbf{C}_>(L)} \cap \overline{\mathbf{C}_>([m] - J_s)} \supseteq -\mathcal{F}(L, J_s)$, and thus $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$. □

Lemma 2.24. *If any subsystem, of cardinality n, of the system $\mathfrak{S}$ is feasible then for the multi-index $J_s \in \mathbf{J}$ of each of its maximal feasible subsystem and for an arbitrary collection of $|\mathcal{N}(J_s)|$ representatives $\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t) : J_t \in \mathcal{N}(J_s)\}$ the inclusion $-\overline{\mathbf{C}_>(J_s)} \subseteq \operatorname{pos}\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t) : J_t \in \mathcal{N}(J_s)\}$ holds.*

Proof. Let $\{\mathcal{F}(L_k, J_s) : k \in [l]\}$ be the set of all one-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$, that is, by definition, $\dim \mathbf{H}(L_k) = 1$ and $L_k \subset J_s$ for each $k \in [l]$; let $\{\boldsymbol{x}_k \in \mathcal{F}(L_k, J_s) - \{\mathbf{0}\} : k \in [l]\}$ be an arbitrary collection of l representatives of these faces. Recall that

$$\operatorname{pos}\{\boldsymbol{x}_k : k \in [l]\} = \overline{\mathbf{C}_>(J_s)} . \tag{2.28}$$

Let us assign to each representative $\boldsymbol{x}_k$, $k \in [l]$, a vector $\boldsymbol{z}_k \in -\mathbf{C}_>(J_s)$ and the multi-index of a MFS $J_{t_k} \in \mathcal{N}(J_s)$ such that $-\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k)$. For this, let us choose the multi-index J_{t_k} in such a way that the inclusion $J_{t_k} \supseteq L_k \cup ([m] - J_s)$ holds. We can do that because, in accordance with Lemma 2.23 (ii), $\mathbf{C}_>(L_k \cup ([m] - J_s)) \neq \emptyset$ for all $k \in [l]$. By Lemma 2.21 (i), $-\boldsymbol{x}_k \in -\mathcal{F}(L_k, J_s) \subset \mathbf{H}(L_k) \subseteq \overline{\mathbf{C}_>(L_k)}$. By the choice of the number t_k, we have $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(J_{t_k}) \subseteq \mathbf{C}_>(L_k)$.

Let us suppose $\boldsymbol{z}_k := -\lambda \boldsymbol{y}_{t_k} + (1+\lambda)(-\boldsymbol{x}_k)$ and show that $\boldsymbol{z}_k \in -\mathbf{C}_>(J_s)$ for a sufficiently small parameter $\lambda > 0$.

Indeed, for any $\lambda > 0$ we have $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle = -\lambda \langle \boldsymbol{a}_i, \boldsymbol{y}_{t_k} \rangle + (1+\lambda)\langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle$, for each $i \in L_k$; that is, $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle < 0$, because $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(L_k)$ and $-\boldsymbol{x}_k \in \mathbf{H}(L_k)$.

Further, for each $i \in J_s - L_k$ we have $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle = \langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle + \lambda(\langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle - \langle \boldsymbol{a}_i, \boldsymbol{y}_{t_k} \rangle) < 0$ for a sufficiently small $\lambda > 0$ because $-\boldsymbol{x}_k \in \mathbf{H}(L_k) \cap -\mathbf{C}_>(J_s - L_k)$. By the choice of $\boldsymbol{z}_k$, the inclusion $-\boldsymbol{x}_k \in [\boldsymbol{y}_{t_k}, \boldsymbol{z}_k]$ holds and, because of $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(L_k)$ and $\boldsymbol{z}_k \in -\mathbf{C}_>(L_k)$, we have $-\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k)$.

Let us pick an arbitrary index $k^* \in [m] - J_s$. Denote $\boldsymbol{H}^* := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_{k^*}, \boldsymbol{x}\rangle = 1\}$. According to Lemma 2.23, $-\boldsymbol{x}_k \in -\mathcal{F}(L_k, J_s) \subset \mathbf{C}_>([m] - J_s)$ and, as a consequence, $-\overline{\mathbf{C}_>(J_s)} = \operatorname{pos}\{-\boldsymbol{x}_k : k \in [l]\} \subset \mathbf{C}_>([m] - J_s) \subset \mathbf{C}_>(k^*)$. Besides, for each $k \in [l]$ we have $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(J_{t_k}) \subset \mathbf{C}_>([m] - J_s) \subseteq \mathbf{C}_>(k^*)$. Thus, for each $k \in [l]$ the inequalities $\langle \boldsymbol{a}_{k^*}, -\boldsymbol{x}_k\rangle > 0$ and $\langle \boldsymbol{a}_{k^*}, \boldsymbol{y}_{t_k}\rangle > 0$ are fulfilled; therefore, without loss of generality we can suppose that the vectors $-\boldsymbol{x}_k$ and $\boldsymbol{y}_{t_k}$, $k \in [l]$, are chosen by their norms in such a way that they belong to the hyperplane $\boldsymbol{H}^*$. But this means that the inclusions $\boldsymbol{z}_k \in \boldsymbol{H}^*$, $k \in [l]$, also hold. Besides, we have

$$\boldsymbol{H}^* \cap -\overline{\mathbf{C}_>(J_s)} = \operatorname{conv}\{-\boldsymbol{x}_k : k \in [l]\},$$
$$-\boldsymbol{x}_k, \boldsymbol{z}_k \in \operatorname{conv}\{-\boldsymbol{x}_k : k \in [l]\}, \quad -\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k), \quad k \in [l]. \tag{2.29}$$

It follows from a geometric argument that the following statement is true: If $\boldsymbol{E} \subseteq \mathbb{R}^n$, as well as $\boldsymbol{x}, \boldsymbol{z} \in \operatorname{conv}\boldsymbol{E}$ and $\boldsymbol{x} \in (\boldsymbol{y}, \boldsymbol{z})$, then $\operatorname{conv}\boldsymbol{E} \subseteq \operatorname{conv}\left((\boldsymbol{E} \cup \{\boldsymbol{y}\}) - \{\boldsymbol{x}\}\right)$. Using this observation and relations (2.29), let us write down the following chain of inclusions:

$$\begin{aligned}
&\operatorname{conv}\{-\boldsymbol{x}_1, \dots, -\boldsymbol{x}_l\} \\
&\quad \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, -\boldsymbol{x}_2, \dots, -\boldsymbol{x}_l\} \subseteq \cdots \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, \boldsymbol{y}_{t_2}, \dots, \boldsymbol{y}_{t_{l-1}}, -\boldsymbol{x}_l\} \\
&\quad\quad \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, \boldsymbol{y}_{t_2}, \dots, \boldsymbol{y}_{t_l}\} \subseteq \operatorname{conv}\{\boldsymbol{y}_t : J_t \in \mathcal{N}(J_s)\}.
\end{aligned} \tag{2.30}$$

Note that (2.28) and (2.30) imply $-\overline{\mathbf{C}_>(J_s)} = \operatorname{pos}\{-\boldsymbol{x}_1, \dots, -\boldsymbol{x}_l\} \subseteq \operatorname{pos}\{\boldsymbol{y}_t : J_t \in \mathcal{N}(J_s)\}$, and the lemma is proved. □

As mentioned earlier, the degrees of the vertices of the graph of MFSs of system (2.26) are important parameters. We present two estimates which will be augmented later by Proposition 2.36.

Theorem 2.25. *Let $J_s \in \mathbf{J}$ be the multi-index of some maximal feasible subsystem of system* (2.26).
(i) *The degree of the vertex J_s in its graph* MFSG($\mathfrak{S}$) *is at least two:* $|\mathcal{N}(J_s)| \geq 2$.
(ii) *If each subsystem, of cardinality n, of system* (2.26) *is feasible then the degree of the vertex J_s in its graph* MFSG($\mathfrak{S}$) *is at least n:* $|\mathcal{N}(J_s)| \geq n$.

Proof. (i) Let $\{\mathcal{F}(L_k, J_s) : k \in [d]\}$ be the set of all $(n-1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$. According to Lemma 2.22, $\mathbf{C}_>([m] - J_s) \neq \emptyset$ and, as a consequence, there exists a maximal feasible subsystem of the system $\mathfrak{S}$ with a multi-index $J_p \supset [m] - J_s$.

Since J_s and J_p are the multi-indices of distinct MFSs, we have $\mathbf{C}_>(J_s) \cap \mathbf{C}_>(J_p) \neq \emptyset$. Then there exists the index k^* of an $(n-1)$-dimensional face $\mathcal{F}(L_{k^*}, J_s)$ of the cone $\mathbf{C}_>(J_s)$ such that $L_{k^*} \subsetneqq J_p$, because otherwise the inclusion $L_k \subseteq J_p$ would hold for each of d such faces, and this would imply the relations $\mathbf{C}_>(J_p) \subseteq \mathbf{C}_>(\bigcup_{k=1}^{d} L_k) = \mathbf{C}_>(J_s)$ and $\mathbf{C}_>(J_p) \cap \mathbf{C}_>(J_s) = \mathbf{C}_>(J_p) \neq \emptyset$.

In the system $\mathfrak{S}$ every subsystem with two inequalities is by convention feasible, and this implies, according to Lemma 2.23 (ii), that for the $(n-1)$-dimensional face $\mathcal{F}(L_{k^*}, J_s)$ we have $\mathbf{C}_>\big(L_{k^*} \cup ([m] - J_s)\big) \neq \emptyset$; as a consequence, there exists a maximal feasible subsystem with a multi-index $J_t \supseteq L_{k^*} \cup ([m] - J_s)$, $J_t \neq J_p$, because $L_{k^*} \subset J_t$ and $L_{k^*} \subsetneq J_p$, that is, $\mathcal{N}(J_s) \supseteq \{J_p, J_t\}$ and thus $|\mathcal{N}(J_s)| \geq 2$.

(ii) Let us pick for the open cone $\mathbf{C}_>(J_t)$ of solutions to each maximal feasible subsystem, with a multi-index $J_t \in \mathcal{N}(J_s)$, one representative $\boldsymbol{y}_t \in \mathbf{C}_>(J_t)$. According to Lemma 2.24, the inclusion $-\overline{\mathbf{C}_>(J_s)} \subseteq \operatorname{pos}\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t)\}$ holds. Since the cone $\overline{\mathbf{C}_>(J_s)}$ is n-dimensional, the relation $|\mathcal{N}(J_s)| \geq n$ holds. ☐

Lemma 2.26. *If $J_s \in \mathbf{J}$ and $J_t \in \mathbf{J}$ are the multi-indices of two distinct maximal feasible subsystems of system* (2.26) *then $J_s \cap J_t \neq \emptyset$.*

Proof. Suppose to the contrary that $J_s \cap J_t = \emptyset$. Let $\mathcal{F}(L, J_t)$ be an $(n-1)$-dimensional face of the cone $\overline{\mathbf{C}_>(J_t)}$. Since the set $\boldsymbol{A}(\mathfrak{S})$ of vectors that determine the system $\mathfrak{S}$ by convention contains no pairs of antipodes, we have $\mathbf{C}_>([m]-J_t) \cap \mathbf{C}_>(L) \neq \emptyset$, according to Lemma 2.23 (ii), that is, the subsystem of the system $\mathfrak{S}$, with the multi-index $([m] - J_t) \cup L$, is feasible; since $L \neq \emptyset$ and $L \cap J_s \neq \emptyset$, this contradicts the maximality of J_s. ☐

Lemma 2.27. *Let a partition $\mathbf{J} = \mathbf{J}' \dot\cup \mathbf{J}''$, $\#\mathbf{J}' > 0$, $\#\mathbf{J}'' > 0$, of the family of the multi-indices of MFSs of system* (2.26) *be given, and let $J' \in \mathbf{J}'$ and $J'' \in \mathbf{J}''$ be the multi-indices of maximal feasible subsystems such that $|J' \cap J''| = \max_{L \in \mathbf{J}',\, M \in \mathbf{J}''} |L \cap M|$. Then the subsystem of the system $\mathfrak{S}$, with the multi-index $([m] - J') \cup ([m] - J'')$, is feasible.*

Proof. By Lemma 2.26, $J' \cap J'' \neq \emptyset$. For the multi-index $J := J' \cap J''$ and for any indices $j' \in J' - J$ and $j'' \in J'' - J$, we have

$$\mathbf{C}_>(J \cup \{j'\} \cup \{j''\}) = \emptyset\,. \tag{2.31}$$

Indeed, suppose to the contrary that $\mathbf{C}_>(J \cup \{j'\} \cup \{j''\}) \neq \emptyset$ for some indices $j' \in J' - J$ and $j'' \in J'' - J$. Then there exists a maximal feasible subsystem with a multi-index $J^* \supseteq J \cup \{j'\} \cup \{j''\}$. Let $J^* \in \mathbf{J}'$. Then $J'' \in \mathbf{J}''$ and $|J^* \cap J''| \geq |J \cup \{j''\}| = |J| + 1$, a contradiction with the maximal cardinality $|J|$ of the multi-index J. The case $J^* \in \mathbf{J}''$, analogously, leads to a contradiction. These contradictions verify (2.31).

Fix two vectors $\boldsymbol{x} \in \mathbf{C}_>(J')$ and $\boldsymbol{y} \in \mathbf{C}_>(J'')$. Let us define the sets

$$\boldsymbol{C}' := \bigcup_{j' \in J' - J} \mathbf{C}_>(J \cup \{j'\}) \quad \text{and} \quad \boldsymbol{C}'' := \bigcup_{j'' \in J'' - J} \mathbf{C}_>(J \cup \{j''\})\,.$$

Note that the inclusions

$$\mathbf{C}_>(J') \subset \boldsymbol{C}' \subset \mathbf{C}_>(J)\,, \quad \mathbf{C}_>(J'') \subset \boldsymbol{C}'' \subset \mathbf{C}_>(J)\,, \quad [\boldsymbol{x}, \boldsymbol{y}] \subset \mathbf{C}_>(J)\,. \tag{2.32}$$

hold.

We also have $[\boldsymbol{x}, \boldsymbol{y}] \cap \overline{\boldsymbol{C}'} = [\boldsymbol{x}, \boldsymbol{z}_1]$ and $[\boldsymbol{x}, \boldsymbol{y}] \cap \overline{\boldsymbol{C}''} = [\boldsymbol{z}_2, \boldsymbol{y}]$, for some vectors $\boldsymbol{z}_1, \boldsymbol{z}_2 \in [\boldsymbol{x}, \boldsymbol{y}]$.

Suppose that $[\boldsymbol{x},\boldsymbol{z}_1]\cap[\boldsymbol{z}_2,\boldsymbol{y}] \neq \emptyset$, and fix a vector $\boldsymbol{z}^* \in [\boldsymbol{x},\boldsymbol{z}_1]\cap[\boldsymbol{z}_2,\boldsymbol{y}] \subset \overline{\boldsymbol{C}'} \cup \overline{\boldsymbol{C}''}$. It follows from the definition of the sets $\overline{\boldsymbol{C}'}$ and $\overline{\boldsymbol{C}''}$ that $\boldsymbol{z}^* \in \overline{\mathbf{C}_>(J\cup\{j'\})} \cap \overline{\mathbf{C}_>(J\cup\{j''\})} \subseteq \overline{\mathbf{C}_>(j')} \cap \overline{\mathbf{C}_>(j'')}$ for some indices $j' \in J'-J$ and $j'' \in J''-J$. Since the set $\boldsymbol{A}(\mathfrak{S})$ of vectors determining system (2.26) by convention contains no pairs of antipodes, we have $\mathbf{C}_>(j')\cap\mathbf{C}_>(j'') \neq \emptyset$. Let us pick a vector $\boldsymbol{v} \in \mathbf{C}_>(j')\cap\mathbf{C}_>(j'')$. Since $[\boldsymbol{x},\boldsymbol{y}] \in \mathbf{C}_>(J)$, the inclusion $\boldsymbol{z}^* \in \mathbf{C}_>(J)$ holds; but then $\boldsymbol{z}^* + \varepsilon\boldsymbol{v} \in \mathbf{C}_>(J')\cap\mathbf{C}_>(j')\cap\mathbf{C}_>(j'')$ for a sufficiently small $\varepsilon > 0$, a contradiction with (2.31); as a consequence, $[\boldsymbol{x},\boldsymbol{z}_1]\cap[\boldsymbol{z}_2,\boldsymbol{y}] = \emptyset$ and thus $(\boldsymbol{z}_1,\boldsymbol{z}_2) \neq \emptyset$ and $(\boldsymbol{z}_1,\boldsymbol{z}_2)\cap(\overline{\boldsymbol{C}'}\cup\overline{\boldsymbol{C}''}) = \emptyset$. Consider a vector $\boldsymbol{w} \in (\boldsymbol{z}_1,\boldsymbol{z}_2)$; for each index $j' \in J'-J$ we have $\boldsymbol{w} \in -\mathbf{C}_>(j')$, because otherwise $\boldsymbol{w} \in \overline{\mathbf{C}_>(j')} \cap \overline{\mathbf{C}_>(J)} = \overline{\mathbf{C}_>(\{j'\}\cup J)} \subseteq \overline{\boldsymbol{C}'}$. Analogously, for each index $j'' \in J''-J$ the inclusion $\boldsymbol{w} \in -\mathbf{C}_>(j'')$ holds. Thus,

$$\boldsymbol{w} \in -\mathbf{C}_>\big((J'-J)\cup(J''-J)\big)\,. \tag{2.33}$$

According to Lemma 2.22, $\boldsymbol{x} \in -\mathbf{C}_>([m]-J')$ and $\boldsymbol{y} \in -\mathbf{C}_>([m]-J'')$. If $([m]-J')\cap([m]-J'') \neq \emptyset$ then we have

$$\boldsymbol{w} \in [\boldsymbol{x},\boldsymbol{y}] \subset -\mathbf{C}_>\big(([m]-J')\cap([m]-J'')\big)\,. \tag{2.34}$$

Besides, the equality

$$\big(([m]-J')\cap([m]-J'')\big)\cup(J'-J)\cup(J''-J) = ([m]-J')\cup([m]-J'') \tag{2.35}$$

holds. It follows from (2.33)–(2.35) that $\boldsymbol{w} \in \mathbf{C}_>\big(([m]-J')\cup([m]-J'')\big) \neq \emptyset$, and the lemma is proved. □

Theorem 2.28. *The graph* MFSG($\mathfrak{S}$) *of system* (2.26) *contains at least one cycle of odd length.*

Proof. Suppose to the contrary that the graph MFSG($\mathfrak{S}$) contains no cycles of odd length. In this case MFSG($\mathfrak{S}$) is bipartite, that is, there exists a partition $\mathbf{J} = \mathbf{J}' \dot\cup \mathbf{J}''$, $\#\mathbf{J}' > 0$, $\#\mathbf{J}'' > 0$, of its vertex set such that the vertices from the parts $\mathbf{J}'$ and $\mathbf{J}''$ are not linked by an edge. According to Lemma 2.27, there exist the multi-indices of MFSs $J' \in \mathbf{J}'$ and $J'' \in \mathbf{J}''$ such that the subsystem with the multi-index $([m]-J')\cup([m]-J'')$ is feasible. As a consequence, there exists a subsystem of the system $\mathfrak{S}$, with a multi-index J, such that $J \supseteq ([m]-J')\cup([m]-J'')$ and thus the pairs $\{J,J'\}$ and $\{J,J''\}$ are edges of the graph MFSG($\mathfrak{S}$). Since $J \in \mathbf{J}'\dot\cup\mathbf{J}'' = [m]$, either the edge $\{J,J'\}$ or the edge $\{J,J''\}$ contradicts the assumption that the graph MFSG($\mathfrak{S}$) is bipartite. □

For the construction of algorithms of extracting maximal feasible subsystems of system (2.26) it is important to know a characterization of the graphs of MFSs of such systems. The answer to this question is well known in the case of the rank 2 systems $\mathfrak{S}$. Before recalling it, we will describe some additional properties of the systems $\mathfrak{S}$.

Let $L \subset [m]$ be the multi-index of a feasible subsystem of system (2.26), that is, $\mathbf{C}_>(L) \neq \emptyset$. The inequality with an index $l \in L$ is called an *inequality implied by the subsystem* with the multi-index $L-\{l\}$ when $\mathbf{C}_>(L-\{l\}) = \mathbf{C}_>(L)$. Let us denote by

$\mathbf{mmi}(L) \subseteq L$ the inclusion-**m**inimal **m**ulti-**i**ndex of the subsystem of the system with the multi-index L such that $\mathbf{C}_>(\mathbf{mmi}(L)) = \mathbf{C}_>(L)$; in other words, the set $\mathbf{mmi}(L)$ is by definition composed of those indices $i \in [m]$ for which the half-spaces $\mathbf{H}(i)$ represent the linear hulls of the $(n-1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(L)}$.

Remark 2.29. *If $L \subset [m]$ is the multi-index of a feasible subsystem of system* (2.26) *then*
(i) $i \in L - \mathbf{mmi}(L) \Longleftrightarrow \mathbf{C}_>(L) = \mathbf{C}_>(L - \{i\})$*;*
(ii) $i \in \mathbf{mmi}(L) \Longleftrightarrow \mathcal{F}(\{i\}, L) \neq \emptyset$, $\dim \operatorname{lin}\left(\mathcal{F}(\{i\}, L)\right) = n - 1$.

Lemma 2.30. *If J_s and J_t are the multi-indices of two distinct MFSs of system* (2.26) *then*

$$\mathbf{mmi}(J_s) \cap J_t \neq \emptyset\,, \tag{2.36}$$

$$\mathbf{mmi}(J_s) \cap ([m] - J_t) \neq \emptyset\,. \tag{2.37}$$

Proof. Let us prove (2.36): suppose to the contrary that $\mathbf{mmi}(J_s) \cap J_t = \emptyset$. Then $\mathbf{mmi}(J_s) \subseteq [m] - J_t$ and $\mathbf{C}_>(J_s) = \mathbf{C}_>([m] - J_t) \supset -\mathbf{C}_>(J_t) \neq \emptyset$. If $-\mathbf{C}_>(J_t) \cap \mathbf{C}_>(J_s) \neq \emptyset$ then $J_s \cap J_t = \emptyset$, a contradiction with Lemma 2.26.

Relation (2.37) is proved similarly. □

Lemma 2.31. *If J_s and J_t are the multi-indices of MFSs of system* (2.26) *such that $J_s \cup J_t = [m]$, that is, $\{J_s, J_t\}$ is an edge of the graph $\mathtt{MFSG}(\mathfrak{S})$, then $\emptyset \neq \mathbf{mmi}(J_s) - J_t \subseteq \mathbf{mmi}([m] - J_t)$.*

Proof. According to Lemma 2.30, $\mathbf{mmi}(J_s) - J_t \neq \emptyset$. Let us consider an arbitrary index $j^* \in \mathbf{mmi}(J_s) - J_t \subseteq J_s - J_t$. Let us show that $j^* \in \mathbf{mmi}([m] - J_t)$. Suppose to the contrary that $j^* \in ([m] - J_t) - \mathbf{mmi}([m] - J_t)$. According to Remark 2.29 (i), $\mathbf{C}_>\left(([m] - J_t) - \{j^*\}\right) = \mathbf{C}_>([m] - J_t)$. Since, by definition, $J_s \cup J_t = [m]$, we have $J_s \supseteq [m] - J_t$. Let us represent the multi-index J_s in the form $J_s = (J_s - ([m] - J_t)) \cup ([m]-J_t)$. Then $\mathbf{C}_>(J_s) = \mathbf{C}_>\left(J_s - ([m] - J_t)\right) \cap \mathbf{C}_>([m]-J_t)$ and, because of $\mathbf{C}_>([m]-J_t) = \mathbf{C}_>\left(([m] - J_t) - \{j^*\}\right)$ and $j^* \in [m] - J_t$, we have $\mathbf{C}_>(J_s) = \mathbf{C}_>(J_s - \{j^*\})$, from where, according to Remark 2.29 (i), we obtain $j^* \in J_s - \mathbf{mmi}(J_s)$, a contradiction with the choice of the index j^*. □

Lemma 2.32. *If $\mathfrak{S}$ is a rank two system* (2.26) *over $\mathbb{R}^2$, then the degree of each vertex of its graph $\mathtt{MFSG}(\mathfrak{S})$ is 2.*

Proof. Let J_s be an arbitrary vertex of the graph $\mathtt{MFSG}(\mathfrak{S})$. Without loss of generality we will suppose that in the graph $\mathtt{MFSG}(\mathfrak{S})$ this vertex is adjacent to the vertices $J_1, J_2, \ldots, J_p$. Since, by Theorem 2.25 (i), we have $p \geq 2$, it suffices to show that $p \leq 2$. Since $J_i \supseteq [m] - J_s$ for each index i, $1 \leq i \leq p$, the inclusions

$$\mathbf{C}_>(J_i) \subseteq \mathbf{C}_>([m] - J_s)\,, \quad 1 \leq i \leq p\,,$$

hold. Further, according to Lemma 2.31, $\emptyset \neq \mathbf{mmi}(J_i) - J_s \subseteq \mathbf{mmi}([m] - J_s)$, for each index i, $1 \leq i \leq p$. For each such an index, let us pick an index $t_i \in \mathbf{mmi}(J_i) \cap ([m] - \mathbf{mmi}(J_t)) \neq \emptyset$. Then the inclusions

$$\emptyset \neq \mathcal{F}(\{t_i\}, J_i) \subseteq \mathcal{F}(\{t_i\}, [m] - J_s)\,, \quad 1 \leq i \leq p\,, \tag{2.38}$$

hold. Let us show that the implications

$$i_1 \neq i_2 \quad \Longrightarrow \quad \mathcal{F}(\{t_{i_1}\}, J_{i_1}) \cap \mathcal{F}(\{t_{i_2}\}, J_{i_2}) \neq \emptyset\,, \quad 1 \leq i_1, i_2 \leq p\,, \tag{2.39}$$

are true. Suppose to the contrary that there exist distinct indices i_1 and i_2, $1 \leq i_1, i_2 \leq \#\mathbf{J}$, such that some vector $\boldsymbol{x}^* \in \mathcal{F}(\{t_{i_1}\}, J_{i_1}) \cap \mathcal{F}(\{t_{i_2}\}, J_{i_2})$ can be chosen, $\boldsymbol{x}^* \neq \mathbf{0}$, that is, $\boldsymbol{x}^* \in \mathbf{H}(t_{i_1}) \cap \mathbf{C}_>(J_{i_1} - \{t_{i_1}\}) \cap \mathbf{H}(t_{i_2}) \cap \mathbf{C}_>(J_{i_2} - \{t_{i_2}\})$. Since $n = 2$, then $\mathbf{H}(i_1) = \mathbf{H}(i_2)$ for $i_1 \neq i_2$; this is impossible because each subsystem of the system $\mathfrak{S}$, with two inequalities, is by definition of rank 2.

Since $\emptyset \neq \mathbf{C}_>([m] - J_s) \subset \mathbb{R}^2$, the boundary of the cone $\overline{\mathbf{C}_>([m] - J_s)}$ can be represented as a union of two rays $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ radiating from the point $\mathbf{0}$; thus, $\mathcal{F}(\{j\}, [m] - J_s) \subseteq \boldsymbol{l}_1 \cup \boldsymbol{l}_2$, for each index $j \in \mathbf{mmi}([m] - J_s)$. For each index i, $1 \leq i \leq p$, the nonempty face $\mathcal{F}(\{t_i\}, J_i)$ represents a ray radiating from the point $\mathbf{0}$ and not containing it. It follows from (2.38) that for each number i, $1 \leq i \leq p$, one of the two equalities $\mathcal{F}(\{t_i\}, J_i) = \boldsymbol{l}_1 - \{\mathbf{0}\}$ and $\mathcal{F}(\{t_i\}, J_i) = \boldsymbol{l}_2 - \{\mathbf{0}\}$ only holds, from where it follows, taking into account (2.39), that $p \leq 2$. □

Proposition 2.33. *Some graph is isomorphic to the graph* MFSG($\mathfrak{S}$) *of a rank two system* $\mathfrak{S}$ *over* $\mathbb{R}^2$ *if and only if this graph represents a simple cycle of odd length* q, $3 \leq q \leq m$.

Proof. The *necessity*. Let an arbitrary system (2.26), with m inequalities, over $\mathbb{R}^2$ be given. It is well known that the system $\mathfrak{S}$ contains an odd number, not exceeding m, of MFSs. Since, by Theorem 2.20, the graph MFSG($\mathfrak{S}$) is connected and, by Lemma 2.32, the degree of each of its vertex equals 2, the graph MFSG($\mathfrak{S}$) represents a simple cycle of odd length $\#\mathbf{J}$.

The *sufficiency* is verified by explicit constructing, for an arbitrary odd q, $3 \leq q \leq m$, a rank 2 system (2.26), with m inequalities, over $\mathbb{R}^2$. Let each vector $\boldsymbol{a}'_i \in \mathbb{R}^2$, $1 \leq i \leq q$, be obtained by rotating the vector $(0, 1) \in \mathbb{R}^2$ through a counterclockwise angle of $2\pi i/q$. The infeasible system $\mathfrak{S}' := \{\langle \boldsymbol{a}'_i, \mathbf{x}\rangle > 0\colon\ i \in [q]\}$, composed of q inequalities, has q maximal feasible subsystems with the multi-indices $J_1 := \{1, 2, \ldots, \lfloor q/2 \rfloor + 1\}$, $J_2 := \{2, 3, \ldots, \lfloor q/2 \rfloor + 2\}, \ldots, J_q := \{q, 1, \ldots, \lfloor q/2 \rfloor\}$. The corresponding multi-indices $\mathbf{mmi}(J_i)$ describing the one-dimensional faces of the closures of the solution cones to the maximal feasible subsystems are the two-element sets $\mathbf{mmi}(J_1) = \{1, \lfloor q/2 \rfloor + 1\}$, $\mathbf{mmi}(J_2) = \{2, \lfloor q/2 \rfloor + 2\}, \ldots, \mathbf{mmi}(J_q) = \{q, \lfloor q/2 \rfloor\}$; note that the index of each inequality of the system $\mathfrak{S}'$ occurs in the multi-indices $\mathbf{mmi}(J_i)$ twice. The graph MFSG($\mathfrak{S}'$) is a simple cycle of length q. Let us choose the value of angular deviation ϵ sufficiently small for the augmented inequality system $\mathfrak{S} := \{\mathfrak{S}',\ \{\langle \boldsymbol{a}_k, \mathbf{x}\rangle > 0\colon\ k \in [m - q]\}\}$ – in which the determining vector $\boldsymbol{a}_k$ is obtained from the vector $\boldsymbol{a}'_1$ by rotating through an angle of $k\epsilon$ – to differ from the initial system $\mathfrak{S}'$ by a deformation of the solution cone to a unique maximal feasible subsystem. By construction, the resulting system $\mathfrak{S}$, with m inequalities and with q maximal feasible subsystems, has the graph MFSG($\mathfrak{S}$) isomorphic to the graph MFSG($\mathfrak{S}'$) and representing a simple cycle of odd length q. □

Recall that a shortest simple path connecting two vertices of a graph is called a *geodesic*, and the *diameter* of the graph is defined as the length of its largest geodesic.

According to Proposition 2.33, the graph of MFSs of a rank 2 system (2.26) has the following properties: any its edge belongs to a simple cycle of length not exceeding m; the graph contains a simple cycle of odd length not exceeding m; the diameter of the graph does not exceed $\lfloor \frac{m}{2} \rfloor$.

It turns out that the graphs of MFSs of the systems $\mathfrak{S}$, of any rank n, have analogous properties. Before verifying this claim, let us turn to the half-spaces defined by the inequalities of system (2.26), and prove an auxiliary statement.

Let us consider the family

$$\{(\mathbf{C}_>(i), \mathbf{C}_<(i)) : i \in [m]\} \tag{2.40}$$

of the ordered pairs of open half-spaces associated with system (2.26), and consider the abstract simplicial complex $([m], \Delta^>)$, on the vertex set $[m]$, defined for the nonempty subfamilies $F \subset [m]$ as follows:

$$F \in \Delta^> \iff \bigcap_{f \in F} (\mathbf{C}_>(f), \mathbf{C}_<(f)) \neq (\emptyset, \emptyset) .$$

If $\boldsymbol{S}$ is a subspace of the space $\mathbb{R}^n$ then the complex $([m], \Delta^>_{\boldsymbol{S}})$ is defined in a similar way:

$$F \in \Delta^>_{\boldsymbol{S}} \iff \bigcap_{f \in F} (\mathbf{C}_>(f), \mathbf{C}_<(f)) \cap (\boldsymbol{S}, \boldsymbol{S}) \neq (\emptyset, \emptyset) ; \tag{2.41}$$

in particular, $([m], \Delta^>) := ([m], \Delta^>_{\mathbb{R}^n})$.

Lemma 2.34. *Suppose that each subsystem of cardinality k, where $3 \le k \le n$, of system* (2.26) *of rank $n \ge 3$ has rank k.*

Consider an arbitrary family $\{J_{s_1}, J_{s_2}, \ldots, J_{s_r}\} \subset \mathbf{J}$ of the multi-indices of its MFSs, where $1 \le r \le \min\{k, n-1\}$.

There exists an $(n-1)$-dimensional subspace $\boldsymbol{R} \subset \mathbb{R}^n$ satisfying the condition

$$L \subset [m],\ |L| = r \implies \dim(\boldsymbol{R} \cap \mathbf{H}(L)) = n - r - 1 .$$

In the graph of the independence system $\mathtt{ISG}([m], \Delta^>)$ there are vertices $J^*_{t_1}, J^*_{t_2}, \ldots, J^*_{t_r}$ such that $J^*_{t_i} = J_{s_i}$, $1 \le i \le r$.

There exists a homomorphism $\psi\colon \mathtt{ISG}([m], \Delta^>_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^>)$ of the graphs of independence systems such that for each facet $J^* \in \mathbf{max}\, \Delta^>_{\boldsymbol{R}}$ the inclusion $\psi(J^*) \supseteq J^*$ holds, and for each i, $1 \le i \le r$, the relations $\psi(J^*_{t_i}) = J_{s_i}$ and $\psi^{-1}(J_{s_i}) = \{J^*_{t_i}\}$ hold.

Proof. Let us define, for any vector $\boldsymbol{x} \in \mathbb{R}^n$, a continuous map $\varphi_{\boldsymbol{x}}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{y} \mapsto \langle \boldsymbol{y}, \boldsymbol{y} \rangle \boldsymbol{x} - \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{y}$. Let us pick unique representatives $\boldsymbol{x}_i \in \mathbf{C}_>(J_{s_i})$, $1 \le i \le r$. Since $r \le n-1$, there exists a vector $\boldsymbol{z}^*$ such that $\|\boldsymbol{z}^*\| = 1$ and $\langle \boldsymbol{x}_i, \boldsymbol{z}^* \rangle = 0$, $1 \le i \le r$. Then $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}^*) = \boldsymbol{x}_i$, $1 \le i \le r$, and thus $\boldsymbol{z}^* \in \bigcap_{1 \le i \le r} \varphi^{-1}_{\boldsymbol{x}_i}(\boldsymbol{x}_i) \subset \boldsymbol{V} := \bigcap_{1 \le i \le r} \varphi^{-1}_{\boldsymbol{x}_i}(\mathbf{C}_>(J_{s_i})) \neq \emptyset$.

The set $\boldsymbol{V}$ is open because of the continuity of the maps $\varphi_{\boldsymbol{x}_i}$ and the openness of the cones $\mathbf{C}_>(J_{s_i})$, $1 \le i \le r$. Family (2.40) satisfies the condition

$$L \subset [m],\ |L| = r \quad \Longrightarrow \quad \dim \mathbf{H}(L) = n - r\,. \tag{2.42}$$

For an arbitrary multi-index $L \subseteq [m]$, let us denote by $(\mathbf{H}(L))^{\perp}$ the orthogonal complement of the subspace $\mathbf{H}(L)$ up to $\mathbb{R}^n$; thus, $\mathbb{R}^n = \mathbf{H}(L) \oplus (\mathbf{H}(L))^{\perp}$. Since $2 \le r \le n-1$, the set $\boldsymbol{U} := \bigcup_{|L|=r} (\mathbf{H}(L))^{\perp}$ is nowhere dense in $\mathbb{R}^n$; as a consequence, $\boldsymbol{V} \cap (\mathbb{R}^n - \boldsymbol{U}) \neq \emptyset$. Let us pick a vector $\boldsymbol{z} \in \boldsymbol{V} \cap (\mathbb{R}^n - \boldsymbol{U})$, $\boldsymbol{z} \neq \mathbf{0}$. Suppose $\boldsymbol{R} := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{z}, \boldsymbol{x} \rangle = 0\}$. Since for any multi-index $L \subseteq [m]$, $|L| = r$, we have $\boldsymbol{z} \notin \boldsymbol{U}$, that is, $\boldsymbol{z} \notin (\mathbf{H}(L))^{\perp}$, then $\boldsymbol{R} \not\supseteq \mathbf{H}(L)$, $L \subseteq [m]$, $|L| = r$, from where, taking into account (2.42), we obtain

$$L \subseteq [m],\ |L| = r \quad \Longrightarrow \quad \dim(\boldsymbol{R} \cap \mathbf{H}(L)) = (n-1) - r\,. \tag{2.43}$$

Consider the family

$$\{(\mathbf{C}_>(i), \mathbf{C}_<(i)) \cap (\boldsymbol{R}, \boldsymbol{R}) :\ i \in [m]\} \tag{2.44}$$

over the $(n-1)$-dimensional subspace $\boldsymbol{R}$, and the corresponding abstract simplicial complex $\Delta^>_{\boldsymbol{R}}$ defined in (2.41). Since $\boldsymbol{z} \in \boldsymbol{V} := \bigcap_{1 \le i \le r} \varphi^{-1}_{\boldsymbol{x}_i}(\mathbf{C}_>(J_{s_i}))$, for each i, $1 \le i \le r$, the inclusions $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \mathbf{C}_>(J_{s_i})$ hold. Besides, $\langle \varphi_{\boldsymbol{x}_i}(\boldsymbol{z}), \boldsymbol{z} \rangle = 0$, $1 \le i \le r$, that is, $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \boldsymbol{R}$, $1 \le i \le r$. Thus, $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \boldsymbol{R} \cap \mathbf{C}_>(J_{s_i})$, $1 \le i \le r$, and comparing families (2.40) and (2.44) verifies that the subset $J_{s_i} \subset [m]$ is a facet of the complex $\Delta^>_{\boldsymbol{R}}$, that is, in the graph $\mathtt{ISG}([m], \Delta^>_{\boldsymbol{R}})$ there are vertices $J^*_{t_1}, J^*_{t_2}, \ldots, J^*_{t_r}$ such that $J_{s_i} = J^*_{t_i}$, $1 \le i \le r$. For any subsets $F \subset [m]$, the implications $\bigcap_{f \in F} (\mathbf{C}_>(f), \mathbf{C}_<(f)) \cap (\boldsymbol{R}, \boldsymbol{R}) \neq (\emptyset, \emptyset) \Rightarrow \bigcap_{f \in F} (\mathbf{C}_>(f), \mathbf{C}_<(f)) \neq (\emptyset, \emptyset)$ are true and thus, according to Proposition 2.1, there exists a homomorphism $\psi : \mathtt{ISG}([m], \Delta^>_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^>)$ such that $\psi(J^*_i) \supseteq J_i$, for each facet $J^*_i \in \mathbf{max}\, \Delta^>_{\boldsymbol{R}}$. The maximality of the sets J_{s_i} and $J^*_{t_i}$, and the equalities $J_{s_i} = J^*_{t_i}$ for all i, $1 \le i \le r$, imply that $\psi(J^*_{t_i}) = J_{s_i}$ and $\psi^{-1}(J_{s_i}) = \{J^*_{t_i}\}$, $1 \le i \le r$. ☐

Proposition 2.35. (i) *Any edge of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *of system* (2.26) *belongs to a simple cycle* (*and, as a consequence, the graph* $\mathtt{MFSG}(\mathfrak{S})$ *has no bridges*) *of length not exceeding* m;

(ii) *the graph* $\mathtt{MFSG}(\mathfrak{S})$ *contains a simple cycle of odd length not exceeding* m;

(iii) *the diameter of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *does not exceed* $\lfloor \frac{m}{2} \rfloor$.

Proof. (i) The proof is by induction on the rank n of system (2.26). The statement is true for $n = 2$ because, according to Proposition 2.33, the graph $\mathtt{MFSG}(\mathfrak{S})$ represents a simple cycle of odd length not exceeding m.

Let us assume that the statement is true for an arbitrary system (2.26) of rank $n-1$ over $\mathbb{R}^{n-1}$, where $n \ge 3$. Let us consider family (2.40) of the ordered pairs of subspaces associated with the rank n system $\mathfrak{S}$ over $\mathbb{R}^n$; this family satisfies condition (2.42). Let us consider an arbitrary edge of the graph $\mathtt{MFSG}(\mathfrak{S})$; without loss of generality we will suppose that this edge is the pair $\{J_1, J_2\}$. Let us set $r := 2 \le \min\{2, n-1\}$ and use Lemma 2.34 for the r multi-indices J_1 and J_2 of MFSs of the system $\mathfrak{S}$ with the corresponding family (2.40). Let $\boldsymbol{R}$ be an $(n-1)$-dimensional subspace of the space $\mathbb{R}^n$; it is assigned to family (2.44) of subspace pairs and the graph of the independence system $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. According to Lemma 2.34, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exist vertices $J^*_{t_1}$ and $J^*_{t_2}$ such that $J^*_{t_1} = J_1$ and $J^*_{t_2} = J_2$. Since $J^*_{t_1} \cup J^*_{t_2} = J_1 \cup J_2 = [m]$, the pair $\{J^*_{t_1}, J^*_{t_2}\}$ is an edge of the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. Then by the induction hypothesis, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exists a simple cycle $(J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}, J^*_{t_1})$, $3 \le p \le m$, containing the edge $\{J^*_{t_1}, J^*_{t_2}\}$. Let $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$ be the homomorphism mentioned in Lemma 2.34. Let us consider the sequence $(\psi(J^*_{t_2}), \psi(J^*_{t_3}), \dots, \psi(J^*_{t_p}), \psi(J^*_{t_1}))$ of the vertices of the graph $\mathtt{ISG}([m], \Delta^{>})$, isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$, which is a $(\psi(J^*_{t_2}) \leftrightarrow \psi(J^*_{t_1}))$-path because of the conservation of vertex adjacency under the homomorphism $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$. Since $\psi^{-1}(\psi(J^*_{t_1})) = \{J^*_{t_1}\}$, $\psi^{-1}(\psi(J^*_{t_2})) = \{J^*_{t_2}\}$, and $J^*_{t_1}, J^*_{t_2} \notin \{J^*_{t_3}, \dots, J^*_{t_p}\}$, then $\psi(J^*_{t_1}), \psi(J^*_{t_2}) \notin \{\psi(J^*_{t_3}), \dots, \psi(J^*_{t_p})\}$.

Let us consider the two possible cases: (1) $\psi(J^*_{t_3}) = \psi(J^*_{t_p})$ and (2) $\psi(J^*_{t_3}) \ne \psi(J^*_{t_p})$.

In the case 1, we have the simple cycle $(\psi(J^*_{t_1}), \psi(J^*_{t_2}), \psi(J^*_{t_3}), \psi(J^*_{t_1})) = (J_1, J_2, \psi(J^*_{t_3}), J_1)$, of length three, containing the edge $\{J_1, J_2\}$.

In the case 2, one can distinguish in the path $(\psi(J^*_{t_3}), \dots, \psi(J^*_{t_p}))$ a simple chain $(\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}))$, where $(\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p})) \subseteq \{\psi(J^*_{t_3}), \psi(J^*_{t_4}), \dots, \psi(J^*_{t_p})\}$, and thus $\psi(J^*_{t_1}), \psi(J^*_{t_2}) \notin (\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}))$. Since $k + 2 \le p - 2$, the sequence $(\psi(J^*_{t_1}), \psi(J^*_{t_2}), \psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}), \psi(J^*_{t_1}))$ represents a simple cycle of length $k + 4 \le p \le m$, and it contains the edge $\{\psi(J^*_{t_1}), \psi(J^*_{t_2})\} = \{J_1, J_2\}$, as was to be proved.

(ii) Recall that the existence of a cycle of odd length in the graph $\mathtt{MFSG}(\mathfrak{S})$ was proved in Theorem 2.28. We will show here, by using a similar argument, that there exists such a cycle of length not exceeding m. In the case of a rank 2 system (2.26) the graph $\mathtt{MFSG}(\mathfrak{S})$ represents such a cycle – see Proposition 2.33. Assume by induction that the statement is true for all systems of rank $n - 1 \ge 2$. Let us fix in $\mathbb{R}^n$ an arbitrary $(n-1)$-dimensional subspace $\boldsymbol{R}$; then, as in the proof of statement (i), let us turn to Lemma 2.34 in the situation where $r := 2 \le \min\{2, n-1\}$. Let us choose in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ some cycle $(J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}, J^*_{t_1})$ of odd length p such that $3 \le p \le m$ which is contained in the graph by the induction hypothesis. We will use the denotations $\mathbf{Z}^* := \{J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}\}$ and $\mathbf{Z} := \{\psi(J^*_{t_1}), \psi(J^*_{t_2}), \dots, \psi(J^*_{t_p})\}$ and show that the induced subgraph $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z} \rangle$ of the graph $\mathtt{ISG}([m], \Delta^{>})$ contains a simple cycle of odd length not exceeding m.

First, assume that the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z} \rangle$ contains no simple cycles of odd length at all, that is, it is bipartite. Then there exists a partition $\mathbf{Z} = \mathbf{Z}_1 \dot\cup \mathbf{Z}_2$, $\#\mathbf{Z}_1 > 0$, $\#\mathbf{Z}_2 > 0$, such that the induced subgraphs $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_1 \rangle$ and $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_2 \rangle$ are edgeless. In this case, we get the partition of the family $\mathbf{Z}^*$ into two nonempty subfamilies $\psi^{-1}(\mathbf{Z}_1) \cap \mathbf{Z}^*$ and $\psi^{-1}(\mathbf{Z}_2) \cap \mathbf{Z}^*$. Under the homomorphism ψ, the edge $\{J^*_{t_i}, J^*_{t_j}\}$ is mapped onto the edge $\{\psi(J^*_{t_i}), \psi(J^*_{t_j})\}$; indeed, the situation where $\psi(J^*_{t_i}) = \psi(J^*_{t_j})$ is impossible – that would mean that $\psi(J^*_{t_i}) \cup \psi(J^*_{t_j}) \ne [m]$, a contradiction with the condition according to which the map ψ is a homomorphism. Then the edgelessness of the subgraphs $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_1 \rangle$ and $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_2 \rangle$ implies that the subgraphs $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \psi^{-1}(\mathbf{Z}_1) \cap \mathbf{Z}^* \rangle$ and $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \psi^{-1}(\mathbf{Z}_2) \cap \mathbf{Z}^* \rangle$ are also edgeless, but this is impossible because the subgraph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \mathbf{Z}^* \rangle$ contains a simple cycle of odd length. As a consequence, the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z} \rangle$, as well as the graph $\mathtt{ISG}([m], \Delta^{>})$ itself which is isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$, contains a simple graph of odd length not exceeding m, because $|\mathbf{Z}| \le |\mathbf{Z}^*| \le p \le m$, as was to be proved.

(iii) It was mentioned in the proof of (i) and (ii) that the statement is true for any rank 2 system $\mathfrak{S}$ – see Proposition 2.33. Assume that it is true for any system $\mathfrak{S}$ of rank $n-1$, where $n-1 \ge 2$. Let us consider family (2.40) and the corresponding graph of the independence system $\mathtt{ISG}([m], \Delta^{>})$ that is isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$. Let J_{s_1} and J_{s_2} be arbitrary vertices of the graph $\mathtt{ISG}([m], \Delta^{>})$. Let us show that they are linked by a simple chain of length not exceeding $\lfloor \frac{m}{2} \rfloor$. Taking into account that family (2.40) satisfies restriction (2.42), let us set $r := 2 \le \min\{2, n-1\}$ and use Lemma 2.34 for family (2.40) and for the distinguished vertices J_{s_1} and J_{s_2} of the graph $\mathtt{ISG}([m], \Delta^{>})$. Let $\boldsymbol{R}$ be an $(n-1)$-dimensional subspace of the space $\mathbb{R}^n$, and (2.44) the corresponding family of subspace pairs, which in its turn is assigned the graph of the independence system $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. Let $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$ be the graph homomorphism mentioned in Lemma 2.34; according to this lemma, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exist vertices $J^*_{t_1}$ and $J^*_{t_2}$ such that $J^*_{t_1} = J_{s_1}$, $J^{*_{t_2}=J_{s_2}}$, and $\psi(J^*_{t_1}) = J_{s_1}$, $\psi(J^{*_{t_2}}) = J_{s_2}$. By the induction hypothesis, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exists a simple $(J^*_{t_1} \leftrightarrow J^*_{t_2})$-chain of length $p \le \lfloor \frac{m}{2} \rfloor$. Under the homomorphism ψ, this chain is mapped onto a $\big(\psi(J^*_{t_1}) \leftrightarrow \psi(J^*_{t_2})\big)$-walk in the graph $\mathtt{ISG}([m], \Delta^{>})$ containing $p \le \lfloor \frac{m}{2} \rfloor$ edges. One can distinguish in this walk a simple $\big(\psi(J^*_{t_1}) \leftrightarrow \psi(J^*_{t_2})\big)$-chain containing at most $p \le \lfloor \frac{m}{2} \rfloor$ edges and linking the vertices J_{s_1} and J_{s_2}, because $\psi(J^*_{t_1}) = J_{s_1}$ and $\psi(J^*_{t_2}) = J_{s_2}$; this completes the proof of the statement. □

When constructing algorithms of extracting maximal feasible subsystems of system (2.26), with the use of its graph $\mathtt{MFSG}(\mathfrak{S})$, those properties of the graph $\mathtt{MFSG}(\mathfrak{S})$ play a significant role which characterize its type of connectedness. It turns out that the graph of MFSs of the system $\mathfrak{S}$ conditionally has a connectedness type, which is stronger than just the connectedness certified by Theorem 2.20.

We will need an auxiliary statement that complements Theorem 2.25 and whose proof is given on page 47; it touches on the degrees of vertices in $\mathtt{MFSG}(\mathfrak{S})$.

Proposition 2.36. *Suppose that for some k, $1 \le k \le n-1$, each subsystem with $k+1$ inequalities of system* (2.26) *is feasible. Then the degree of any vertex J_s in the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is at least $k+1$.*

Proposition 2.37. *If the rank of each subsystem with 3 inequalities of system* (2.26) *equals 3, then its graph* $\mathtt{MFSG}(\mathfrak{S})$ *is 2-connected.*

Proof. According to Proposition 2.36, the graph $\mathtt{MFSG}(\mathfrak{S})$ has at least four vertices: $\#\mathbf{J} \ge 4$; it suffices to show that $\mathtt{MFSG}(\mathfrak{S})$ does not contain a cutvertex, that is a vertex $J \in \mathbf{J}$ such that the subgraph $\mathtt{MFSG}(\mathfrak{S})\langle \mathbf{J} - \{J\} \rangle$ is disconnected.

Suppose to the contrary that the graph of MFSs of system (2.26) has a cutvertex, say the vertex J_q, that is, the graph $\mathtt{MFSG}(\mathfrak{S})\langle \mathbf{J} - \{J_q\}\rangle$ is disconnected. Without loss of generality we will suppose that the vertices J_1 and J_2 belong in the subgraph $\mathtt{MFSG}(\mathfrak{S})\langle \mathbf{J} - \{J_q\}\rangle$ to distinct connected components. According to relation (2.37) of Lemma 2.30, $\mathbf{mmi}(J_1) \cap ([m] - J_q) \neq \emptyset$ and $\mathbf{mmi}(J_2) \cap ([m] - J_q) \neq \emptyset$. Let us choose arbitrary indices $j_1 \in \mathbf{mmi}(J_1) \cap ([m] - J_q)$ and $j_2 \in \mathbf{mmi}(J_2) \cap ([m] - J_q)$. For the $(n-1)$-dimensional faces $\mathcal{F}(\{j_1\}, J_1)$ and $\mathcal{F}(\{j_2\}, J_2)$ of the cones $\overline{\mathbf{C}_>(J_1)}$ and $\overline{\mathbf{C}_>(J_2)}$, respectively, we have

$$\mathbf{H}(J_1) \cap \mathbf{C}_>(J_1 - \{j_1\}) \neq \emptyset \quad \text{and} \quad \mathbf{H}(J_2) \cap \mathbf{C}_>(J_2 - \{j_2\}) \neq \emptyset\,. \tag{2.45}$$

Let us consider the families

$$\big\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_1), -\mathbf{C}_>(i) \cap \mathbf{H}(j_1)) :\ i \in [m] - \{j_1\}\big\}\,,$$
$$\big\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_2), -\mathbf{C}_>(i) \cap \mathbf{H}(j_2)) :\ i \in [m] - \{j_2\}\big\}$$

and

$$\big\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2), -\mathbf{C}_>(i) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)) :\ i \in [m] - \{j_1, j_2\}\big\}\,; \tag{2.46}$$

they are assigned the abstract simplicial complexes $\Delta^>_{\mathbf{H}(j_1)}$, $\Delta^>_{\mathbf{H}(j_2)}$ and $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$, respectively, defined in (2.41).

Since the implications

$$|L| = 3 \quad \Longrightarrow \quad \dim \mathbf{H}(L) = n - 3 \tag{2.47}$$

are by condition true for all multi-indices $L \subseteq [m]$, we conclude with the help of Theorem 2.20 that the graphs of the independence systems $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)})$ and $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_2)})$, isomorphic to the graphs of MFSs of some rank $n-1$ linear inequality systems, are connected.

Let us show that the graph of the independence system $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$ is not edgeless. Let J_s be some facet of the complex $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$. Fix an arbitrary vector $\boldsymbol{x}^* \in \mathbf{C}_>(J_s) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)$. Then $-\boldsymbol{x}^* \in \mathbf{C}_>\big(([m] - \{j_1, j_2\}) - J_s\big) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)$, because otherwise, under the assumption that $-\boldsymbol{x}^* \in -\overline{\mathbf{C}_>(j_0)}$ for some index $j_0 \in ([m] - \{j_1, j_2\}) - J_s$, we would obtain $\mathbf{C}_>(J_s) \cap \mathbf{C}_>(j_0) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2) \neq \emptyset$, taking into account (2.47) – a contradiction with the maximality of the feasible subsystem with the multi-index J_s. As a consequence, the subfamily, with the multi-index $([m] - \{j_1, j_2\}) - J_s$, of family (2.46) has an intersection different from $(\emptyset, \emptyset)$, and thus there exists a facet J_t of the complex $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$ such that $J_t \supset ([m] - \{j_1, j_2\}) - J_s$. Thus, $\{J_s, J_t\}$ is an edge of the graph $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$.

Note that, according to Lemma 2.21 (ii), the implications

$$L \subseteq [m] - \{j_1\}\,,\ \ \mathbf{C}_>(L) \cap \mathbf{H}(j_1) \neq \emptyset \implies \mathbf{C}_>(L \cup \{j_1\}) \neq \emptyset\,,$$

$$L \subseteq [m] - \{j_2\}\,,\ \ \mathbf{C}_>(L) \cap \mathbf{H}(j_2) \neq \emptyset \implies \mathbf{C}_>(L \cup \{j_2\}) \neq \emptyset\,,$$

$$L \subseteq [m] - \{j_1, j_2\}\,,\ \ \mathbf{C}_>(L) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2) \neq \emptyset \implies \mathbf{C}_>(L \cup \{j_1, j_2\}) \neq \emptyset$$

hold.

These implications imply that there exist maps $\psi_1 \colon \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)} \to \mathbf{max}\,\Delta^>$, $\psi_2 \colon \mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^>$, $\psi_3 \colon \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}$ and $\psi_4 \colon \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)}$ such that the relations

$$\begin{aligned} J \in \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)} &\implies \psi_1(J) \supseteq J \cup \{j_1\}\,,\\ J \in \mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)} &\implies \psi_2(J) \supseteq J \cup \{j_2\}\,,\\ J \in \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} &\implies \psi_3(J) \supseteq J \cup \{j_1\}\,,\\ J \in \mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} &\implies \psi_4(J) \supseteq J \cup \{j_2\} \end{aligned} \tag{2.48}$$

hold.

The maps ψ_1, ψ_2, ψ_3, and ψ_4 are homomorphisms of the corresponding graphs of independence systems. Since the graphs of the independence systems $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)})$ and $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_2)})$ are connected, the subgraphs $\mathtt{ISG}([m], \Delta^>)\langle\psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)})\rangle$ and $\mathtt{ISG}([m], \Delta^>)\langle\psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})\rangle$ are also connected.

Now let $\{J_s, J_t\}$ be an edge of the graph $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$, that is, $J_s \cup J_t = [m] - \{j_1, j_2\}$; then $\psi_3(J_s) \cup \psi_4(J_t) = [m]$ and thus $\psi_1(\psi_3(J_s)) \cup \psi_2(\psi_4(J_t)) = [m]$. As a consequence, the vertex $\psi_1(\psi_3(J_s)) \in \psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)})$ is adjacent in the graph $\mathtt{ISG}([m], \Delta^>)$ to the vertex $\psi_2(\psi_4(J_t)) \in \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})$, and thus the subgraph $\mathtt{ISG}([m], \Delta^>)\langle\psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})\rangle$ is connected. According to (2.48), any vertex from $\psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})$ contains either the index j_1 or the index j_2, and because of $j_1, j_2 \notin J_q$, we have $J_q \notin \psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})$, that is, the vertices from $\psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})$ belong to the same connected component of the subgraph $\mathtt{ISG}([m], \Delta^>)\langle\mathbf{J} - \{J_q\}\rangle$. It follows from (2.45) that there exist facets $J_{t_1} \in \mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)})$ and $J_{t_2} \in \mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_2)})$ such that $J_{t_1} \supseteq J_1 - \{j_1\}$ and $J_{t_2} \supseteq J_2 - \{j_2\}$. Because of the maximality of the feasible subsystems with the multi-indices J_1 and J_2, we see, taking into account (2.48), that $\psi(J_{t_1}) = J_1$ and $\psi(J_{t_2}) = J_2$; thus, $J_1, J_2 \in \psi_1(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^>_{\mathbf{H}(j_2)})$, a contradiction with the assumption that the vertices J_1 and J_2 in the graph $\mathtt{ISG}([m], \Delta^>)\langle\mathbf{J} - \{J_q\}\rangle$ belong to distinct connected components. This contradiction proves the proposition. □

Proof of Proposition 2.36. As a matter of fact, we will present the proofs of several independent statements that lead to the result formulated in the proposition:

(A) *Suppose that each subsystem with $k + 1$ inequalities, where $1 \le k \le n - 1$, of system* (2.26) *is feasible; let $J_s \in \mathbf{J}$ be the multi-index of some of its MFS, and $\mathcal{F}(L, J_s)$ an arbitrary $(n - k)$-dimensional face of the cone $\overline{\mathbf{C}_>(J_s)}$.*

The inclusion

$$-\mathfrak{F}(L, J_s) \subset \mathbf{C}_>([m] - J_s) \tag{2.49}$$

and the relation $\mathbf{C}_>([m] - J_s) \cap \mathbf{C}_>(L) \neq \emptyset$ *hold.*

▷ *Proof.* Let us first show that inclusion (2.49) holds. Suppose to the contrary that there exists an index $j_0 \in [m] - J_s$ such that $-\mathfrak{F}(L, J_s) \not\subset \mathbf{C}_>(j_0)$. Recall that the complement, up to the index set $[m]$, of the multi-index of any maximal feasible subsystem of the system $\mathfrak{S}$ is the multi-index of a *feasible* subsystem. Hence $-\mathfrak{F}(L, J_s) \subset -\overline{\mathbf{C}_>(J_s)} \subseteq \overline{\mathbf{C}_>([m] - J_s)} \subseteq \overline{\mathbf{C}_>(j_0)}$. Thus, two cases are only possible: (1) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0) \neq \emptyset$ and (2) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0) = \emptyset$ and, as a consequence, $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(j_0)$.

(1) Let us consider the first case, and pick two vectors $\boldsymbol{x}^* \in -\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0)$ and $\boldsymbol{y}^* \in -\mathfrak{F}(L, J_s) \cap \mathbf{H}_>(j_0)$. Under $\lambda > 0$ we have the inclusion $\boldsymbol{z}^* := -\lambda \boldsymbol{x}^* + (1 + \lambda)\boldsymbol{y}^* \in -\mathbf{C}_>(j_0)$. Since $\boldsymbol{x}^*, \boldsymbol{y}^* \in -\mathfrak{F}(L, J_s)$ and the set $-\mathfrak{F}(L, J_s)$ is convex and open with respect to the subspace $\mathbf{H}(L)$, then also $\boldsymbol{z}^* \in -\mathfrak{F}(L, J_s) \subseteq -\overline{\mathbf{C}_>(J_s)}$, for a sufficiently small $\lambda > 0$. Thus, $\boldsymbol{z}^* \in -\mathbf{C}_>(j_0) \cap -\mathbf{C}_>(J_s)$. As a consequence, $\mathbf{C}_>(j_0) \cap \mathbf{C}_>(J_s) \neq \emptyset$; this contradicts the maximality of the feasible subsystem with the multi-index J_s.

(2) Let us consider the second case, that is, $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(j_0)$. Since the set $-\mathfrak{F}(L, J_s)$ is open with respect to the subspace $\mathbf{H}(L)$, then $\mathbf{H}(L) \subseteq \mathbf{H}(j_0)$, that is, the rank of the subsystem with the multi-index $L \cup \{j_0\}$ equals the rank of the subsystem with the multi-index L, namely k. Since in the system $\mathfrak{S}$ each subsystem with $k+1$ inequalities is by convention feasible, the rank k subsystem with the multi-index $L \cup \{j_0\}$ is also feasible, that is, $\mathbf{C}_>(L \cup \{j_0\}) \neq \emptyset$. By definition, $\mathfrak{F}(L, J_s) \subset \mathbf{C}_>(J_s - L)$; on the other hand, by applying Lemma 2.21 (i) to the multi-index $L \cup \{j_0\}$, we obtain $\mathfrak{F}(L, J_s) \subset \mathbf{H}(L) = \mathbf{H}(L \cup \{j_0\}) \subseteq \overline{\mathbf{C}_>(L \cup \{j_0\})}$. Thus, $\mathbf{C}_>(J_s - L) \cap \overline{\mathbf{C}_>(L \cup \{j_0\})} \supseteq \mathfrak{F}(L, J_s) \neq \emptyset$ or $\mathbf{C}_>(J_s \cup \{j_0\}) \neq \emptyset$; this contradicts the maximality of the feasible subsystem with the multi-index J_s.

Thus, inclusion (2.49) is proved.

Since $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(L) \subseteq \overline{\mathbf{C}_>(L)}$, then $\overline{\mathbf{C}_>(L)} \cap \mathbf{C}_>([m] - J_s) \supseteq -\mathfrak{F}(L, J_s) \neq \emptyset$, and thus $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$, and this completes the proof of statement (A). ◁

(B) *Let* $\mathbf{A} := \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\} \subset \mathbb{R}^n$, $p \geq 2$; *pick two vectors* $\boldsymbol{x}^*, \boldsymbol{z}^* \in \operatorname{pos} \mathbf{A}$, $\boldsymbol{x}^* \neq \boldsymbol{z}^*$, $\boldsymbol{x}^* \in (\boldsymbol{y}^*, \boldsymbol{z}^*)$. *Let* $\boldsymbol{b} \in \mathbb{R}^n$ *be a vector such that* $\langle \boldsymbol{y}^*, \boldsymbol{b}\rangle > 0$ *and* $\langle \boldsymbol{a}, \boldsymbol{b}\rangle \geq \mathbf{0}$, *for each vector* $\boldsymbol{a} \in \mathbf{A}$.
Then $\operatorname{pos} \mathbf{A} \subseteq \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

▷ *Proof.* Let us show that $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

If $\boldsymbol{z}^* \in \mathbf{A}$ then $\boldsymbol{z}^* \in \mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\}$ because $\boldsymbol{z}^* \neq \boldsymbol{x}^*$.

If $\boldsymbol{x}^* \notin \mathbf{A}$ then $\operatorname{pos}(\mathbf{A} - \{\boldsymbol{x}^*\}) = \operatorname{pos} \mathbf{A}$ and $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

Now let $z^* \notin \mathbf{A}$ and $x^* \in \mathbf{A}$; specifically, let us set $x^* = a_1$. The expression

$$z^* = \alpha_1 x^* + \sum_{i=2}^{p} \alpha_i a_i \,, \quad \alpha_i \geq 0 \,, \quad 1 \leq i \leq p \,, \tag{2.50}$$

is true.

If $\alpha_1 = 0$ then $z^* \in \operatorname{pos}(\mathbf{A} - \{x^*\}) \subseteq \operatorname{pos}(\mathbf{A} \cup \{y^*\} - \{x^*\})$.

Suppose $\alpha_1 > 0$. Since $x^* = \mu z^* + (1-\mu) y^*$, for some μ, $0 < \mu < 1$, then (2.50) implies that

$$z^*(1 - \alpha_1\mu) = \alpha_1(1-\mu) y^* + \sum_{i=2}^{p} \alpha_i a_i \,, \quad \alpha_1 > 0 \,, \quad 0 < \mu < 1 \,. \tag{2.51}$$

By multiplying scalarly both sides of expression (2.51) by the vector b, we obtain

$$(1 - \alpha_1\mu)\langle z^*, b\rangle = \alpha_1(1-\mu)\langle y^*, b\rangle + \sum_{i=2}^{p} \alpha_i \langle a_i, b\rangle \,. \tag{2.52}$$

The right-hand side of (2.52) is positive because $\alpha_1(1-\mu)\langle y^*, b\rangle > 0$, and for all i, $2 \leq i \leq p$, the relations $\langle a_i, b\rangle \geq 0$ are fulfilled.

Since $z^* \in \operatorname{pos}\mathbf{A}$, we have $\langle z^*, b\rangle \geq 0$ and, as a consequence, $1 - \alpha_1\mu > 0$. By dividing both sides of (2.51) by $1 - \alpha_1\mu$, we obtain $z^* \in \operatorname{pos}(\mathbf{A} \cup \{y^*\} - \{x^*\})$; this inclusion implies that $x^* \in (z^*, y^*) \subset \operatorname{pos}(\mathbf{A} \cup \{y^*\} - \{x^*\})$ and, as a consequence, $\operatorname{pos}\mathbf{A} \subseteq \operatorname{pos}(\mathbf{A} \cup \{y^*\} - \{x^*\})$. Statement (B) is proved. ◁

Recall two definitions. Let $M \subset \mathbb{R}^n$ be a convex body. A point $x^* \in \operatorname{bd} M$ is said to be *lightened from the outside* by a source $y^* \in \mathbb{R}^n$ if there exists a point $z^* \in \operatorname{int} M$ such that $x^* \in (y^*, z^*)$. The set $B \subset \operatorname{bd} M$ is *lightened from the outside* by a source family $N \subset \mathbb{R}^n$ when each point is lightened from the outside by at least one source $y^* \in N$.

(C) *Let us consider system* (2.26). *Let $M \subset [m]$, $M \neq \emptyset$, $\mathbf{C}_>(M) \neq \emptyset$, $L \neq \emptyset$, and let $\mathcal{F}(L, M)$ be a face of dimension r, $0 \leq r \leq n-1$, of the cone $\overline{\mathbf{C}_>(M)}$. A point $x^* \in \mathcal{F}(L, M)$ is lightened from the outside by a source $y^* \in \mathbb{R}^n$ if and only if $y^* \in -\mathbf{C}_>(L)$.*

▷ *Proof*. The *necessity*. Let the point $x^* \in \mathcal{F}(L, M) \neq \emptyset$ be lightened from the outside by the source y^*, that is, there exists a point $z^* \in \mathbf{C}_>(M) \subseteq \mathbf{C}_>(L)$ such that $x^* \in (y^*, z^*)$, that is, $x^* = \alpha z^* + (1-\alpha) y^*$, where $0 < \alpha < 1$. Further, $x^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L)$, $z^* \in \mathbf{C}_>(M)$; as a consequence, for $y^* = \frac{1}{1-\alpha} x^* - \frac{\alpha}{1-\alpha} z^*$ we have $\langle a_i, y^*\rangle = \frac{1}{1-\alpha}\langle a_i, x^*\rangle - \frac{\alpha}{1-\alpha}\langle a_i, z^*\rangle < 0$, for each index $i \in L$, that is, $y^* \in -\mathbf{C}_>(L)$.

The *sufficiency*. Let $y^* \in -\mathbf{C}_>(L)$, $x^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L)$. Let us consider a vector $z^* := -\varepsilon y^* + (1+\varepsilon) x^*$, where $\varepsilon > 0$. Then $-\varepsilon y^* \in \mathbf{C}_>(L)$ and $(1+\varepsilon) x^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L) \subseteq \overline{\mathbf{C}_>(L)}$; as a consequence, $z^* \in \mathbf{C}_>(L)$ for any $\varepsilon > 0$, and $z^* \in \mathbf{C}_>(M - L)$ for a sufficiently small $\varepsilon > 0$; therefore, $z^* \in \mathbf{C}_>(M)$ for a sufficiently small $\varepsilon > 0$. Since $x^* = \frac{1}{1+\varepsilon} z^* + \frac{\varepsilon}{1+\varepsilon} y^*$, where $\varepsilon > 0$, we have $x^* \in (y^*, z^*)$, that is, the point x^* is lightened from the outside by the source y^*. ◁

(D) *Suppose that for some* k, $1 \le k \le n-1$, *each subsystem with* $k+1$ *inequalities of system* (2.26) *is feasible; let* J_s *be the multi-index of some of its maximal feasible subsystem,* q *the degree of the vertex* J_s *in the graph* $\mathtt{MFSG}(\mathfrak{S})$, $\mathcal{N}(J_s) := \{J_{s_1}, \dots, J_{s_q}\}$ *the neighborhood of the vertex* J_s *in the graph* $\mathtt{MFSG}(\mathfrak{S})$, *and* $\boldsymbol{y}_i^* \in -\mathbf{C}_>(J_{s_i})$ *some representatives of the sets* $-\mathbf{C}_>(J_{s_i})$, $1 \le i \le q$.
Each $(n-k)$*-dimensional face* $\mathfrak{F}(L, J_s)$ *of the cone* $\overline{\mathbf{C}_>(J_s)}$ *is lightened from the outside by the source family* $\boldsymbol{N} := \{\boldsymbol{y}_i^* : \ 1 \le i \le q\}$.

▷ *Proof*. Under the hypothesis of statement (D) for an $(n-k)$-dimensional face $\mathfrak{F}(L, J_s)$ of the cone $\overline{\mathbf{C}_>(J_s)}$, we see that $\mathbf{C}_>([m] - J_s) \cap \mathbf{C}_>(L) \neq \emptyset$, in accordance with statement (A). As a consequence, there exists a MFS with a multi-index $J_p \supseteq ([m] - J_s) \cup L$, and since $J_p \cup J_s = [m]$, then $J_p \in \{J_{s_i} : \ 1 \le i \le q\}$. Specifically, let $J_p = J_{s_1}$. Then $\boldsymbol{y}_{s_1}^* \in -\mathbf{C}_>(J_{s_1}) = -\mathbf{C}_>(J_p) \subseteq -\mathbf{C}_>(L)$ and, according to statement (C), the face $\mathfrak{F}(L, J_s)$ is lightened from the outside by the source $\boldsymbol{y}_{s_1}^* \in \boldsymbol{N}$; this completes the proof. ◁

(E) *Let us consider system* (2.26). *Let* $\emptyset \neq M \subset [m]$, $\mathbf{C}_>(M) \neq \emptyset$, $r := \dim \mathbf{H}(M) < n-1$, *and suppose that all* $(r+1)$*-dimensional faces of the cone* $\overline{\mathbf{C}_>(M)}$ *are lightened from the outside by a source family* $\boldsymbol{N}$ *and, besides, there exists a vector* $\boldsymbol{b} \in \mathbb{R}^n$ *such that* $\langle \boldsymbol{y}^*, \boldsymbol{b} \rangle > 0$, *for each* $\boldsymbol{y}^* \in \boldsymbol{N}$, *and* $\langle \boldsymbol{c}, \boldsymbol{b} \rangle \ge 0$, *for each* $\boldsymbol{c} \in \overline{\mathbf{C}_>(M)}$.
Then $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \boldsymbol{N})$.

▷ *Proof*. Let vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_r \in \mathbf{H}(M)$ represent a basis of the space $\mathbf{H}(M)$, and let us consider the vector $\boldsymbol{b}_0 := -\boldsymbol{b}_1 - \boldsymbol{b}_2 - \cdots - \boldsymbol{b}_r$. We set $\boldsymbol{B} := \{\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_r\}$. Let us choose, for each $(r+1)$-dimensional face $\mathfrak{F}(L, M)$ of the cone $\overline{\mathbf{C}_>(M)}$, one representative $\boldsymbol{x}^* \in \mathfrak{F}(L, M)$, and let us form the set $\{\boldsymbol{x}_j : \ 1 \le j \le l\}$ containing precisely these points; let l be the number of the $(r+1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(M)}$; note that $l > 0$ because $\dim \mathbf{H}(M) = r$ and $\mathbf{C}_>(M) \neq \emptyset$. It is well known that

$$\mathbf{C}_>(M) = \operatorname{pos}\left(\boldsymbol{B} \cup \{\boldsymbol{x}_j^* : \ 1 \le j \le l\}\right) . \tag{2.53}$$

By convention, the set $\{\boldsymbol{x}_j^* : \ 1 \le j \le l\}$ is lightened from the outside by the source family $\boldsymbol{N}$. Let us assign to each point $\boldsymbol{x}_j^*$, $1 \le j \le l$, a source $\boldsymbol{y}_j^*$ that lighten $\boldsymbol{x}_j^*$ from the outside. Then it follows from the definition that for each number j, $1 \le j \le l$, there exists a point $\boldsymbol{z}_j^* \in \mathbf{C}_>(M)$ such that

$$\boldsymbol{x}_j^* \in (\boldsymbol{z}_j^*, \boldsymbol{y}_j^*) . \tag{2.54}$$

By the hypothesis of the statement, we have

$$\langle \boldsymbol{c}, \boldsymbol{b} \rangle \ge 0 , \quad \forall \boldsymbol{c} \in \mathbf{C}_>(M) \quad \text{and} \quad \langle \boldsymbol{y}_j^*, \boldsymbol{b} \rangle > 0 , \quad 1 \le j \le l . \tag{2.55}$$

Besides,

$$\boldsymbol{x}_j^* \neq \boldsymbol{y}_j^* , \quad 1 \le i, j \le l \quad \text{and} \quad \boldsymbol{x}_j^* \notin \mathbf{H}(M) , \quad 1 \le i \le l . \tag{2.56}$$

By using statement (C) and relations (2.53), (2.54), and (2.56), let us write down the chain of inclusions

$$\begin{aligned}\operatorname{pos}(\mathbf{H}(M)\cup\{\boldsymbol{x}_1^*,\boldsymbol{x}_2^*,\dots,\boldsymbol{x}_{l-1}^*,\boldsymbol{x}_l^*\}) &\subseteq \operatorname{pos}(\mathbf{H}(M)\cup\{\boldsymbol{y}_1^*,\boldsymbol{x}_2^*,\dots,\boldsymbol{x}_{l-1}^*,\boldsymbol{x}_l^*\})\\ \subseteq\cdots\subseteq \operatorname{pos}(\mathbf{H}(M)\cup\{\boldsymbol{y}_1^*,\boldsymbol{y}_2^*,\dots,\boldsymbol{y}_{l-1}^*,\boldsymbol{x}_l^*\}) &\subseteq \operatorname{pos}(\mathbf{H}(M)\cup\{\boldsymbol{y}_1^*,\boldsymbol{y}_2^*,\dots,\boldsymbol{y}_{l-1}^*,\boldsymbol{y}_l^*\})\\ &\subseteq \operatorname{pos}(\mathbf{H}(M)\cup\boldsymbol{N})\,. \qquad (2.57)\end{aligned}$$

Relations (2.53) and (2.57) imply the inclusion $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}(\mathbf{H}(M)\cup\boldsymbol{N})$, which completes the proof. ◁

Let J_s be the multi-index of an arbitrary maximal feasible subsystem of system (2.26), q the degree of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ of that system, $\mathcal{N}(J_s) := \{J_{s_1},\dots,J_{s_q}\}$ the neighborhood of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$, and $\boldsymbol{y}_i^* \in -\mathbf{C}_>(J_{s_i})$ some representatives of the open cones $-\mathbf{C}_>(J_{s_i})$, $1 \le i \le q$. Let r_0 be the least integer such that all r_0-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$ are lightened from the outside by the source set $\{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$. Since, according to statement (D), all $(n-k)$-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$ are lightened from the outside by the source family $\{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$, then $r_0 \le n-k \in [n-1]$.

The following assertion is true:

(F) *Under the hypothesis of Proposition* 2.36, *there exists a cone* $\overline{\mathbf{C}_>(M)}$, $M \subseteq J_s$, *such that* $\dim\mathbf{H}(M) = r_0 - 1$, *all r_0-dimensional faces of the cone $\overline{\mathbf{C}_>(M)}$ are lightened from the outside by a source family $\boldsymbol{N} := \{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$, and there exists a vector $\boldsymbol{b} \in \mathbb{R}^n$ such that $\langle \boldsymbol{y}^*, \boldsymbol{b}\rangle > 0$ for each $\boldsymbol{y}^* \in \boldsymbol{N}$, and $\langle \boldsymbol{c}, \boldsymbol{b}\rangle \ge 0$ for each $\boldsymbol{c} \in \mathbf{C}_>(M)$.*

▷ *Proof*. According to Lemma 2.26, $J_{s_i} \cap J_s \ne \emptyset$, for each i, $1 \le i \le q$; therefore, $\overline{\mathbf{C}_>(J_{s_i})} \cap -\mathbf{C}_>(J_s) \ne \emptyset$, $1 \le i \le q$. As a consequence, $\{\boldsymbol{y}_i^* :\ 1 \le i \le q\} \cap -\mathbf{C}_>(J_s) = \emptyset$, and therefore, because of statement (C), the face $\mathcal{F}(J_s, J_s) = \mathbf{H}(J_s)$ of dimension $\dim\mathbf{H}(J_s)$ is not lightened from the outside by the source family $\{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$. Since the face $\mathcal{F}(J_s, J_s) = \mathbf{H}(J_s)$ is of dimension that is minimal among all faces of the cone $\overline{\mathbf{C}_>(M)}$, then $r_0 > \dim\mathbf{H}(J_s)$ and, by the definition of the quantity r_0, there exists an (r_0-1)-dimensional face $\mathcal{F}(M, J_s)$ of the cone $\overline{\mathbf{C}_>(J_s)}$, which is not lightened from the outside by the source family $\{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$. Let us show that the cone $\overline{\mathbf{C}_>(M)}$ satisfies statement (E). Let us show that

$$\mathcal{F}(M, J_s) \cap -\mathbf{C}_>([m]-J_s) = \emptyset\,. \qquad (2.58)$$

Assume the converse. Then, because of $\mathcal{F}(M, J_s) \subset \mathbf{H}(M)$, we have $\mathbf{H}(M)\cap\mathbf{C}_>([m]-J_s) \ne \emptyset$ and, according to Lemma 2.21 (ii), $\mathbf{C}_>(M) \cap \mathbf{C}_>([m]-J_s) \ne \emptyset$, from where we conclude that there exists a maximal feasible subsystem with a multi-index $J_p \supseteq M \cup ([m]-J_s)$. Since $J_p \cup J_s = [m]$, then $J_p \in \{J_{s_1},\dots,J_{s_q}\}$. Specifically, let $J_p = J_{s_1}$. Then the face $\mathcal{F}(M, J_s)$ is lightened from the outside by a source $\boldsymbol{y}_{s_1}^* \in -\mathbf{C}_>(J_{s_1}) \subseteq -\mathbf{C}_>(M)$; this contradicts the choice of the face $\mathcal{F}(M, J_s)$. This contradiction proves (2.58).

On the other hand, $\mathcal{F}(M, J_s) \subset \overline{\mathbf{C}_>(J_s)} \subset -\overline{\mathbf{C}_>([m] - J_s)}$. Taking into account (2.58), we have $\mathcal{F}(M, J_s) \subseteq -\overline{\mathbf{C}_>([m]-J_s)}-(-\mathbf{C}_>([m]-J_s))$. Since $-\overline{\mathbf{C}_>([m]-J_s)}-(-\mathbf{C}_>([m]-J_s)) \subset \bigcup_{j\in[m]-J_s} \mathbf{H}(j)$, the inclusion $\mathcal{F}(M, J_s) \subset \bigcup_{j\in[m]-J_s} \mathbf{H}(j)$ holds, and thus there exists an index $j_0 \in [m] - J_s$ such that $\mathcal{F}(M, J_s) \cap \mathbf{H}(j_0) \neq \emptyset$. Let us show that $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$. Suppose to the contrary that $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) \neq \emptyset$. Then we derive from the relations $\mathcal{F}(M, J_s) \cap \mathbf{H}(j_0) = \mathbf{H}(M) \cap \mathbf{C}_>(J_s - M) \cap \mathbf{H}(j_0) \neq \emptyset$ and $\mathbf{C}_>(M) \cup \{j_0\} \neq \emptyset$, in accordance with Lemma 2.21 (ii), that $\mathbf{C}_>(M) \cap \mathbf{C}_>(J_s - M) \cap \mathbf{C}_>(j_0) = \mathbf{C}_>(J_s \cup \{j_0\}) \neq \emptyset$, $j_0 \in [m] - J_s$ – a contradiction with the maximality of the feasible subsystem with the multi-index J_s; thus, $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$. Since $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$ and the sets $\mathbf{C}_>(M)$ and $\mathbf{C}_>(j_0)$ are open in $\mathbb{R}^n$, then $\mathbf{C}_>(M) \cap \overline{\mathbf{C}_>(j_0)} = \emptyset$; as a consequence, $\mathbf{C}_>(M) \subset -\mathbf{C}_>(j_0)$. Since $\mathbf{C}_>(M) \subset -\mathbf{C}_>(j_0)$ and $-\mathbf{C}_>(J_{s_i}) \subset -\mathbf{C}_>([m] - J_s) \subseteq -\mathbf{C}_>(j_0)$, $1 \leq i \leq q$, the vector $\boldsymbol{b} \in \mathbb{R}^n$ in statement (F) can be replaced with the vector $-\boldsymbol{a}_{j_0}$ for which $-\mathbf{C}_>(j_0) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle -\boldsymbol{a}_{j_0}, \boldsymbol{x}\rangle > 0\}$.

Now suppose $L_0 \subset M$, and let $\mathcal{F}(L_0, M)$ be an arbitrary r_0-dimensional face of the cone $\overline{\mathbf{C}_>(M)}$; such faces in the cone $\overline{\mathbf{C}_>(M)}$ do exist because $\dim \mathbf{H}(M) = r_0 - 1$ and $\mathbf{C}_>(M) \neq \emptyset$. Let us show that then $\mathcal{F}(L_0, J_s)$ represents an r_0-dimensional face of the cone $\overline{\mathbf{C}_>(J_s)}$, thus, $\mathcal{F}(L_0, J_s) = \mathbf{H}(L_0) \cap \mathbf{C}_>(J_s - L_0) \neq \emptyset$. Let us pick two vectors $\boldsymbol{x}^* \in \mathcal{F}(M, J_s) = \mathbf{H}(M) \cap \mathbf{C}_>(J_s - M) = \mathbf{H}(L_0) \cap \mathbf{H}(M - L_0) \cap \mathbf{C}_>(J_s - M) \neq \emptyset$ and $\boldsymbol{y}^* \in \mathcal{F}(L_0, M) = \mathbf{H}(L_0) \cap \mathbf{C}_>(M - L_0) \neq \emptyset$. Since $L_0 \subset M \subset [m]$, then for the vector $\boldsymbol{z}^* := \boldsymbol{x}^* + \varepsilon \boldsymbol{y}^*$ we have $\boldsymbol{z}^* \in \mathbf{H}(L)$, for any ε; $\boldsymbol{z}^* \in \mathbf{C}_>(M - L_0)$ for any $\varepsilon > 0$, and $\boldsymbol{z}^* \in \mathbf{C}_>([m] - M)$ for a sufficiently small ε; as a consequence, $\boldsymbol{z}^* \in \mathbf{H}(L_0) \cap \mathbf{C}_>(M - L_0) \cap \mathbf{C}_>([m] - M) = \mathbf{H}(L_0) \cap \mathbf{C}_>(J_s - L_0) = \mathcal{F}(L_0, J_s) \neq \emptyset$ for a sufficiently small $\varepsilon > 0$. Since $\mathcal{F}(L_0, J_s)$, being an r_0-dimensional face of the cone $\overline{\mathbf{C}_>(J_s)}$, is lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \leq i \leq q\}$, according to statement (C), $-\mathbf{C}_>(L_0) \cap \{\boldsymbol{y}_i^* : 1 \leq i \leq q\} \neq \emptyset$; as a consequence, according to statement (C), the face $\mathcal{F}(L_0, M)$ of the cone $\overline{\mathbf{C}_>(M)}$ is lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \leq i \leq q\}$; this completes the proof of statement (F). ◁

Now we can complete the proof of Proposition 2.36. For the cone $\overline{\mathbf{C}_>(M)}$ from statement (F), according to statement (E), we have $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}\left(\mathbf{H}(M) \cup \{\boldsymbol{y}_i^* : 1 \leq i \leq q\}\right)$. Since $\dim \mathbf{H}(M) = r_0 - 1$ and the cone $\overline{\mathbf{C}_>(M)}$ is n-dimensional, then the rank of the set $\{\boldsymbol{y}_i^* : 1 \leq i \leq q\}$ is at least $n - (r_0 - 1)$; as a consequence, $q \geq n - (r_0 - 1)$, and $q \geq k + 1$ because $r_0 \leq n - k$. Proposition 2.36 is proved. □

Recall some basic properties of 2-connected graphs:

Proposition 2.38. *Let* **G** *be a simple connected graph. Then the following assertions are equivalent:*
(1) *the graph* **G** *is* 2*-connected;*
(2) *any two vertices of the graph* **G** *belong to some simple cycle;*
(3) *any vertex and any edge of the graph* **G** *belong to some simple cycle;*
(4) *any two edges belong to a simple cycle;*

(5) *for any two vertices a and b, and for any edge E, there exists a simple* $(a \leftrightarrow b)$*-chain, containing E;*
(6) *for any three vertices a, b and c, there exists a simple* $(a \leftrightarrow b)$*-chain going through c.*

The following statement summarizes Propositions 2.37 and 2.38:

Corollary 2.39. *If the rank of each subsystem with three inequalities of system* (2.26) *equals* 3*, then the following assertions are equivalent:*
(1) *The multi-indices of any two MFSs* J_{i_1} *and* J_{i_2} *of the system* $\mathfrak{S}$ *belong to a simple cycle of the graph* $\texttt{MFSG}(\mathfrak{S})$*;*
(2) *the multi-index of any MFS* J_{i_1} *and the multi-indices of any two MFSs* J_{i_2} *and* J_{i_3}*, such that* $J_{i_2} \cup J_{i_3} = [m]$*, belong to a simple cycle of the graph* $\texttt{MFSG}(\mathfrak{S})$*;*
(3) *the multi-indices of any four MFSs* $J_{i_1}, J_{i_2}, J_{i_3}$*, and* J_{i_4}*, such that* $J_{i_1} \cup J_{i_2} = J_{i_3} \cup J_{i_4} = [m]$*, belong to a simple cycle of the graph* $\texttt{MFSG}(\mathfrak{S})$*;*
(4) *for the multi-indices of any two MFSs* J_{i_1} *and* J_{i_2}*, and for the multi-indices of any pair of MFSs* J_{i_3} *and* J_{i_4}*, such that* $J_{i_3} \cup J_{i_4} = [m]$*, there exists a simple* $(J_{i_1} \leftrightarrow J_{i_2})$*-chain of the graph* $\texttt{MFSG}(\mathfrak{S})$ *containing the pair of multi-indices* J_{i_3} *and* J_{i_4}*;*
(5) *for the multi-indices of any three MFSs* J_{i_1}, J_{i_2}*, and* J_{i_3}*, there exists a simple* $(J_{i_1} \leftrightarrow J_{i_2})$*-chain of the graph* $\texttt{MFSG}(\mathfrak{S})$ *going through the vertex* J_{i_3}*.*

2.4 The hypergraph of maximal feasible subsystems of an infeasible system of linear inequalities

An analysis of covers of the index set of the inequalities of system (2.26) by *arbitrary* families of the multi-indices of its MFSs leads to a natural generalization of the notion of graph of MFSs, which represented the research subject in Section 2.3:

The *hypergraph* $\texttt{MFSH}(\mathfrak{S})$ *of maximal feasible subsystems* (*hypergraph of MFSs*) of the system $\mathfrak{S}$ is defined as follows:
– *the vertex set of the hypergraph* $\texttt{MFSH}(\mathfrak{S})$ *is the family* $\mathbf{J}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$*;*
– *the hyperedge family of the hypergraph* $\texttt{MFSH}(\mathfrak{S})$ *is the family of all the unordered collections* $\mathfrak{J} \subseteq \mathbf{J}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$ *that cover the index set of the inequalities of the system:*

$$\bigcup_{J \in \mathfrak{J}} J = [m] \,.$$

The properties of the hypergraph of MFSs of a *rank* 2 system $\mathfrak{S}$ *over* $\mathbb{R}^2$ are well studied and thus augment the information from Proposition 2.33 on the graph of MFSs of this system:

Let $\mathbf{J} := \{J_1, \ldots, J_{\mathrm{q}}\}$; recall that the number q of maximal feasible subsystems of the system $\mathfrak{S}$ is odd, that is, $\mathrm{q} = 2t + 1$ for some t. Let us consider a $\{0, 1\}$-matrix $\boldsymbol{M}$ of size $m \times \mathtt{q}$ whose (j, i)th entry m_{ji} by definition is

$$m_{ji} := \begin{cases} 1\,, & \text{if } j \in J_i\,, \\ 0\,, & \text{if } j \notin J_i\,. \end{cases}$$

Let us re-index the inequalities and multi-indices of MFSs of the system $\mathfrak{S}$ in such a way that the matrix $\boldsymbol{M}$ gets a presentation suitable for consideration. For this to be done, let us assign to each inequality of the system $\mathfrak{S}$ a directing unit vector $\boldsymbol{c}_j$ of the line $\{\mathbf{x} \in \mathbb{R}^2 :\ \langle \boldsymbol{a}_j, \mathbf{x}\rangle = 0\}$, by choosing between the two possible vectors the unique vector such that, when moving along the line in the direction prescribed by it, the half-plane $\{\mathbf{x} \in \mathbb{R}^2 :\ \langle \boldsymbol{a}_j, \mathbf{x}\rangle > 0\}$ is left on the right-hand side. Without loss of generality, we will suppose that the multi-indices $J_1, \dots, J_{\mathtt{q}}$ of maximal feasible subsystems of the system $\mathfrak{S}$ are indexed in the ascending order with respect to the polar angles of the corresponding vectors $\boldsymbol{c}_j$, supposing that the index 1 is assigned to the directing vector of the left boundary of the solution cone to the maximal feasible subsystem with the multi-index J_1.

Under the chosen numeration of the inequalities and multi-indices of MFSs of the system $\mathfrak{S}$, the matrix $\boldsymbol{M}$ obtains the following form:

$$\boldsymbol{M} := \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ \hdotsfor{9} \\ 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 \\ \hdotsfor{9} \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 \\ \hdotsfor{9} \\ \hdotsfor{9} \\ 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \\ \hdotsfor{9} \\ 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

The index of each inequality is included in the multi-indices of precisely $t + 1$ MFSs; since in the matrix $\boldsymbol{M}$ there are precisely $\mathtt{q} = 2t + 1$ pairwise distinct rows, then the indices of the inequalities that compose the system $\mathfrak{S}$, are partitioned into $\mathtt{q}$ equivalence classes: the inequalities with indices j_1 and j_2 are included in the multi-indices of the same MFSs if and only if they are representatives of the same class (respectively when the rows of the matrix $\boldsymbol{M}$, with the indices j_1 and j_2, coincide.) Let us index the equivalence classes of the inequalities of the system $\mathfrak{S}$, in natural order, by the integers $1, \dots, \mathtt{q}$.

Let us consider the hypergraph $(\mathbf{J}, \mathcal{E}) := \mathtt{MFSH}(\mathfrak{S})$, on the vertex set $\mathbf{J}$, with the hyperedge family $\mathcal{E}$, of a rank 2 system $\mathfrak{S}$. For investigating the structure of the hyperedge family $\mathcal{E}$, it suffices to leave for consideration the index of one inequality for each equivalence class, by considering – instead of the initial matrix $\boldsymbol{M}$ – the square $\{0, 1\}$-matrix $\boldsymbol{M}'$ of size $\mathtt{q} \times \mathtt{q}$:

$$
\boldsymbol{M}' := \begin{array}{c} \\ \\ \\ \\ t+1 \\ \\ \\ \\ \\ \end{array}
\begin{pmatrix}
1 & 0 & \cdots & \cdots & \overset{t+1}{0} & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & \ddots & \ddots & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 & 1 \\
1 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 \\
0 & 1 & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix} .
$$

According to the description of the matrix $\boldsymbol{M}'$, the vertex J_1 is, for example, included into two-element hyperedges $\{J_1, J_{t+1}\}$ and $\{J_1, j_{t+2}\}$. The following assertion poses a condition on the indices of the vertices that are included in some vertex subset $U \subset \mathbf{J}$, necessary and sufficient for the inclusion $U \in \mathcal{E}$ to hold. When we write $U := \{J_{i_1}, J_{i_2}, \dots, J_{i_s}\}$ below we will mean that the ordering $i_1 < i_2 < \cdots < i_s$ is the case.

Proposition 2.40. *A vertex subset $U := \{J_{i_1}, \dots, J_{i_s}\}$ of the hypergraph $(\mathbf{J}, \mathcal{E}) :=$ $\mathtt{MFSH}(\mathfrak{S})$ is its hyperedge, that is, $U \in \mathcal{E}$, if and only if for each $k \in [s]$ the condition*

$$(i_{(k \pmod{s})+1} - i_k) \pmod{\mathfrak{q}} \le t + 1$$

is satisfied.

Proof. The *sufficiency* follows from the structure off the matrix $\boldsymbol{M}'$.

The *necessity*. Let $U \in \mathcal{E}$. Then $\bigcup_{k=1}^{s} J_{i_k} = [m]$, by the definition of the hypergraph $\mathtt{MFSH}(\mathfrak{S})$. Let us show that, for example, $i_2 - i_1 \le t + 1$. Note that into the maximal feasible subsystem with a multi-index J_k are included all the inequalities-representatives of the classes only with the numbers: k, $(k \pmod{\mathfrak{q}})+1, \dots, (k+(t-1)) \pmod{\mathfrak{q}} + 1$. Let us consider an arbitrary inequality with an index τ from the class $((i_1 + t) \pmod{\mathfrak{q}}) + 1$. We have $\tau \notin J_{i_1}$ and, as a consequence, there exists $k \in \{2, 3, \dots, s\}$ such that $\tau \in J_{i_k}$. Thus, all the inequalities from the specified class are included in the MFS with the multi-index J_{i_k}, that is, either $i_k = ((i_1 + t) \pmod{\mathfrak{q}}) + 1$ or there exists $e \in \{0, 1, \dots, t-1\}$ such that $((i_k + e) \pmod{\mathfrak{q}}) + 1 = ((i_1 + t) \pmod{\mathfrak{q}}) + 1$. In the first case, $i_k - 1 = (i_k - 1) \pmod{\mathfrak{q}} = (i_1 + t) \pmod{\mathfrak{q}}$, hence $i_k - i_1 = (i_k - i_1) \pmod{\mathfrak{q}} = (t+1) \pmod{\mathfrak{q}} = t+1$. In the second case, $i_k - i_1 = (i_k - i_1) \pmod{\mathfrak{q}} = (t-e) \pmod{\mathfrak{q}} = t - e \le t$. Since $i_2 - i_1 \le i_k - i_1$ then $i_2 - i_1 \le t + 1$. The proposition is proved. ☐

In particular, it follows from Proposition 2.40 that if the number $\mathfrak{q} := \#\mathbf{J}$ of maximal feasible subsystems of the system $\mathfrak{S}$ is quite large, then the hyperedge family $\mathcal{E}$ of the hypergraph $\mathtt{MFSH}(\mathfrak{S})$ has hyperedges that contain no two-element hyperedges as subsets.

Notes

The graphs of independence systems that describe the covers of the vertex sets of any abstract simplicial complexes by pairs of their facets – the research subject in Section 2.1 – were studied in works [48, 53]. They are a natural generalization of the notion of the graph of MFSs of a finite infeasible system of linear inequalities.

Homomorphisms (simplicial maps) of simplicial complexes are basic tools of combinatorial topology, see, for example, [121, 130, 132].

In Section 2.1, we use the standard terminology (open and closed sets, connectedness, density, continuous maps, and so on) of general topology which is presented in various texts; we will point out at just a few sources: [4, 6, 11, 22, 38, 44, 73, 77].

The connectedness regarded from different points of view is one of the most important sides of the description of graphs of general kind, see [13, 28, 32, 67, 104, 136, 147, 156, 166].

Formulating Remark 2.13, we use the information on algebraic varieties from [23].

The notion of hypergraph of an independence system discussed in Section 2.2 reproduces the construction of the hypergraph of maximal feasible subsystems of an infeasible system of constraints, see, for example, [76, 98] and references mentioned in these surveys. Proposition 2.19 is a reformulated Theorem 3.1 from [76] which in its turn is borrowed from [74].

Linear inequality systems are a fundamental subject of pure and applied mathematical research, see, for example, [17, 39–43, 146]; recall that in Section 2.3 we address an analysis of the combinatorial properties of a particular class of infeasible systems of the form (2.26) because of their significance in the simulation of the contradictory problems of pattern recognition.

The notion of graph of MFSs of a finite infeasible system of linear inequalities was introduced by the second and third authors of work [58] as a result of a generalization of constructions presented in [148]. Works in [47, 51, 58] are devoted to a detailed study of the properties of the graph of MFSs.

The proof of Theorem 2.20 describes one of several approaches to the verification of the connectedness of the graph of MFSs of infeasible system (2.26). The first result in this direction was the derivation, by the second author of work [58], of Theorem 2.20 from the assertion, proved by him, on the connectedness of the square of the graph of MFSs. Another justification of the statement is provided by the proof of Theorem 2 from [58]. The proof of Theorem 2.20 given in Section 2.3 follows the works [48, 53].

Theorem 2.25 (i) is proved in work [148] and reproduced in [58] as Theorem 1.

The proof of Theorem 2.28 is, in particular, based on the well-known observation that a graph is bipartite if and only if all its simple cycles are of even length, see, for example, [67, 165].

One component of the proof of Proposition 2.33 is an important proposition presented in [94], according to which the number of MFSs of an infeasible rank 2 system (2.26) over $\mathbb{R}^2$ is odd and it does not exceed m.

In the proof of Proposition 2.35 (iii) we mention the observation verified in [165], according to which in a walk between two vertices of a graph one can distinguish a simple chain linking them.

In section (A) of the proof of Proposition 2.36, we use the assertion that the complement of any MFS of infeasible system (2.26) is feasible; the proof of this fact is presented in [148].

The terminology related to the lightening of convex bodies, which we recall on page 49 is borrowed from [20].

The list of basic properties of bipartite graphs presented in Proposition 2.38 is the content of Theorem 34.1 from [104].

The properties of the hypergraph of MFSs of an infeasible rank 2 system (2.26) over $\mathbb{R}^2$, described in Section 2.4, are borrowed from survey [76]. Proposition 2.40 reproduces Proposition 3.1 from [76], which in its turn can be found in [74].