3 Polytopes, positive bases, and inequality systems

In this chapter, we study some combinatorial and structural properties of convex polytopes, positive bases of vector spaces, and infeasible systems of linear inequalities.

It was shown in the previous chapters that infeasible systems of linear inequalities inherit their significant properties from other fundamental mathematical constructions. For example, being infeasible systems with the monotonicity property, systems of linear inequalities can additionally be described in the language of abstract simplicial complexes. The connectedness of the graphs of maximal feasible subsystems of such systems follows from the fundamental topological property of the connectedness of the space $\mathbb{R}^n$.

Infeasible systems of linear inequalities are our main subject of consideration in the present and subsequent chapters. Vector spaces and convex polytopes play in a sense a subordinate role for the following reason: the combinatorial properties of rank n infeasible systems with m homogeneous strict liner inequalities over $\mathbb{R}^n$ can be investigated via the properties of an m-point subset of the space $\mathbb{R}^{m-n-1}$. Moreover, under appropriate circumstances, the properties of inequality systems turn out to be equivalent to the properties of $(m-n-1)$-dimensional convex polytopes. In an analysis of infeasible systems of linear inequalities by means of a study of polytopes, the notions of faces and diagonals of polytopes play an important role. In an investigation of the construction of the family of minimal infeasible subsystems, it is important to understand the structure of positive bases of $\mathbb{R}^n$.

3.1 Faces and diagonals of convex polytopes

In plane geometry, the notion of *diagonal* plays a role similar to that played by the notion of side of a polygon. In higher dimensions, sides of polygons are generalized to faces playing an important role in problems of combinatorial classification of polytopes. The notion of diagonal deserving the same attention has less successful d-dimensional fate.

In this section, we consider three possible generalizations of the notion of *diagonal* to an arbitrary d-dimensional case. Each of them can be taken as the basis of some combinatorial classification of polytopes, and in the case of the so-called G-diagonals one obtains a classification agreeing with the conventional one defined by the structure of faces.

We use the standard notation: aff for the affine hull, pos for the positive hull, conv for the convex hull, ri for the relative interior, rbd for the relative boundary, vert for the vertex set, and dim for the affine dimension.

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Three notions of diagonals and their relationships

We consider convex and bounded polytopes only. The polytopes of dimension d are called, for brevity, d-polytopes. Speaking of the faces of a polytope, we always mean *proper faces*, that is, the faces different from the empty set and the entire polytope.

Recall that a *cyclic d-polytope* is the convex hull of a finite m-subset, $m > d$, of points of the *moment curve* $(t, t^2, \dots, t^d)$, where $t \in \mathbb{R}$, $t \neq 0$.

A *k-neighborly polytope* is a polytope such that any of its k-subsets of vertices are the vertex sets of some faces. It is well known that the cyclic d-polytope is $\lfloor \frac{d}{2} \rfloor$-neighborly.

We will say that a polytope $\mathcal{P}$ is obtained from polytopes $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ by the operation of *cross*, and use in this case the notation $\mathcal{P} = \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, when $\dim \operatorname{aff} \mathcal{P} = \sum_{j\in[k]} \dim \operatorname{aff} \mathcal{P}_j$ and the intersection $\bigcap_{j\in[k]} \operatorname{ri} \mathcal{P}$ is nonempty and, as a consequence, it represents a unique point.

Let us consider two auxiliary statements:

Lemma 3.1. *If $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, then the faces of the polytope $\mathcal{P}$ are precisely all the sets of the form $\boldsymbol{F} = \operatorname{conv}(\boldsymbol{F}_{j_1} \cup \boldsymbol{F}_{j_2} \cup \cdots \cup \boldsymbol{F}_{j_s})$, where $\boldsymbol{F}_j$ is a face of $\mathcal{P}_j$; here $\boldsymbol{F}_j \neq \mathcal{P}_j$, $j \in \{j_1, j_2, \dots, j_s\} \subset [k]$.*

Proof. Since $\mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k = (\mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_{k-1}) \perp \mathcal{P}_k$, it suffices to prove the lemma for the case $k = 2$. Let $\boldsymbol{F}_i$ be faces of the polytopes $\mathcal{P}_i$, $i \in [2]$, and thus of the polytope $\mathcal{P}$. Let us prove that $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is also a face of $\mathcal{P}$. Let $\boldsymbol{H}_i$ be a supporting hyperplane of $\mathcal{P}_i$ in $\operatorname{aff} \mathcal{P}_i$, $\boldsymbol{F}_i = \boldsymbol{H}_i \cap \mathcal{P}_i$, $i \in [2]$. Note that $\dim \operatorname{aff}(\boldsymbol{H}_1 \cup \boldsymbol{H}_2) = d - 1$. Let $\boldsymbol{H} := \operatorname{aff}(\boldsymbol{H}_1 \cup \boldsymbol{H}_2)$. Taking into account that $\boldsymbol{H}$ is a hyperplane in $\operatorname{aff} \mathcal{P}_i$, we see that $\boldsymbol{H}_i = \boldsymbol{H} \cap \operatorname{aff} \mathcal{P}_i$, $i \in [2]$; thus, the hyperplane $\boldsymbol{H}$ supports both $\mathcal{P}_1$ and $\mathcal{P}_2$ and, because of $\operatorname{ri} \mathcal{P}_1 \cap \operatorname{ri} \mathcal{P}_2 \neq \emptyset$, it also supports $\mathcal{P}$. It is clear that $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2) = \boldsymbol{H} \cap \mathcal{P}$. Now suppose that $\boldsymbol{F}$ is a face of $\mathcal{P}$, and $\boldsymbol{H}$ is a hyperplane supporting $\mathcal{P}$ in $\mathbb{R}^d$, such that $\boldsymbol{F} = \boldsymbol{H} \cap \mathcal{P}$. Let $V_i := \operatorname{vert}(\boldsymbol{F} \cap \mathcal{P}_i)$, $i \in [2]$. We see that $\boldsymbol{F}_i = \operatorname{conv} V_i$ is a face of $\mathcal{P}_i$ because $\operatorname{conv} V_i = \boldsymbol{H} \cap \mathcal{P}_i$, and $\mathcal{P}_i \not\subset \boldsymbol{H}$ because $\{z\} = \operatorname{ri} \mathcal{P}_1 \cap \operatorname{ri} \mathcal{P}_2 \subset \operatorname{ri} \mathcal{P} \not\subset \boldsymbol{H}$. ☐

Lemma 3.2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polytopes; let $\boldsymbol{H}_1 := \operatorname{aff} \mathcal{P}_1$ and $\boldsymbol{H}_2 := \operatorname{aff} \mathcal{P}_2$ be skew planes, and $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are faces of the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is a face of the polytope $\operatorname{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)$.*

Proof. Let $d_1 := \dim \boldsymbol{H}_1$ and $d_2 := \dim \boldsymbol{H}_2$. Let us choose a d_i-simplex $\mathcal{S}_i \supset \mathcal{P}_i$ containing the set $\boldsymbol{F}_i = \operatorname{rbd} \mathcal{S}_i \cap \mathcal{P}_i$ inside of some of its face $\boldsymbol{Q}_i$, $i \in [2]$. The statement now follows from the observation that $\operatorname{conv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ is a simplex, and $\operatorname{conv}(\boldsymbol{Q}_1 \cup \boldsymbol{Q}_2)$ is its face. ☐

Let $\mathcal{P}$ be a polytope, $D \subsetneq \operatorname{vert} \mathcal{P}$. We will say that the set D (or the set $\operatorname{conv} D$ – it will be clear from the context) is

- *an* A*-diagonal, if $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$, but for any proper subset $D' \subsetneq D$ the set $\operatorname{conv} D'$ is a face of the polytope $\mathcal{P}$;*

– *a* G-*diagonal, if* $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$*, but any proper subset* $D' \subsetneq D$ *lies in some proper face of the polytope* $\mathcal{P}$*;*
– *an* F-*diagonal, if* $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} = \operatorname{ri} \operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$.

We will denote the family of all A-, G- and F-diagonals of the polytope $\mathcal{P}$ by $\mathcal{D}_{\mathrm{A}}(\mathcal{P})$, $\mathcal{D}_{\mathrm{G}}(\mathcal{P})$, and $\mathcal{D}_{\mathrm{F}}(\mathcal{P})$, respectively; the family of all r-dimensional diagonals (or r-diagonals) will be denoted by $\mathcal{D}^r_{\mathrm{A}}(\mathcal{P})$, $\mathcal{D}^r_{\mathrm{G}}(\mathcal{P})$, and $\mathcal{D}^r_{\mathrm{F}}(\mathcal{P})$, respectively. The following statement follows immediately from the definition of diagonals:

Proposition 3.3. *For* A-, G-, *and* F-*diagonals of a d-polytope $\mathcal{P}$ the following relations hold:*

$$\mathcal{D}^r_{\mathrm{A}}(\mathcal{P}) \subset \mathcal{D}^r_{\mathrm{G}}(\mathcal{P}) \subset \mathcal{D}^r_{\mathrm{F}}(\mathcal{P}), \quad r \in [d-1]\,; \tag{3.1}$$

$$\mathcal{D}^0_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^0_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^0_{\mathrm{F}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{F}}(\mathcal{P}) = \emptyset\,; \tag{3.2}$$

$$\mathcal{D}^1_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{F}}(\mathcal{P})\,. \tag{3.3}$$

Another observation on the relationship between the notions of F- and G-diagonals:

Proposition 3.4. *An* F-*diagonal D of a polytope $\mathcal{P}$ is a* G-*diagonal if and only if $\mathcal{P}$ is a simplex.*

Proposition 3.5. *Let $\mathcal{P}$ be a pyramid, namely* $\mathcal{P} := \operatorname{conv}(\{v\} \cup \mathcal{P})$*, for some polyhedral basis $\mathcal{P}'$ and a point $v \notin \operatorname{aff} \mathcal{P}'$. Then*

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \emptyset\,; \tag{3.4}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \{D = \{v\} \cup D' :\ D' \in \mathcal{D}_{\mathrm{G}}(\mathcal{P}')\}\,; \tag{3.5}$$

$$\mathcal{D}_{\mathrm{F}}(\mathcal{P}) = \{D = \{v\} \cup D' :\ D' \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}')\}\,. \tag{3.6}$$

Proof. If D is a diagonal of the polytope $\mathcal{P}$, of any type, then $v \in D$, because otherwise the convex hull $\operatorname{conv} D$ would lie in a face of $\mathcal{P}$. By a similar argument, in all three cases the relation

$$(D - \{v\}) \cap \operatorname{ri} \mathcal{P}' \neq \emptyset \tag{3.7}$$

holds.

Now in the case (3.4) the set $\operatorname{conv}(D - \{v\})$, being a face of the polytope $\mathcal{P}$, must coincide with $\mathcal{P}'$, that is, $\operatorname{conv} D = \mathcal{P}$, and thus $D = \operatorname{vert} \mathcal{P}$, a contradiction.

For (3.5), the set $D' = D - \{v\}$ is inclusion-minimal with respect to property (3.7) if and only if D is minimal with respect to the property $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$; relation (3.5) is thus proved.

As to relation (3.6), it should be mentioned that the fulfillment of $\operatorname{rbd} D' \cap \operatorname{ri} \mathcal{P}' \neq \emptyset$ is equivalent to the fulfillment of $\operatorname{rbd} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$. □

Proposition 3.6. *Let* $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$. *Then*

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \bigcup_{j\in[k]} \mathcal{D}_{\mathrm{A}}(\mathcal{P}_j) \cup \{\mathcal{P}_i : \ \mathcal{P}_i \textit{ is a simplex}\} \, ; \tag{3.8}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \bigcup_{j\in[k]} \mathcal{D}_{\mathrm{G}}(\mathcal{P}_j) \cup \{\mathcal{P}_i : \ \mathcal{P}_i \textit{ is a simplex}\} \, ; \tag{3.9}$$

$$\mathcal{D}_{\mathrm{F}}(\mathcal{P}) = \{D \subset \operatorname{vert} \mathcal{P} : \ \operatorname{conv} D = \operatorname{conv} D_{j_1} \perp \operatorname{conv} D_{j_2} \perp \cdots \perp \operatorname{conv} D_{j_m}\} \, , \tag{3.10}$$

where either $D_{j_i} \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}_{j_i})$ *or* $D_{j_i} = \operatorname{vert} \mathcal{P}_{j_i}$*; besides,* $\{j_i : D_{j_i} = \operatorname{vert} \mathcal{P}_{j_i}\} \subsetneq [k]$.

Proof. In the cases (3.8) and (3.9), if D is a diagonal (of the corresponding type) of the polytope $\mathcal{P}$, and $D_j = D \cap \operatorname{vert} \mathcal{P}_j$, then, in view of Lemma 3.1, we have $\operatorname{conv} D_j \cap \operatorname{ri} \mathcal{P} \neq \emptyset$ for some j. Hence, because of the minimality property of diagonals of these types, we obtain $D = D_j$, and D_j is a diagonal of $\mathcal{P}_j$ or, in the case where $\mathcal{P}_j$ is a simplex, $D_j = \mathcal{P}_j$.

In the case (3.10), let us denote $\{\boldsymbol{z}\} := \bigcap_{j\in[k]} \operatorname{ri} \mathcal{P}_j$. For $j \in [l]$, let us denote $D_j := D \cap \operatorname{vert} \mathcal{P}_j \neq \emptyset$, and show that, under $l \geq 2$, we have $\boldsymbol{z} \in \mathbf{ri\,conv}\, \boldsymbol{D_j}$, $j \in [l]$. Indeed, if $\boldsymbol{z} \notin \operatorname{ri} \operatorname{conv} D_1$, then $\operatorname{ri} \operatorname{conv} D_1 \cap \operatorname{ri} \operatorname{conv} (D - D_1) = \emptyset$, $\operatorname{conv} D_1 \subset \operatorname{rbd} \operatorname{conv} D$, and $\operatorname{conv} (D - D_1) \subset \operatorname{rbd} \operatorname{conv} D$. Hence, by the definition of F-diagonals, $\operatorname{conv} D_1 \subset \operatorname{rbd} \mathcal{P}$, $\operatorname{conv} D_2 \subset \operatorname{rbd} (\mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k)$, and $D_1 \subset \boldsymbol{F}_1$, $D_2 \subset \boldsymbol{F}_2$, for some faces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ of the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, respectively; but, by Lemma 3.1, the set $\boldsymbol{F} := \operatorname{conv} (\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is a face of the polytope $\mathcal{P}$ and it contains D, a contradiction with the inclusion $D \in \mathcal{D}_{\mathrm{F}}(\mathcal{P})$. Thus, under $l \geq 2$, $\operatorname{conv} D = \operatorname{conv} D_1 \perp \operatorname{conv} D_2 \perp \cdots \perp \operatorname{conv} D_l$ and, in view of Lemma 3.1, $\operatorname{rbd} \operatorname{conv} D \subset \operatorname{rbd} \mathcal{P}$ implies the inclusion $\operatorname{rbd} \operatorname{conv} D_j \subset \operatorname{rbd} \mathcal{P}_j$, that is, $D_j \in \mathcal{D}_{\mathrm{F}}(\mathcal{P})$ or $D_j = \mathcal{P}_j$. The case $l = 1$, that is, $D_j = D$, is obvious. □

Proposition 3.7. *Each vertex of an arbitrary polytope, which is not a simplex, is contained in at least one its* G*-diagonal*

We now ascertain the relationships between the following properties:
C_1: a polytope $\mathcal{P}$ is cyclic;
C_2: the vertex set $\operatorname{vert} \mathcal{P}$ is in general position in $\operatorname{aff} \mathcal{P}$;
C_3: the polytope $\mathcal{P}$ is simplicial;
C_4: $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P})$;
C_5: $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{G}}(\mathcal{P})$;
C_6: $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P})$.

Proposition 3.8. *Defining, for any polytope* $\mathcal{P}$*,*

$$\mathrm{c}_{ij} := \begin{cases} 1, & \textit{if}\ \mathrm{C}_i \Rightarrow \mathrm{C}_i \, , \\ 0, & \textit{otherwise} \, , \end{cases}$$

we have $(\mathrm{c}_{ij})_{i,j\in[6]} = \begin{pmatrix} 1&1&1&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&1&0&1&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$.

Proof. The binary relation C, where $i\mathrm{C}j \Leftrightarrow \mathrm{c}_{ij} = 1$, is transitive; we will use below this property without special mention.

The equalities $\mathrm{c}_{ii} = 1$ for $i \in [6]$, and $\mathrm{c}_{23} = \mathrm{c}_{45} = \mathrm{c}_{46} = 1$, $\mathrm{c}_{32} = 0$, are easily verified.

Example 3.9. *Let $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2$, where $\mathcal{P}_1$ is a one-dimensional polytope, $\mathcal{P}_2$ is a $(d-1)$-dimensional simplicial polytope, which is not a simplex. It is easily seen that $\mathcal{P}$ is a simplicial polytope, and $\mathcal{P}_2$ is its* F*-diagonal, which is not a* G*-diagonal* (*Propositions* 3.4 *and* 3.6.) ◁

Example 3.9 yields $\mathrm{c}_{36} = 0$, hence $\mathrm{c}_{34} = 0$. Since any vertex subset of a simplex forms its face, it follows immediately from the definitions that $\mathrm{c}_{35} = 1$; hence $\mathrm{c}_{25} = 1$. Taking into account Proposition 3.6, Example 3.9 implies $\mathrm{c}_{56} = \mathrm{c}_{54} = 0$.

Proposition 3.4 yields $\mathrm{c}_{26} = 1$; this observation and the equality $\mathrm{c}_{25} = 1$ imply $\mathrm{c}_{24} = 1$.

Example 3.10. *Let $\mathcal{P}$ be a pyramid whose basis $\mathcal{P}'$ is in general position in $\mathbb{R}^{d-1} :=$ $\operatorname{aff} \mathcal{P}'$.* ◁

Because of the above-proved equality $\mathrm{c}_{26} = 1$, we have $\mathcal{D}_{\mathrm{G}}(\mathcal{P}') = \mathcal{D}_{\mathrm{F}}(\mathcal{P}')$, from where, by Propositions 3.5 and 3.7, it follows that $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P}) \neq \emptyset$; at the same time, by Proposition 3.5, $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \emptyset$; thus, $\mathrm{c}_{65} = \mathrm{c}_{64} = 0$. It is known that if $\mathcal{P}$ is a cyclic polytope then $\operatorname{vert} \mathcal{P}$ is a set in general position in $\mathbb{R}^d$, that is, $\mathrm{c}_{12} = 1$. We now have $\mathrm{c}_{1j} = 1$, $3 \le j \le 6$. In order to prove that $\mathrm{c}_{21} = 0$, let us consider Example 3.9 in the situation where $\mathcal{P}_2$ is a simplex. In this case, the polytope whose vertices are obviously in general position has the one-dimensional diagonal $\mathcal{P}_1$, and for this reason it is not 2-neighborly, and it cannot be cyclic. We now have $\mathrm{c}_{i1} = 0$, $3 \le i \le 6$.

It remains to show that $\mathrm{c}_{43} = \mathrm{c}_{42} = \mathrm{c}_{53} = \mathrm{c}_{52} = \mathrm{c}_{62} = 0$. For this, it suffices to check the property $\mathrm{c}_{43} = 0$.

Example 3.11. *Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Let us define the sets*

$$\mathcal{P}_1 := \operatorname{conv}\{\pm \boldsymbol{e}_i : \ i \in [d-1]\},$$
$$\mathcal{P}_2 := \operatorname{conv}\Big\{\boldsymbol{v} := \sum_{i \in [d-1]} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d : \ \alpha_i \in \{-1, 1\}, \ i \in [d-1]\Big\},$$
$$\mathcal{P} := \operatorname{conv}(\mathcal{P}_1 \cup \mathcal{P}_2), \quad \boldsymbol{H}_1 := \operatorname{aff} \mathcal{P}_1, \quad \boldsymbol{H}_2 := \operatorname{aff} \mathcal{P}_2 .$$

We will call a pair of vertices $\{\boldsymbol{u}, \boldsymbol{v}\}$, $\boldsymbol{u} \in \operatorname{vert} \mathcal{P}_1$, $\boldsymbol{v} \in \operatorname{vert} \mathcal{P}_2$, diagonal *if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -1$, that is, for $\boldsymbol{u} = \alpha_k \boldsymbol{e}_k$, $\alpha_k \in \{-1, 1\}$, we must have*

$$\boldsymbol{v} = -\alpha_k \boldsymbol{e}_k + \sum_{\substack{i \in [d-1], \\ i \neq k}} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d .$$

Let us show that the following two assertions are true:

(1) *any diagonal pair of vertices $\{\boldsymbol{u}, \boldsymbol{v}\}$ forms a diagonal of the polytope $\mathcal{P}$ – it is obvious that $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathcal{D}_{\mathrm{A}}(\mathcal{P}) \subset \mathcal{D}_{\mathrm{G}}(\mathcal{P}) \subset \mathcal{D}_{\mathrm{F}}(\mathcal{P})$;*

(2) *any inclusion-maximal vertex subset of the polytope $\mathcal{P}$, containing no diagonal pairs, forms some face of $\mathcal{P}$.*

In order to prove (1) *assume the converse. Let $\boldsymbol{u} := \boldsymbol{e}_1$, $\boldsymbol{v} := -\boldsymbol{e}_1 + \sum_{i=2}^{d-1} \tilde{\alpha}_i \boldsymbol{e}_i + \boldsymbol{e}_d$, where $\tilde{\alpha}_i \in \{-1, 1\}$, $2 \leq i \leq d-1$. Further, let $\boldsymbol{\beta} := (\beta_1, \beta_2, \ldots, \beta_k)$, and let the equation $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$ determine a supporting hyperplane of $\mathcal{P}$ that contains the points $\boldsymbol{u}$ and $\boldsymbol{v}$. Since $\mathcal{P}$ contains the origin, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle \leq 1$ for all $\boldsymbol{x} \in \mathcal{P}$. We thus obtain*

$$\langle \boldsymbol{\beta}, \boldsymbol{v} \rangle = -\beta_1 + \sum_{2 \leq i \leq d-1} \tilde{\alpha}_i \beta_i + \beta_d = 1 \,, \tag{3.11}$$

$$\langle \boldsymbol{\beta}, \boldsymbol{u} \rangle = \beta_1 = 1 \,, \tag{3.12}$$

and for the vertex $\boldsymbol{w} := \boldsymbol{e}_1 + \sum_{2 \leq i \leq d-1} \tilde{\alpha}_i \boldsymbol{e}_i + \boldsymbol{e}_d$, we have

$$\langle \boldsymbol{\beta}, \boldsymbol{w} \rangle = \beta_1 + \sum_{2 \leq i \leq d-1} \tilde{\alpha}_i \beta_i + \beta_d \leq 1 \,. \tag{3.13}$$

Taking into account (3.12), *relations* (3.11) *and* (3.13) *lead to a contradiction.*

We now prove assertion (2). *Let $\mathcal{S}$ be an inclusion-maximal subset of vertices of the polytope $\mathcal{P}$ that contains no diagonal pairs. It is clear that it suffices to prove the case where the sets $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_1$ and $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_2$ are both nonempty.*

It follows from the definition of diagonal pairs that $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_1$ contains no pair $\{\boldsymbol{e}_j, -\boldsymbol{e}_j\}$ for any j. Thus, without loss of generality we suppose that

$$\mathcal{S} := \Big\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_k, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_k + \sum_{k+1 \leq i \leq d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d : \\ \alpha_i \in \{-1, 1\},\ k+1 \leq i \leq d-1 \Big\} \,.$$

Let us define $\boldsymbol{\beta} := (\beta_1, \beta_2, \ldots, \beta_d)$ as follows:

$$\beta_i := \begin{cases} 1, & \text{if } i \in [k] \,, \\ 0, & \text{if } k+1 \leq i \leq d-1 \,, \\ 1-k, & \text{if } i = d \,. \end{cases}$$

Note that for all vectors $\boldsymbol{x} \in \mathcal{S}$ the equality $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$ holds.

Now let $\boldsymbol{x} \in \operatorname{vert} \mathcal{P} - \mathcal{S}$. If $\boldsymbol{x} \in \operatorname{vert} \mathcal{P}_1$, then

$$\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = \begin{cases} -1, & \text{if } \boldsymbol{x} = -\boldsymbol{e}_j,\ j \leq k \,, \\ 0, & \text{if } \boldsymbol{x} = \pm \boldsymbol{e}_j,\ k+1 \leq j \leq d-1 \,. \end{cases}$$

But if $\boldsymbol{x} \in \operatorname{vert} \mathcal{P}_2 - \mathcal{S}$, that is, $\boldsymbol{x} = \sum_{0 \leq i \leq d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d$, where $\alpha_i \in \{-1, 1\}$, $i \in [d-1]$, and among $\alpha_1, \alpha_2, \ldots, \alpha_k$ there are numbers different from 1, *then $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = \sum_{i \in [k]} \alpha_i + (1-k) < 1$. Thus, for any vector $\boldsymbol{x} \in \operatorname{vert} \mathcal{P} - \mathcal{S}$, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle < 1$, and for any vector $\boldsymbol{x} \in \mathcal{S}$, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$, that is, $\operatorname{conv} \mathcal{S}$ is a face of the polytope $\mathcal{P}$.* ◁

Example 3.11 provides a nonsimplicial polytope $\mathcal{P}$ such that $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{A}}(\mathcal{P})$, and it thus proves the equality $c_{53} = 0$. For the polytope $\mathcal{P}$, we have $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) \neq \mathcal{D}_{\mathrm{F}}(\mathcal{P})$. For example,

$$\mathcal{S} := \Big\{\boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_1 + \sum_{2\le i\le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d, -\boldsymbol{e}_1 + \sum_{2\le i\le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d\Big\} \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}) - \mathcal{D}_{\mathrm{A}}(\mathcal{P})\,;$$

this follows from the above-proved assertions (1) and (2).

The following example proves the stronger property $c_{43} = 0$.

Example 3.12. *The desired polytope $\mathcal{Q}$ is obtained from $\mathcal{P}$ (see Example 3.11) by the replacement of the vertices $\boldsymbol{u}_i \in \operatorname{vert}\mathcal{P}_1$ by sufficiently close vertices $\boldsymbol{w}_i$, $i \in [2(d-1)]$, lying in the hyperplane $\boldsymbol{H}_1$. Let U_i be an ϵ-neighborhood of the point $\boldsymbol{u}_i \in \operatorname{vert}\mathcal{P}$ in the plane $\boldsymbol{H}_1 := \operatorname{aff}\mathcal{P}_1$, and $\boldsymbol{w}_i \in U_i$ an arbitrary point, $i \in [2(d-1)]$. Let us set $\mathcal{Q} := \operatorname{conv}(\mathcal{P}_2 \cup \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2(d-1)}\})$.*

We will call $\{\boldsymbol{w}_j, \boldsymbol{v}_i\} \subset \operatorname{vert}\mathcal{Q}$ a diagonal pair *for $\mathcal{P}$ if $\{\boldsymbol{u}_j, \boldsymbol{v}_i\}$ is a diagonal pair for the polytope $\mathcal{P}$.*

Note that the properties (1) and (2) of diagonal pairs $\mathcal{P}$ also remain true for diagonal pairs $\mathcal{Q}$ under the condition that $\epsilon > 0$ is sufficiently small. It deserves explanation (concerning (2)) that an inclusion-maximal vertex set containing no diagonal pairs and coinciding with neither $\operatorname{vert}\mathcal{P}_2$ nor $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2(d-1)}\}$ has the form $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k, \boldsymbol{w}_1 + \boldsymbol{w}_2 + \dots + \boldsymbol{w}_k + \sum_{k+1\le i\le d-1} \alpha_i \boldsymbol{u}_i + \boldsymbol{e}_d : \alpha_i \in \{-1, 1\}, k+1 \le i \le d-1\}$, that is, it is composed of the points $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k$ and other points lying in a $(d-k-1)$-dimensional face of the cube $\mathcal{P}_2$, and it thus necessarily lies in some hyperplane of $\mathbb{R}^d$. The latter, because of the small value of ϵ, will be close to the corresponding hyperplane that supports $\mathcal{P}$ and, as a consequence, it also supports $\mathcal{Q}$.

Now we describe an inductive approach to the choice of $\boldsymbol{w}_i \in U_i$, $i \in [2(d-1)]$, which leads to a desired polytope $\mathcal{Q}$.

Suppose $\boldsymbol{w}_1 := \boldsymbol{u}_1$. Let the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r$ be already chosen. Let us consider all the planes spanned by the subsets of the set $\operatorname{vert}\mathcal{P}_2 \cup \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r\}$ and not containing $\boldsymbol{H}_1$. By the Baire category theorem, their union does not cover $\boldsymbol{H}_1$. Let us pick for $\boldsymbol{w}_{r+1} \in U_{r+1}$ an arbitrary uncovered point.

Let us now show that the polytope $\mathcal{Q}$ does satisfy the prescribed condition $\mathcal{D}_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}_{\mathrm{A}}(\mathcal{Q})$.

Let $\mathcal{S} := \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_l, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$ be an arbitrary F-diagonal of the polytope $\mathcal{Q}$. In view of the above argument, it must contain a diagonal pair, say the pair $\{\boldsymbol{w}_s, \boldsymbol{w}_t\}$.

We have chosen the vectors $\boldsymbol{w}_i$, $i \in [2(d-1)]$, in such a way that $\boldsymbol{E}_1 := \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_l\}$ and $\boldsymbol{E}_2 := \operatorname{aff}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots \boldsymbol{v}_m\}$ are skew planes. Indeed, let $\boldsymbol{E}_1 := \boldsymbol{w}_1 + \boldsymbol{L}_1$, $\boldsymbol{E}_2 := \boldsymbol{w}_2 + \boldsymbol{L}_2$, where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are linear subspaces of the space $\mathbb{R}^d$. Suppose to the contrary that $\boldsymbol{L}$ is a one-dimensional linear subspace contained in $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$. Further, let j be a number such that $\operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots \boldsymbol{w}_j\} \not\supset \boldsymbol{w}_1 + \boldsymbol{L} \subset \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j, \boldsymbol{w}_{j+1}\}$. This means that $\boldsymbol{w}_{j+1} \in \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j\} + \boldsymbol{L} \subset \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$, a contradiction with the choice of $\boldsymbol{w}_{j+1}$.

Now, if $\mathcal{S} \neq \{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *then, by Lemma* 3.2, $\operatorname{conv}\{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *is a face of* $\operatorname{conv}\mathcal{S}$ *and, by the definition of* F*-diagonals,* $\operatorname{conv}\{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *lies in some face of the polytope* $\mathfrak{Q}$*. The obtained contradiction yields* $\mathcal{S} = \{\boldsymbol{w}_s, \boldsymbol{v}_t\}$*, that is,* $\mathcal{D}_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}^1_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}^1_{\mathrm{A}}(\mathcal{Q}) = \mathcal{D}_{\mathrm{A}}(\mathcal{Q})$. ◁

Thus, $c_{43} = 0$ is established. □

Diagonals and combinatorial classification of polytopes

Two polytopes $\mathcal{P}$ and $\mathfrak{Q}$, whose face lattices are isomorphic, by definition have the same combinatorial type. The relation "to have the same combinatorial type" is an equivalence relation, and it determines a classification on the set of all polytopes. A similar classification is also possible on the basis of the notion of diagonal.

Proposition 3.13. *Let* $\mathcal{L}$ *be the face lattice of a bounded convex polytope* $\mathcal{P}$ *(which is not a simplex) with the atom set* $\mathcal{L}^{\mathrm{a}} := \{\{\boldsymbol{v}\} : \boldsymbol{v} \in \operatorname{vert}\mathcal{P}\}$ *and the coatom set* $\mathcal{L}^{\mathrm{c}} := \{H \subset \operatorname{vert}\mathcal{P} : \operatorname{conv} H$ *is a facet of* $\mathcal{P}\}$*. The following relations hold:*

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathfrak{B}\left(\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}} : H \in \mathcal{L}^{\mathrm{c}}\}^{\perp}\right), \tag{3.14}$$

$$\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}} : H \in \mathcal{L}^{\mathrm{c}}\}^{\perp} = \mathfrak{B}(\mathcal{D}_{\mathrm{G}}(\mathcal{P})); \tag{3.15}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathbf{min}\, \mathcal{D}_{\mathrm{F}}(\mathcal{P}), \tag{3.16}$$

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \left\{D \in \mathcal{D}_{\mathrm{G}}(\mathcal{P}) : D - \{\boldsymbol{u}\} \in \mathcal{L},\ \forall \boldsymbol{u} \in D\right\}, \tag{3.17}$$

where, as earlier, $\mathfrak{B}(\cdot)$ *denotes the blocker of a set family;* $\mathfrak{I}(H) := \{Y \in \mathcal{L} : Y \preceq H\}$ *is the order ideal of the lattice* $\mathcal{L}$ *generated by its element* H*;* $\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}} : H \in \mathcal{L}^{\mathrm{c}}\}^{\perp} := \{\mathcal{L}^{\mathrm{a}} - (\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}}) : H \in \mathcal{L}^{\mathrm{c}}\}$.

Proof. Assertions (3.14), (3.16), and (3.17) follow immediately from the definitions. Relation (3.15) follows easily from (3.14), taking into account Proposition 1.1. □

One says that a nonempty family $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ of nonempty subsets of a finite set $\bigcup_{i\in[\alpha]} A_i$ is *combinatorially isomorphic* to a family $\mathcal{B} := \{B_1, B_2, \ldots, B_\alpha\}$ of subsets of a set $\bigcup_{i\in[\alpha]} B_i$, if there exists a one-to-one map $\varphi : \bigcup_{i\in[\alpha]} A_i \to \bigcup_{i\in[\alpha] B_i}$ such that for each $i \in [\alpha]$ it holds $\varphi(A_i) = B_i$.

We will say that *polytopes* $\mathcal{P}$ *and* $\mathfrak{Q}$ *have the same* A-, G- *or* F*-diagonal types, if the families* $\mathcal{D}_{\mathrm{A}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{A}}(\mathfrak{Q})$, $\mathcal{D}_{\mathrm{G}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{G}}(\mathfrak{Q})$, $\mathcal{D}_{\mathrm{F}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{F}}(\mathfrak{Q})$*, respectively, are combinatorially isomorphic.*

It is obvious that the relation "to have the same diagonal combinatorial type" is also an equivalence relation, and it determines a combinatorial classification on the set of all polytopes.

Given a polytope $\mathcal{P}$, let us denote by $\mathcal{F}(\mathcal{P})$ the class of all the polytopes whose combinatorial type coincides with that of $\mathcal{P}$, and denote by $\mathcal{F}_{\mathrm{A}}(\mathcal{P})$ the class of all the polytopes whose A-diagonal type coincides with that of $\mathcal{P}$; the denotations $\mathcal{F}_{\mathrm{G}}(\mathcal{P})$ and $\mathcal{F}_{\mathrm{F}}(\mathcal{P})$ have the analogous meaning.

Proposition 3.14. *For an arbitrary polytope $\mathcal{P}$ the relations*

$$\mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}(\mathcal{P}) = \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})$$

hold.

Proof. Note that if a subset family of a certain type always determines uniquely a subset family of another type by means of operations that are invariant with respect to one-to-one maps, then the combinatorial equivalence on the basis of the combinatorial isomorphism of subset families of the first type implies the combinatorial equivalence on the basis of the combinatorial isomorphism of subset families of the second type; in other words, the combinatorial types defined by subsets of the second type are "wider." In view of the above argument, we obtain from Proposition 3.13:

$$\begin{aligned}
(3.14) \quad &\Longrightarrow \quad \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})\,,\\
(3.15) \quad &\Longrightarrow \quad \mathcal{F}(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P})\,,\\
(3.16) \quad &\Longrightarrow \quad \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}(\mathcal{P})\,,\\
(3.17) \quad &\Longrightarrow \quad \mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P})\,.
\end{aligned}$$

□

It is known that the facial structure of a simplicial polytope is determined (in our terminology) by the structure of the family of its A-diagonals. We can now interpret this statement as a corollary of the fact that $\mathcal{F}_G(\mathcal{P}) = \mathcal{F}(\mathcal{P})$ and, moreover, for the simplicial polytope $\mathcal{P}$ we have $\mathcal{D}_A(\mathcal{P}) = \mathcal{D}_G(\mathcal{P})$. A stronger assertion is also true:

Proposition 3.15. *If $\mathcal{P}$ is a simplicial polytope then $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}(\mathcal{P}) = \mathcal{F}_G(\mathcal{P})$.*

Proof. In view of the above argument, it suffices to show that if $\mathcal{Q} \in \mathcal{F}_A(\mathcal{P})$ then $\mathcal{Q}$ is simplicial. For this, it suffices to verify that every face of the polytope $\mathcal{Q}$ has at most d vertices. If $V \subset \operatorname{vert} \mathcal{Q}$ determines a face of $\mathcal{Q}$ then V contains no A-diagonals of the polytope $\mathcal{Q}$, therefore $W \subset \operatorname{vert} \mathcal{Q}$ – the image of the set V, under the bijection that realizes a combinatorial isomorphism of the families of A-diagonals, does not contain A-diagonals of the polytope $\mathcal{P}$, and it thus lies in some face of $\mathcal{P}$. Hence, $|W| = |V| \le d$, because of the simpliciality of $\mathcal{P}$.

There are instances of polytopes for which $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}_G(\mathcal{P}) = \mathcal{F}(\mathcal{P}) = \mathcal{F}_F(\mathcal{P})$.

Example 3.16. *Let $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are both simplices. Let us show that $\mathcal{Q} \in \mathcal{F}_A(\mathcal{P})$ implies $\mathcal{Q} \in \mathcal{F}_F(\mathcal{P})$. By Proposition* 3.6 (*relation* (3.8)), $\mathcal{D}_A(\mathcal{P}) = \{\operatorname{vert} \mathcal{P}_1, \operatorname{vert} \mathcal{P}_2\}$. *Let* $\operatorname{vert} \mathcal{Q} = V_1 \cup V_2$, *where V_1, V_2 are* A-*diagonals of $\mathcal{Q}$ corresponding to* A-*diagonals* $\operatorname{vert} \mathcal{P}_1$ *and* $\operatorname{vert} \mathcal{P}_2$ *of the polytope $\mathcal{P}$. It easily follows from the definitions that in this situation we have* $\mathcal{Q} = \operatorname{conv} V_1 \perp \operatorname{conv} V_2$ (*where the convex hulls are both simplices*) *and now, by Proposition* 3.6 (*relation* (3.10)), $\mathcal{D}_F(\mathcal{Q}) = \{V_1, V_2\}$, *that is,* $\mathcal{Q} \in \mathcal{F}_F(\mathcal{P})$. ◁

Thus, the simpliciality of the polytope $\mathcal{P}$ implies that $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}_G(\mathcal{P})$ (Proposition 3.15). Besides (Example 3.16), we have instances of (simplicial) polytopes for which $\mathcal{F}_G(\mathcal{P}) = \mathcal{F}_F(\mathcal{P})$. The latter, though, seems to be rather an exception than a rule, even in the case of simplicial polytopes. At least, Example 3.9 provides simplicial polytopes $\mathcal{P}$ with $\mathcal{F}_G(\mathcal{P}) \neq \mathcal{F}_F(\mathcal{P})$. Indeed, it suffices to move, by a small perturbation, the vertices of the polytope $\mathcal{P}$ into general position for obtaining a polytope $\mathcal{Q}$ with the isomorphic face lattice (i.e., $\mathcal{Q} \in \mathcal{F}_G(\mathcal{P})$), but without F-diagonals that are not G-diagonals – at the same time, $\mathcal{P}$ has such an F-diagonal, namely vert $\mathcal{P}_2$ (i.e., $\mathcal{Q} \notin \mathcal{F}_F(\mathcal{P})$).

A particular case of Example 3.9, a 3-polytope $\mathcal{P}_0$ ($\mathcal{P}_2$ is a planar polygon), allows us to construct a series of examples of polytopes such that in the chain $\mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})$ both inclusions are strict.

Example 3.17. *Let us consider in the space $\mathbb{R}^d$, where $d \geq 4$, two polytopes with the equal number $2d$ of vertices, namely $\mathcal{Q}_1$, a prism whose basis is a $(d-1)$-simplex, and $\mathcal{P}$, a $(d-3)$-fold pyramid over a 3-polytope $\mathcal{P}_0$. It is easily verified that for both polytopes the families of A-diagonals are empty and, thus, $\mathcal{Q}_1 \in \mathcal{F}_A(\mathcal{P})$, however, it is obvious that $\mathcal{Q}_1$ and $\mathcal{P}$ are not combinatorially equivalent and, thus, $\mathcal{Q}_1 \notin \mathcal{F}_G(\mathcal{P})$.* ◁

Let us now take into consideration a polytope $\mathcal{Q}_2$ obtained from $\mathcal{P}$ by the above-mentioned small perturbation of the vertices of $\mathcal{P}_0$ that moves them into general position in $\mathbb{R}^3$ without a change in the facial structure of $\mathcal{P}_0$. By Propositions 3.4 and 3.5, the polytope $\mathcal{Q}_2$ has no F-diagonals different from G-diagonals; at the same time, $\mathcal{P}$ has such a diagonal, a $(d-3)$-fold pyramid over the planar polygon $\mathcal{P}_2$. Thus, $\mathcal{Q}_2 \in \mathcal{F}_G(\mathcal{P})$, but $\mathcal{Q}_2 \notin \mathcal{F}_F(\mathcal{P})$ and, thus, $\mathcal{F}_A(\mathcal{P}) \supsetneq \mathcal{F}_G(\mathcal{P}) \supsetneq \mathcal{F}_F(\mathcal{P})$. □

3.2 Positive bases of linear spaces

A *positive basis* (*PB*) $\boldsymbol{B}$ of a linear space $\boldsymbol{L}$ is defined as an inclusion-minimal subset of $\boldsymbol{L}$ whose positive hull (i.e., the inclusion-minimal convex cone, with the apex at the origin $\mathbf{0} \in \boldsymbol{L}$, which contains $\boldsymbol{B}$) coincides with $\boldsymbol{L}$.

We study positive bases in $\mathbb{R}^n$, in particular, from the point of view of the combinatorial structure of two special subset families, the so-called minimal sub-bases and maximal one-sided subsets.

It is well known that for positive bases $\boldsymbol{B}$ of the space $\mathbb{R}^n$ the inequalities $n+1 \leq |\boldsymbol{B}| \leq 2n$ hold.

A positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ is called *minimal* when $|\boldsymbol{B}| = n+1$, and *maximal* when $|\boldsymbol{B}| = 2n$.

A subset $\boldsymbol{B}'$ of a positive basis $\boldsymbol{B}$ is called a *sub-basis* of the basis $\boldsymbol{B}$, if $\boldsymbol{B}'$ is a positive basis of the linear hull $\operatorname{lin} \boldsymbol{B}'$ of the set $\boldsymbol{B}'$. A sub-basis $\boldsymbol{B}' \subset \boldsymbol{B}$ is called a *minimal sub-basis*, if $\boldsymbol{B}'$ is a minimal positive basis of $\operatorname{lin} \boldsymbol{B}'$, that is, $\operatorname{pos} \boldsymbol{B}' = \operatorname{lin} \boldsymbol{B}'$ and $|\boldsymbol{B}'| = \dim \operatorname{lin} \boldsymbol{B}' + 1$.

A positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ is called a *strict positive basis* (*SPB*), if for any of its disjoint subsets $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ it holds $\operatorname{pos}\boldsymbol{B}_1 \cap \operatorname{pos}\boldsymbol{B}_2 = \{\mathbf{0}\}$. A positive basis $\boldsymbol{B}$ consisting of $n + r$ points in $\mathbb{R}^n$ is a SPB if and only if there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot\cup \boldsymbol{B}_2 \dot\cup \cdots \dot\cup \boldsymbol{B}_r$, where $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots, \boldsymbol{B}_r$ are pairwise disjoint minimal sub-bases of the positive basis of $\boldsymbol{B}$. In addition, the space is represented as the direct sum $\mathbb{R}^n = \operatorname{lin}\boldsymbol{B}_1 + \operatorname{lin}\boldsymbol{B}_2 + \cdots + \operatorname{lin}\boldsymbol{B}_r$. In particular, minimal and maximal positive bases are strict positive bases.

In the general case, the following assertion is true:

Proposition 3.18. *Let $\boldsymbol{B}$ be a positive basis consisting of $n + r$ points in $\mathbb{R}^n$. Then there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot\cup \boldsymbol{B}_2 \dot\cup \cdots \dot\cup \boldsymbol{B}_r$ satisfying the following conditions:*
(1) $|\boldsymbol{B}_i| \geq |\boldsymbol{B}_{i+1}| \geq 2$, $i \in [r]$.
(2) $\operatorname{pos}(\boldsymbol{B}_1 \dot\cup \boldsymbol{B}_2 \dot\cup \cdots \dot\cup \boldsymbol{B}_j)$ *is a linear subspace of dimension* $|\boldsymbol{B}_1 \dot\cup \boldsymbol{B}_2 \dot\cup \cdots \dot\cup \boldsymbol{B}_j| - j$, $j \in [r]$.

We will call the set $\boldsymbol{X} \subset \mathbb{R}^n$ *one-sided* if it is contained entirely in an open half-space bounded by a hyperplane that contains $\mathbf{0}$. The inclusion-maximal one-sided subsets of some set will be called its *maximal one-sided subsets*.

In the study of positive bases, the notion of diagram of a positive basis turned out to be very useful.

Let us consider tuples of vectors $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r}) \subset \mathbb{R}^n$ and $\boldsymbol{E} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+r}) \subset \mathbb{R}^r$. The tuple $\boldsymbol{E}$ is called a *linear representation* of the tuple $\boldsymbol{B}$ if the $(n+r) \times (n+r)$ matrix $\mathbf{C}$ whose ith row is the vector $(b_{i1}, b_{i2}, \dots, b_{in}, e_{i1}, e_{i2}, \dots, e_{ir})$, where $(b_{i1}, b_{i2}, \dots, b_{in}) =: \boldsymbol{b}_i$ and $(e_{i1}, e_{i2}, \dots, e_{ir}) =: \boldsymbol{e}_i$, is nonsingular, and each of its first n columns is orthogonal to any of its last r columns. If the tuple $\boldsymbol{B}$ is spanned positively by the space $\mathbb{R}^n$ then the set of points from the tuple $\boldsymbol{E}$ is one-sided. Let $\mathbf{H}$ be a hyperplane that strictly separates the one-element set $\{\mathbf{0}\}$ from the convex hull $\operatorname{conv}\boldsymbol{E}$. Let us denote by $\overline{\boldsymbol{b}}_i$ the intersection point of the hyperplane $\mathbf{H}$ and the ray $\operatorname{pos}\{\boldsymbol{e}_i\}$. The tuple $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \dots, \overline{\boldsymbol{b}}_{n+r})$ is called a *diagram of the positive basis* $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r})$. We will use the following properties of the diagrams of positive bases:

Proposition 3.19. (i) *Each point of a diagram $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \dots, \overline{\boldsymbol{b}}_{n+r})$, which is a vertex of the polytope* $\operatorname{conv}\overline{\boldsymbol{B}}$, *occurs in the tuple $\overline{\boldsymbol{B}}$ at least twice, and* $\dim\overline{\boldsymbol{B}} = r - 1$.
(ii) *Any tuple satisfying conditions listed in* (i) *is a diagram of some positive basis $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r})$ of $\mathbb{R}^n$.*
(iii) *A set $\boldsymbol{B}$ is a strict positive basis if and only if* $\operatorname{conv}\overline{\boldsymbol{B}}$ *is a simplex, and every point $\overline{\boldsymbol{b}}_i$ from the tuple $\overline{\boldsymbol{B}}$ coincides with one of its vertices.*
(iv) *A subset $\boldsymbol{B}' := \{\boldsymbol{b}_i : i \in I \subseteq [n+r]\}$ of a positive basis $\boldsymbol{B}$ is a minimal sub-basis if and only if in a diagram $\overline{\boldsymbol{B}}$ the subtuple $\overline{\boldsymbol{B}} - \overline{\boldsymbol{B}'}$ coincides with $\boldsymbol{F} \cap \overline{\boldsymbol{B}}$ for some facet $\boldsymbol{F}$ of the polytope* $\operatorname{conv}\overline{\boldsymbol{B}}$.
(v) *A subset $\boldsymbol{B}'$ of a positive basis $\boldsymbol{B}$ is maximal one-sided if and only if the subtuple $\overline{\boldsymbol{B}} - \overline{\boldsymbol{B}'}$ is a* G-*diagonal of the tuple $\overline{\boldsymbol{B}}$.*

We will need below the following auxiliary statement:

Proposition 3.20. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be d-dimensional polytopes, $\mathcal{P}_1 \subset \mathcal{P}_2$, $\operatorname{vert}\mathcal{P}_1 \neq \operatorname{vert}\mathcal{P}_2$. Then a certain proper face of the polytope $\mathcal{P}_1$ has a nonempty intersection with the interior of the polytope $\mathcal{P}_2$.*

The maximal one-sided subsets of a positive basis

Let $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r})$ be a positive basis of the space $\mathbb{R}^n$, and $\{\boldsymbol{B}_i : i \in [k]\}$ the family of maximal one-sided subsets of the set $\boldsymbol{B}$. We will use the notation $\alpha(\boldsymbol{B}) := \max_{i\in[k]} |\boldsymbol{B}_i|$ and $\beta(\boldsymbol{B}) := \min_{i\in[k]} |\boldsymbol{B}_i|$.

Let us characterize the strict positive bases in terms of their one-sided subsets:

Proposition 3.21. *A positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ is a strict positive basis if and only if any one-sided subset $\boldsymbol{B}' \subset \boldsymbol{B}$ contains at most n elements, that is, $\alpha(\boldsymbol{B}) = \beta(\boldsymbol{B}) = n$.*

Proof. The *necessity*. If $\boldsymbol{B}$ is an SPB, then all points of the tuple $\boldsymbol{B}$ are situated in the vertices of the $(r-1)$-dimensional simplex $\operatorname{conv}\overline{\boldsymbol{B}}$; as a consequence, any G-diagonal of the tuple $\boldsymbol{B}$ contains r elements, and therefore (see Proposition 3.19 (v)) any one-sided subset of $\boldsymbol{B}$ contains at most n elements.

The *sufficiency*. Let any one-sided subset of a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ contain at most n elements. Suppose to the contrary that $\boldsymbol{B}$ is not a SPB. Then two cases are possible:

(a) $\operatorname{conv}\overline{\boldsymbol{B}}$ is not a simplex;
(b) $\operatorname{conv}\overline{\boldsymbol{B}}$ is a simplex, and there exists a point $\overline{\boldsymbol{b}} \in \overline{\boldsymbol{B}}$, but $\overline{\boldsymbol{b}} \notin \operatorname{vert}\operatorname{conv}\overline{\boldsymbol{B}}$.

In the case (a), in the tuple $\overline{\boldsymbol{B}}$ there exists a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset [n+r]$, consisting of $k \leq r-1$ points, that is, $|N| = k \leq r-1$. Indeed, let $\overline{\boldsymbol{B}}(M)$ be a tuple, consisting of r points, of affine dimension $r-1 = \dim\overline{\boldsymbol{B}}$, that is, $\operatorname{conv}\overline{\boldsymbol{B}}(M)$ is a simplex. Since $\operatorname{conv}\overline{\boldsymbol{B}}$ is not a simplex, then (see Proposition 3.20) at least one face of the set $\operatorname{conv}\overline{\boldsymbol{B}}(M)$ has a nonempty intersection with the interior of the set $\operatorname{conv}\overline{\boldsymbol{B}}$. As a consequence, this face contains a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset M$, where $|N| < |M| = r$.

In the case (b), let us consider a tuple $\overline{\boldsymbol{B}}(M)$, where $M \subset [n+r]$, consisting of r points, of affine dimension $r-1$, which contains the point $\overline{\boldsymbol{b}}$. Since $\overline{\boldsymbol{b}} \notin \operatorname{vert}\operatorname{conv}\overline{\boldsymbol{B}}$, then (see Proposition 3.20) at least one face of the simplex $\operatorname{conv}\overline{\boldsymbol{B}}(M)$ has a nonempty intersection with the interior of the simplex $\operatorname{conv}\overline{\boldsymbol{B}}$. As a consequence, this face contains a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset M$, where $|N| < |M| = r$.

Thus, in both cases, $\overline{\boldsymbol{B}}$ contains a G-diagonal with less than r points from the tuple and, as a consequence (see Proposition 3.19 (v)), in the positive basis $\boldsymbol{B}$ there exists a maximal one-sided subset of cardinality at least $n+1$, a contradiction. □

Proposition 3.22. *Let $\boldsymbol{B}$ be a positive basis of the space $\mathbb{R}^n$ that consists of $n+r$ points. Then*

$$n \le \alpha(\boldsymbol{B}) \le \begin{cases} n+r-1 & \text{if } r \in [n-1]\,, \\ 0 & \text{if } r = n\,. \end{cases}$$

Moreover, if $r \in [n-1]$ then for each s, such that $n \le s \le n+r-1$, there exists a positive basis of the space $\mathbb{R}^n$ that consists of $n+r$ points, such that $\alpha(\boldsymbol{B}) = s$.

Proof. The inequality $n \le \alpha(\boldsymbol{B}) \le n+r-1$ is obvious. If $r = n$ then $\boldsymbol{B}$ is a maximal positive basis; as a consequence, $\boldsymbol{B}$ is a SPB and, by Proposition 3.21, we have $\alpha(\boldsymbol{B}) = n$.

Now suppose $1 \le r \le n-1$ and $n \le s \le n+r-1$. Let us construct a positive basis, with $n+r$ points, of $\mathbb{R}^n$, such that $\alpha(\boldsymbol{B}) = s$. For this, it suffices to choose for $\overline{\boldsymbol{B}}$ a tuple of points $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \dots, \overline{\boldsymbol{b}}_{n+r})$ from $\mathbb{R}^{r-1}$ such that $\operatorname{conv}\overline{\boldsymbol{B}}$ is an $(r-1)$-dimensional simplex whose every vertex occurs in the tuple $\overline{\boldsymbol{B}}$ at least twice, and the remaining $n+r-2r = n-r \ge 1$ points lie in the relative interior of an $(s-n)$-dimensional face of $\operatorname{conv}\overline{\boldsymbol{B}}$. Then all G-diagonals of the tuple $\overline{\boldsymbol{B}}$ have dimension at least $(r-1)-(s-n)$ and, in addition, there exists a G-diagonal of dimension $(r-1)-(s-n)$, that is, it consists of $r-s+n$ points. Then $\alpha(\boldsymbol{B}) = n+r-(r-s+n) = s$. ☐

Proposition 3.23. *Let s and d be nonnegative integers such that $s \le d$. There exists a d-dimensional polytope with $2d-s+1$ vertices, all G-diagonals of which are s-dimensional.*

Proof. If $s := 1$ then a polytope with the desired property is the convex hull of the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_d, -\boldsymbol{x}_1, -\boldsymbol{x}_2, \dots, -\boldsymbol{x}_d\}$, where $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_d\}$ is a linear basis of $\mathbb{R}^d$. We proceed by induction on s. Suppose that in the space $\mathbb{R}^{d-1}$, $d \ge s$, by the induction hypothesis, there exists a polytope $\mathcal{P}$ with $2(d-1)+1-(s-1)$ vertices, all G-diagonals of which are $(s-1)$-dimensional. Let us embed the polytope $\mathcal{P}$ into a hyperplane $\mathbf{H} \in \mathbb{R}^d$ and pick an arbitrary point $\boldsymbol{x} \notin \mathbf{H}$. Then the polytope $\operatorname{conv}(\mathcal{P} \dot\cup \{\boldsymbol{x}\})$, with $2d+1-s$ vertices, has dimension d, and all its G-diagonals are s-dimensional. ☐

Proposition 3.24. *Let $\boldsymbol{B}$ be a positive basis, with $n+r$ points, of $\mathbb{R}^n$. Then*

$$n \le \beta(\boldsymbol{B}) \le \begin{cases} n & \text{if } r \in \{1, 2, n-1, n\}\,, \\ n+r-2 & \text{if } 2 \le r < n-1\,. \end{cases}$$

If $n+r \ge 4(r-1)$ then for any s, such that $n \le s \le n+r-2$, there exists a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$, with $n+r$ points, such that $\beta(\boldsymbol{B}) = s$.

Proof. If $r := 1$ or $r := n$, then the set $\boldsymbol{B}$ is a SPB and, by Proposition 3.21, we have $\beta(\boldsymbol{B}) = n$. Suppose $r := n-1$. Then for a diagram $\overline{\boldsymbol{B}}$ we have $\dim\overline{\boldsymbol{B}} = n-2$, $|\operatorname{vert}\operatorname{conv}\overline{\boldsymbol{B}}| \le \frac{n+r}{2} = n - \frac{1}{2}$, that is, $\operatorname{conv}\overline{\boldsymbol{B}}$ is a simplex. As a consequence, in $\overline{\boldsymbol{B}}$ there exists a G-diagonal consisting of $n-1$ points, and thus $\beta(\boldsymbol{B}) = n$ (see Proposition 3.19 (v)). If $r \ge 2$, then there does not exist a tuple $\overline{\boldsymbol{B}}$, with $n+r$ points, of dimension $\dim\overline{\boldsymbol{B}} = r-1 \ge 1$, such that all its G-diagonals are zero-dimensional. As a consequence, in this case $\beta(\boldsymbol{B}) \le n+r-2$. Since $\operatorname{lin}\boldsymbol{B} = \mathbb{R}^n$, then $\beta(\boldsymbol{B}) \ge n$. Now suppose

$n + r \geq 4(r - 1)$, $n \leq s \leq n + r - 2$ and $r \geq 2$. In the space $\mathbb{R}^d$, where $d := r - 1$, one can construct (see Proposition 3.23) a d-dimensional polytope $\mathcal{P}$, with $2d + 1 - s'$ vertices, all G-diagonals of which are s'-dimensional, for any $s' \in [d]$. Suppose $s' := n + d - s$. Let us construct a tuple $\boldsymbol{E}$ consisting of $n + r$ points of the space $\mathbb{R}^n$, in which every point from the vertex set vert $\mathcal{P}$ occurs at least twice, and vert conv $\boldsymbol{B}$ = vert $\mathcal{P}$. The tuple $\boldsymbol{E}$ is a diagram of some positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ (see Proposition 3.19 (ii)). Since all G-diagonals of the tuple $\boldsymbol{E}$ are s'-dimensional, then $\beta(\boldsymbol{B}) = n + r - (s' + 1) = s$ (see Proposition 3.19 (v)). ☐

Simplicial representation of a positive basis

Proposition 3.25. *Let* $\mathbf{H}$ *be a hyperplane of the space* $\mathbb{R}^n$ *that does not contain the origin* $\mathbf{0}$, *and let* $\boldsymbol{B}', \boldsymbol{B}''$ *be finite subsets of points from* $\mathbf{H}$. *Suppose* $\boldsymbol{B} := \boldsymbol{B}' \cup -\boldsymbol{B}''$. *Then the set* $\boldsymbol{B}$ *is a positive basis of the space* $\mathbb{R}^n$ *if and only if*

$$\dim(\boldsymbol{B}' \cup \boldsymbol{B}'') = n - 1 , \tag{3.18}$$

$$\operatorname{ri}\operatorname{conv}\boldsymbol{B}' \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}'' \neq \emptyset , \tag{3.19}$$

and for any $\boldsymbol{B}_1' \subset \boldsymbol{B}_1$, $\boldsymbol{B}_2' \subset \boldsymbol{B}_2$, *such that* $\boldsymbol{B}_1' \cup \boldsymbol{B}_2' \neq \boldsymbol{B}_1 \cup \boldsymbol{B}_2$, *conditions* (3.18) *and* (3.19) *are not satisfied simultaneously.*

Proof. It suffices to show that properties (3.18) and (3.19) are equivalent to the claim that the set $\boldsymbol{B}' \cup -\boldsymbol{B}''$ spans positively the space $\mathbb{R}^n$. Suppose to the contrary that properties (3.18) and (3.19) are fulfilled. Assume that $\boldsymbol{B}' \cup -\boldsymbol{B}''$ does not span positively the space $\mathbb{R}^n$. Then pos $(\boldsymbol{B}' \cup -\boldsymbol{B}'') \neq \mathbb{R}^n$, and there exists a hyperplane $\boldsymbol{\Gamma}$ that supports the convex hull pos $(\boldsymbol{B}' \cup -\boldsymbol{B}'')$ at the point $\mathbf{0}$. It is clear that $\mathbf{H} \cap \boldsymbol{\Gamma} \neq \emptyset$, because otherwise we would come to a contradiction with the inclusion $\boldsymbol{B}', \boldsymbol{B}'' \subset \mathbf{H}$. Let $\boldsymbol{\Gamma}^+$ and $\boldsymbol{\Gamma}^-$ be two half-spaces bounded by the hyperplane $\boldsymbol{\Gamma}$ and, specifically, let us suppose that pos $(\boldsymbol{B}' \cup -\boldsymbol{B}'') \subset \boldsymbol{\Gamma}^+$. Suppose $\boldsymbol{E} := \boldsymbol{\Gamma} \cap \mathbf{H}$, $\boldsymbol{E}^+ := \boldsymbol{\Gamma}^+ \cap \mathbf{H}$, $\boldsymbol{E}^- := \boldsymbol{\Gamma}^- \cap \mathbf{H}$. Then for the half-planes $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ the inclusions $\boldsymbol{B}' \subset \boldsymbol{E}^+$ and $\boldsymbol{B}'' \subset \boldsymbol{E}^-$ hold, that is, the plane $\boldsymbol{E}$ separates the set $\boldsymbol{E}^+$ from the set $\boldsymbol{E}^-$ in the hyperplane $\mathbf{H}$, but this is impossible in view of (3.18) and (3.19).

Now suppose that $\boldsymbol{B}' \cup -\boldsymbol{B}''$ spans positively the space $\mathbb{R}^n$. This implies that the sets $\boldsymbol{B}'$ and $\boldsymbol{B}''$ are not separated in the hyperplane $\mathbf{H}$ by an $(n-2)$-dimensional plane, but this implies (3.18) and (3.19). ☐

Thus, with a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ can be put in correspondence a pair $(\boldsymbol{B}', \boldsymbol{B}'')$, if one takes an arbitrary hyperplane $\mathbf{H}$, such that $\mathbf{0} \notin \mathbf{H}$, and $\{\gamma\boldsymbol{b} : \gamma \in \mathbb{R}\} \cap \mathbf{H} \neq \emptyset$ for each $\boldsymbol{b} \in \boldsymbol{B}$; then one sets $\boldsymbol{B}' := \{\gamma\boldsymbol{B} : \gamma > 0\} \cap \mathbf{H}$, $\boldsymbol{B}'' := \{-\gamma\boldsymbol{B} : \gamma > 0\} \cap \mathbf{H}$. In such a situation the positive bases $\boldsymbol{B}$ and $\boldsymbol{B}' \cup -\boldsymbol{B}''$ coincide up to positive factors.

We will call the pair $(\boldsymbol{B}', \boldsymbol{B}'')$ a *representation* of the positive basis $\boldsymbol{B}$.

A representation $(\boldsymbol{B}', \boldsymbol{B}'')$ of a positive basis $\boldsymbol{B}$ in which the convex hulls conv $\boldsymbol{B}'$ and conv $\boldsymbol{B}''$ are simplices will be called a *simplicial representation*.

Proposition 3.26. *Let $\boldsymbol{B}$ be a positive basis, consisting of $n+r$ points, of $\mathbb{R}^n$. There exists a linear basis $\boldsymbol{B}' \subset \boldsymbol{B}$ of $\mathbb{R}^n$ strictly separated in $\mathbb{R}^n$ from its complement up to $\boldsymbol{B}$ by a hyperplane that contains the origin $\mathbf{0}$. Moreover, the number of such distinct linear bases is at least 2^r.*

Proof. We proceed by induction on n and r. If $n := 1$ and $r := 1$ then the statement is obvious. Now suppose that $n > 1$ and $r > 1$. Let us use a partition $\boldsymbol{B} = \boldsymbol{E}_1 \dot\cup \boldsymbol{E}_2 \dot\cup \cdots \dot\cup \boldsymbol{E}_r$ with the properties guaranteed by Proposition 3.18. Let us denote $\boldsymbol{E} := \boldsymbol{E}_1 \dot\cup \boldsymbol{E}_2 \dot\cup \cdots \dot\cup \boldsymbol{E}_{r-1}$ and $p := |\boldsymbol{B}_r|$, where $p \geq 2$. Note that the convex hull $\operatorname{conv} \boldsymbol{E}_r$ is a simplex; moreover, the intersection $\operatorname{aff} \boldsymbol{E} \cap \operatorname{aff} \boldsymbol{E}_r$ consists of a unique point $\boldsymbol{z} \in \operatorname{ri} \operatorname{conv} \boldsymbol{E}_r$, and the subspace $\boldsymbol{L} := \operatorname{pos} \boldsymbol{B}$ has dimension $n - p - 1$. By the induction hypothesis, there exists a linear basis $\boldsymbol{E}' \subset \boldsymbol{E}$ strictly separable in $\boldsymbol{L}$ from $\boldsymbol{E} - \boldsymbol{E}'$ by some subspace $\boldsymbol{L}_1 \subset \boldsymbol{L}$ of dimension $n - p - 2$. In view of the above argument, any subset $\boldsymbol{E}'_r \subset \boldsymbol{E}_r$, which consists of $p - 1$ points, together with the set $\boldsymbol{E}' \subset \boldsymbol{L}$, form a linear basis of $\mathbb{R}^n$. Let us show that $\boldsymbol{E}' \cup \boldsymbol{E}'_r$ is strictly separated from its complement up to $\boldsymbol{B}$. Let $\mathbf{H}_1$ be a plane of dimension $p - 2$ containing the point $\boldsymbol{z}$ that strictly separates $\boldsymbol{E}'_r$ from $\boldsymbol{E}_r - \boldsymbol{E}'_r$ in $\operatorname{aff} \boldsymbol{E}_r$. Let us consider the hyperplane $\mathbf{H} := \operatorname{aff}(\boldsymbol{L} \cup \mathbf{H}_1) = \operatorname{lin}(\boldsymbol{L} \cup \mathbf{H}_1)$ in $\mathbb{R}^n$, and consider a subspace $\boldsymbol{L}_2$, which is contained in it, of dimension $n - 2$, with the conditions $\boldsymbol{L}_1 \subseteq \boldsymbol{L}_2$, $\boldsymbol{L}_2 \cap \boldsymbol{E} = \emptyset$. By a sufficiently small rotation of the hyperplane $\mathbf{H}$ around $\boldsymbol{L}_2$ in a relevant direction, we obtain a hyperplane $\mathbf{H}^*$, such that $\mathbf{0} \in \mathbf{H}^*$, which strictly separates the linear basis $\boldsymbol{E}' \cup \boldsymbol{E}'_r$ from its complement up to $\boldsymbol{B}$. It remains to note that, by the induction hypothesis, $\boldsymbol{E}'$ can be chosen in at least 2^{r-1} ways, and $\boldsymbol{E}'_r$ can be chosen in precisely $p \geq 2$ ways; thus, there exist at least 2^r desired linear bases. □

Proposition 3.27. *Any positive basis of the space $\mathbb{R}^n$ has a simplicial representation.*

Proof. According to Proposition 3.26, one can distinguish in the positive basis $\boldsymbol{B}$ a linear basis $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that the sets $\boldsymbol{B}_0$ and $\boldsymbol{B} - \boldsymbol{B}_0$ are strictly separated by a hyperplane $\mathbf{H}$ that contains the origin $\mathbf{0}$. Let $\boldsymbol{c}$ be a normal vector of $\mathbf{H}$, and $\langle \boldsymbol{c}, \boldsymbol{b} \rangle > 0$ for all $\boldsymbol{b} \in \boldsymbol{B}_0$, and also $\langle \boldsymbol{c}, \boldsymbol{b} \rangle < 0$ for all $\boldsymbol{b} \in \boldsymbol{B} - \boldsymbol{B}_0$. Suppose $\mathbf{H}_1 := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = 1\}$. Then the pair $(\boldsymbol{B}', \boldsymbol{B}'')$, where $\boldsymbol{B}' := \{\lambda \boldsymbol{B}_0 : \lambda > 0\} \cap \mathbf{H}_1$, $\boldsymbol{B}'' := \{-\lambda(\boldsymbol{B} - \boldsymbol{B}_0) : \lambda > 0\} \cap \mathbf{H}_1$, is in fact a desired representation. Indeed, since $\boldsymbol{B}_0$ is linear basis, this implies that $\operatorname{conv} \boldsymbol{B}'$ is a simplex of dimension $n-1$. Suppose to the contrary that $\operatorname{conv} \boldsymbol{B}''$ is not a simplex. Let $\boldsymbol{x}_0 \in \operatorname{ri} \operatorname{conv} \boldsymbol{B}' \cap \operatorname{ri} \operatorname{conv} \boldsymbol{B}''$. By Carathéodory's theorem, there exists a subset $\boldsymbol{B}''_1 \subseteq \boldsymbol{B}''$ such that $\boldsymbol{x}_0 \in \operatorname{conv} \boldsymbol{B}''_1$, and $\operatorname{conv} \boldsymbol{B}''_1$ is a simplex, that is, $\boldsymbol{B}''_1 \subsetneq \boldsymbol{B}''$. Since $\dim \boldsymbol{B}' = n - 1$, then $\operatorname{ri} \operatorname{conv} \boldsymbol{B}' \cap \operatorname{ri} \operatorname{conv} \boldsymbol{B}''_1 \neq \emptyset$ and, in addition, $\dim(\boldsymbol{B}'_1 \cup \boldsymbol{B}'') = n-1$, $\boldsymbol{B}'_1 \cup \boldsymbol{B}''_1 \neq \boldsymbol{B}' \subset \boldsymbol{B}''$. We have come to a contradiction with Proposition 3.25. □

Regular positive bases

We will call a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ *regular* if for some of its simplicial representation $(\boldsymbol{B}', \boldsymbol{B}'')$ the inclusison $\boldsymbol{B}'' \subset \operatorname{conv} \boldsymbol{B}'$ holds.

Proposition 3.28. *Let $\boldsymbol{B}$ be a positive basis, consisting of $n+r$ points, of $\mathbb{R}^n$. The following assertions are equivalent:*
(i) *$\boldsymbol{B}$ is a regular positive basis.*
(ii) *$\beta(\boldsymbol{B}) = n$.*
(iii) *$\boldsymbol{B}$ has precisely r minimal sub-bases.*
(iv) *The set $\operatorname{conv} \overline{\boldsymbol{B}}$ is a simplex.*
(v) *In every minimal sub-basis there are at least two points that are not contained in the other minimal sub-bases.*
(vi) *The family of minimal sub-bases of the positive basis $\boldsymbol{B}$ forms an inclusion-minimal cover of the set $\boldsymbol{B}$.*

Proof. (i) $\Rightarrow$ (iii). Let $(\boldsymbol{B}', \boldsymbol{B}'')$ be a simplicial representation of the positive basis $\boldsymbol{B}$, $\boldsymbol{B}'' \subset \operatorname{conv} \boldsymbol{B}'$, and $\boldsymbol{E} \subseteq \boldsymbol{B}$. Suppose $\boldsymbol{E}^+ := \{\lambda \boldsymbol{E} \colon \lambda > 0\} \cap \boldsymbol{B}'$ and $\boldsymbol{E}^- := \{-\lambda \boldsymbol{E} \colon \lambda > 0\} \cap \boldsymbol{B}''$. It follows from Proposition 3.25 that $\boldsymbol{E}$ is a minimal sub-basis if and only if the sets $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ are inclusion-minimal with respect to the property $\operatorname{ri} \operatorname{conv} \boldsymbol{E}^+ \cap \operatorname{ri} \operatorname{conv} \boldsymbol{E}^- \neq \emptyset$. Since the vertex set $\boldsymbol{B}'$ of the simplex $\operatorname{conv} \boldsymbol{B}'$ is affinely independent, then for every point $\boldsymbol{b} \in \operatorname{conv} \boldsymbol{B}'$ there exists a unique subset $\boldsymbol{B_b} \subseteq \boldsymbol{B}'$ such that $\boldsymbol{b} \in \operatorname{ri} \operatorname{conv} \boldsymbol{B_b}$; besides, if $\boldsymbol{C} \subsetneqq \operatorname{conv} \boldsymbol{B}'$ and $\boldsymbol{x} \in \operatorname{ri} \operatorname{conv} \boldsymbol{C}$, then $\boldsymbol{B_b} \subseteq \boldsymbol{B_x}$ for any $\boldsymbol{b} \in \boldsymbol{C}$. Summarizing the above said, we conclude that the set $\boldsymbol{E} \subseteq \boldsymbol{B}$ is a minimal sub-basis of the regular positive basis $\boldsymbol{B}$ if and only if $\boldsymbol{E}^- = \{\boldsymbol{e}\}$, $\boldsymbol{E}^+ = \boldsymbol{B_e}$, for some vector $\boldsymbol{e} \in \boldsymbol{B}''$. Thus, $\boldsymbol{B}$ has precisely $|\boldsymbol{B}''| = r$ minimal sub-bases.

(iii) $\Rightarrow$ (iv). The set $\operatorname{conv} \overline{\boldsymbol{B}}$ in $\mathbb{R}^{r-1}$ has (see Proposition 3.19 (iv)) precisely r facets and, because of $\operatorname{aff} \overline{\boldsymbol{B}} = \mathbb{R}^{r-1}$, this means that $\operatorname{conv} \overline{\boldsymbol{B}}$ is a simplex.

(iv) $\Rightarrow$ (ii). In the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$ there are G-diagonals consisting of precisely r points (all vertices of the simplex) and, as a consequence (see Proposition 3.19 (v)), in $\boldsymbol{B}$ there is a maximal one-sided subset with n points, that is, $\beta(\boldsymbol{B}) = n$.

(ii) $\Rightarrow$ (i). Let $\boldsymbol{B}'$ be a maximal one-sided subset, and $|\boldsymbol{B}'| = n$. Note that the set $\boldsymbol{B}'$ is linearly independent; it remains to show that $-(\boldsymbol{B} - \boldsymbol{B}') \subset \operatorname{pos} \boldsymbol{B}'$. Suppose to the contrary that $\boldsymbol{b} \in \boldsymbol{B} - \boldsymbol{B}'$ and $-\boldsymbol{b} \notin \operatorname{pos} \boldsymbol{B}'$. Since $\operatorname{pos} \boldsymbol{B}'$ is an acute cone, then there exists a hyperplane that contains the origin $\boldsymbol{0}$ and strictly separates the vector $-\boldsymbol{b}$ from $\boldsymbol{B}'$, and thus the point $\boldsymbol{b}$ and the set $\boldsymbol{B}'$ lie in the same open half-space, a contradiction with the maximality of the one-sided set $\boldsymbol{B}'$.

(iv) $\Rightarrow$ (v). Let $\boldsymbol{E}$ be a minimal sub-basis of $\boldsymbol{B}$. It is assigned a facet $\boldsymbol{F}$ of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$ such that $\boldsymbol{b} \in \boldsymbol{E} \iff \overline{\boldsymbol{b}} \notin \boldsymbol{F}$ for any vector $\boldsymbol{b} \in \boldsymbol{B}$ (see Proposition 3.19 (iv)). Let us choose a vector $\boldsymbol{x} \in \operatorname{vert} \operatorname{conv} \overline{\boldsymbol{B}}$ such that $\boldsymbol{x} \notin \boldsymbol{F}$. In the positive basis $\boldsymbol{B}$ there exist two distinct points $\boldsymbol{b}$ and $\boldsymbol{e}$ such that $\overline{\boldsymbol{b}} = \overline{\boldsymbol{e}} = \boldsymbol{x}$ (see Proposition 3.19 (i)). Since the vector $\boldsymbol{x}$ is contained in all other facets of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$, then the only minimal sub-basis containing the points $\boldsymbol{b}$ and $\boldsymbol{e}$ is the sub-basis $\boldsymbol{E}$, see Proposition 3.19 (iv).

(v) ⇒ (vi). Easily verified.

(vi) ⇒ (iv). Suppose to the contrary that the set conv $\overline{\boldsymbol{B}}$ is not a simplex; then the number of its facets N satisfies the inequality $N \geq r + 1$. Because of the minimality of the cover by minimal sub-bases and the characterization of minimal sub-bases in the language of diagrams (see Proposition 3.19 (iv)), any $N - 1 \geq r$ facets of conv $\overline{\boldsymbol{B}}$ have a nonempty intersection. Hence, by Helly's theorem applied to the space $\mathbb{R}^{r-1}$, we conclude that all N facets have a nonempty intersection, a contradiction. □

Corollary 3.29. *Any SPB is regular.*

Proof. By Proposition 3.21, for some strict positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ the equality $\beta(\boldsymbol{B}) = \alpha(\boldsymbol{B}) = n$ holds. □

Corollary 3.30. *If $n \in [4]$ then any positive basis of the space $\mathbb{R}^n$ is regular. Besides, for an arbitrary n, if $r \in \{1, 2, n-1, n\}$ then any positive basis of the space $\mathbb{R}^n$, consisting of $n + r$ points, is regular.*

Proof. In view of Proposition 3.24, in the listed cases we have $\beta(\boldsymbol{B}) = n$. □

Corollary 3.31. *For each $n \geq 5$, in the space $\mathbb{R}^n$ there exists a positive basis that is not regular.*

Proof. In view of Proposition 3.24, for each $n \geq 5$, in $\mathbb{R}^n$ there exists a positive basis $\boldsymbol{B}$, consisting of $n + 3 \geq 4(3-1)$ points, for which $\beta(\boldsymbol{B}) = n + 1$. □

Proposition 3.32. *Let $\boldsymbol{B}$ and $\boldsymbol{E}$ be positive bases, consisting of $n + r$ points, of the space $\mathbb{R}^n$. The family of minimal sub-bases of the positive basis $\boldsymbol{B}$ is combinatorially isomorphic to the family of minimal sub-bases of the positive basis $\boldsymbol{E}$ if and only if the family of maximal one-sided subsets of the set $\boldsymbol{B}$ is combinatorially isomorphic to the family of maximal one-sided subsets of the set $\boldsymbol{E}$.*

Proof. It suffices to recall that for any positive basis of the space $\mathbb{R}^n$, the family of its minimal sub-bases and the family of its maximal one-sided subsets uniquely determine each other. □

Proposition 3.33. *Let $I_1, I_2, \dots, I_r \subset [n]$. The family $\{I_1, I_2, \dots, I_r\}$ forms an inclusion-minimal cover of the set $[n]$ if and only if there exists a regular positive basis $\boldsymbol{B} := \{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n, \dots, \boldsymbol{b}_{n+r}\}$ of the space $\mathbb{R}^n$ with the family of minimal sub-bases $\{\boldsymbol{B}(I_1 \cup \{n+1\}), \boldsymbol{B}(I_2 \cup \{n+2\}), \dots, \boldsymbol{B}(I_r \cup \{n+r\})\}$, where $\boldsymbol{B}(I') = \{\boldsymbol{b}_i : i \in I'\}$. Besides, $\boldsymbol{B}$ is a strict positive basis if and only if $I_1, I_2, \dots, I_r$ are pairwise disjoint subsets.*

Proof. The *sufficiency* follows from Proposition 3.28 (v).

The *necessity*. Let $\mathbf{H}$ be a hyperplane in the space $\mathbb{R}^n$ that does not contain the origin $\mathbf{0}$. Let $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n$ be the vertices of some simplex in $\mathbf{H}$. For each index $i \in [r]$, let us pick a point $\boldsymbol{e}_i \in \operatorname{ri}\operatorname{conv}\{\boldsymbol{b}_j : j \in I_i\}$. Let us denote $\boldsymbol{B}' := \{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n\}$ and $\boldsymbol{B}'' := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_r\}$. Then $\operatorname{ri}\operatorname{conv}\boldsymbol{B}' \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}'' \neq \emptyset$, because $I_1 \cup I_2 \cup \dots \cup I_r = [n]$. Suppose $\boldsymbol{B}'_1 \subseteq \boldsymbol{B}'$ and $\boldsymbol{B}''_1 \subseteq \boldsymbol{B}''$. If $\boldsymbol{B}'_1 = \boldsymbol{B}'$ and $\boldsymbol{B}''_1 \neq \boldsymbol{B}''$, then $\operatorname{ri}\operatorname{conv}\boldsymbol{B}'_1 \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}''_1 = \emptyset$, because of the minimality of the cover $\{I_1, I_2, \dots, I_r\}$ of the set $[n]$. Suppose $\boldsymbol{B}_1' \neq \boldsymbol{B}'$. Then $\boldsymbol{B}_1'$ lies in some facet $\boldsymbol{F}$ of the simplex $\operatorname{conv} \boldsymbol{B}'$. Suppose $\mathbf{H}_1 := \operatorname{aff} \boldsymbol{F}$, where $\dim \mathbf{H}_1 = n-2$. Note that $\operatorname{ri}\operatorname{conv} \boldsymbol{B}_1' \cap \operatorname{ri}\operatorname{conv} \boldsymbol{B}_1'' \neq \emptyset$ implies that $\boldsymbol{B}_1' \cup \boldsymbol{B}_1'' \subset \mathbf{H}_1$, that is, $\dim(\boldsymbol{B}_1' \cup \boldsymbol{B}_2'') = n-2$. Thus, the conditions from Proposition 3.25 are fulfilled; therefore, the set $\boldsymbol{B} := \boldsymbol{B}_1' \cup -\boldsymbol{B}_1''$ is a positive basis of the space $\mathbb{R}^n$. The pair $(\boldsymbol{B}_1', \boldsymbol{B}_1'')$ is a simplicial representation of the positive basis $\boldsymbol{B}$, see the proof of Proposition 3.27. The positive basis $\boldsymbol{B}$ is regular, because, by construction, $\boldsymbol{B}'' \subset \operatorname{conv} \boldsymbol{B}'$. Let us denote $\boldsymbol{b}_{n+1} := -\boldsymbol{e}_i$, for $i \in [r]$. The regular positive basis $\boldsymbol{B}$ has the family of minimal subbases $\{\{\boldsymbol{b}_{n+i}\} \cup \{\boldsymbol{b}_j : j \in I_i\} : i \in [r]\}$, as we saw in the proof of the implication (i) $\Rightarrow$ (iii) of Proposition 3.28. The first assertion of the proposition is thus proved. The second assertion of the proposition, which concerns SPBs, is a known fact. □

3.3 Polytopes and infeasible systems of inequalities

We continue the study, initiated in Section 2.3, of infeasible systems (2.26) of homogeneous strict linear inequalities of rank r over the real Euclidean space $\mathbb{R}^r$. The subject of our investigation is the system of the more general form

$$\mathsf{S} := \{\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0 : \ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r; \ \|\boldsymbol{a}_i\| = 1, \ i \in [m]\}, \tag{3.20}$$

with the set of its determining vectors $\boldsymbol{A}(\mathsf{S}) := \{\boldsymbol{a}_i : \ i \in [m]\}$.

We use the notation $\mathbf{J}$ to denote the family of the multi-indices of maximal feasible subsystems of the system S, and the notation $\mathbf{I}$ to denote the family of the multi-indices of its minimal infeasible subsystems. The characters $\mathfrak{q}$ and $\mathfrak{p}$ denote the number of multi-indices in the families $\mathbf{J}$ and $\mathbf{I}$, respectively; $\mathfrak{q} := \#\mathbf{J}$, $\mathfrak{p} := \#\mathbf{I}$.

We will need two statements:

Lemma 3.34. *System* (3.20) *is infeasible if and only if* $\sum_{i \in [m]} \lambda_i \boldsymbol{a}_i = \mathbf{0}$, *for some nonnegative numbers* $\lambda_1, \dots, \lambda_m$, *at least one of which is not* 0.

Lemma 3.35. *Let* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *be convex sets in* $\mathbb{R}^r$ *such that* $\operatorname{aff}(\boldsymbol{V} \cup \boldsymbol{W}) = \mathbb{R}^r$. *The sets* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *can be separated by a hyperplane if and only if* $\operatorname{ri} \boldsymbol{V} \cap \operatorname{ri} \boldsymbol{W} = \emptyset$.

An inequality $\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0$ of the system S is called *essential* if it does not belong to at least one its MFS. The system S is called *irreducible* if all its inequalities are essential.

Proposition 3.36. *A system* S *is irreducible if and only if the set* $\operatorname{pos} \boldsymbol{A}(\mathsf{S})$ *is a linear subspace.*

Proof. The *necessity.* Let system (3.20) be irreducible. Let us show that the set $\boldsymbol{K} := \operatorname{pos} \boldsymbol{A}(\mathsf{S})$ is a linear subspace. Assume the converse. Let us assign to an index subset $L \subseteq [m]$ the subset $\boldsymbol{A}_L(\mathsf{S}) := \{\boldsymbol{a}_i : \ i \in L\}$ of the corresponding vectors that define a subsystem, and suppose $J_0 := \{i \in [m] : \ \boldsymbol{a}_i \in \boldsymbol{K} \cap -\boldsymbol{K}\}$ and $J' := [m] - J_0 = \{i \in [m] : \boldsymbol{a}_i \notin \boldsymbol{K} \cap -\boldsymbol{K}\}$. Since the set $\boldsymbol{K} \cap -\boldsymbol{K}$ is a linear subspace, according to our assumption,

$\boldsymbol{K} \neq \boldsymbol{K} \cap -\boldsymbol{K}$, and thus $|J'| > 0$, $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) \leq r - 1$. Let us denote by $\boldsymbol{K}^*$ the polar of the cone $\boldsymbol{K}$, that is, the set $\boldsymbol{K}^* := \{\boldsymbol{x} \in \mathbb{R}^r : \langle \boldsymbol{g}, \boldsymbol{x} \rangle \geq 0, \forall \boldsymbol{g} \in \boldsymbol{K}\}$. Taking into account the relation $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) + \dim \boldsymbol{K}^* = r$, we have $\dim \boldsymbol{K}^* \geq 1$. For an arbitrary vector $\boldsymbol{b} \in \operatorname{ri} \boldsymbol{K}^*$ the relation $0 \leq \langle \boldsymbol{K} \cap -\boldsymbol{K}, \boldsymbol{b} \rangle \leq 0$ holds, therefore, $\langle \boldsymbol{A}_{J_0}(\mathsf{S}), \boldsymbol{b} \rangle = 0$.

Let us show that $\langle \boldsymbol{A}_{J'}(\mathsf{S}), \boldsymbol{b} \rangle > 0$. Suppose to the contrary that there exists an index $s \in J'$ such that $\langle \boldsymbol{a}_s, \boldsymbol{b} \rangle = 0$. Since $(\boldsymbol{K} \cap -\boldsymbol{K}) + \operatorname{lin} \boldsymbol{K}^* = \mathbb{R}^r$, then $\boldsymbol{a}_s = \boldsymbol{z}_1 + \boldsymbol{z}_2$, where $\boldsymbol{z}_1 \in \boldsymbol{K} \cap -\boldsymbol{K}$, $\boldsymbol{z}_2 \in \operatorname{lin} \boldsymbol{K}^*$, and $\boldsymbol{z}_2 \neq \boldsymbol{0}$ in view of $\boldsymbol{a}_s \notin \boldsymbol{K} \cap -\boldsymbol{K}$. But then $\langle \boldsymbol{a}_s, \boldsymbol{b} - \epsilon \boldsymbol{z}_2 \rangle = -\epsilon \langle \boldsymbol{z}_2, \boldsymbol{z}_2 \rangle < 0$ for any $\epsilon > 0$ and, in addition, $\boldsymbol{b} - \epsilon \boldsymbol{z}_2 \in \operatorname{lin} \boldsymbol{K}^*$, which contradicts the choice of the point $\boldsymbol{b} \in \boldsymbol{K}^*$. Further, since $\langle \boldsymbol{A}_{J_0}(\mathsf{S}), \boldsymbol{b} \rangle = 0$ and $\langle \boldsymbol{A}_{J'}(\mathsf{S}), \boldsymbol{b} \rangle > 0$, the equality $\sum_{i \in [m]} \lambda_i \boldsymbol{a}_i = 0$ implies the equality $\lambda_i = 0$ for all indices $i \in J'$. This means, with respect to Lemma 3.34, that the inequalities with the indices from J' are included in none of the IISs of the system S; as a consequence, they belong to all MFSs of the system S, a contradiction with the irreducibility of this system.

The *sufficiency*. Let $\boldsymbol{K} := \operatorname{pos} \boldsymbol{A}(\mathsf{S})$ be a linear subspace of $\mathbb{R}^n$. Let us show that system (3.20) is irreducible. It suffices to check that the inequality $\langle \boldsymbol{a}_1, \mathbf{x} \rangle > 0$ is essential. Since $\boldsymbol{K}$ is a linear subspace, then $-\boldsymbol{a}_1 = \sum_{i \in [m]} \lambda_i \boldsymbol{a}_i$ for some factors $\lambda_i \geq 0$, $i \in [m]$. The latter equality can be rewritten in the form $-\boldsymbol{a}_1 = \sum_{i=2}^m \lambda_i' \boldsymbol{a}_i$, where $\lambda_i' \geq 0$, $2 \leq i \leq m$. Let us choose among all such equalities some equality $-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i$ with the minimal number of indices in the set L. Let us show that the subsystem with the multi-index L of the system S is feasible. Assume the converse. Then, by Lemma 3.34, $\sum_{i \in L} \gamma_i \boldsymbol{a}_i = \boldsymbol{0}$ for some numbers $\gamma_i \geq 0$, $i \in L$, among which at least one number is positive. Suppose $\epsilon := \min\{\frac{\alpha_i}{\gamma_i} : i \in L, \gamma_i > 0\}$. Then

$$-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i - \sum_{i \in L} \epsilon \gamma_i \boldsymbol{a}_i = \sum_{i \in L} (\alpha_i - \epsilon \gamma_i) \boldsymbol{a}_i \,,$$

where $\alpha_i - \epsilon \gamma_i \geq 0$, by the choice of ϵ, for each index $i \in L$, and $\alpha_i - \epsilon \gamma_i = 0$ for some index $i \in L$. But this contradicts the minimality of L. As a consequence, the subsystem with the multi-index L is feasible. On the other hand, the subsystem with the multi-index $L \cup \{1\}$ is infeasible because $-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i$, where $\alpha_i \geq 0$ for all $i \in L$. Thus, the inequality $\langle \boldsymbol{a}_1, \mathbf{x} \rangle > 0$ is essential because it does not belong to the MFS that contains the feasible subsystem with the multi-index L. The proposition is proved. □

It follows from Proposition 3.36 that the union of two irreducible subsystems of the system S is also its irreducible subsystem; therefore, the system S has an inclusion-maximal irreducible infeasible subsystem; let J_0 be the multi-index of this subsystem. Then the families $\{J_0 \cap J_s : s \in [\mathfrak{q}]\}$ and $\{I_s : s \in [\mathfrak{p}]\}$ are the families of the multi-indices of MFSs and IISs of the irreducible system $\{\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0, \ i \in J_0\}$, respectively, if and only if $\{J_s : s \in [\mathfrak{q}]\}$ and $\{I_s : s \in [\mathfrak{p}]\}$ are the families of the multi-indices of MFSs and IISs of system (3.20), respectively. Thus, in the study of the combinatorial properties of the infeasible systems of linear inequalities we can restrict ourselves to the irreducible systems.

Combinatorial properties of polytopes and infeasible systems of linear inequalities

In the combinatorial theory of polytopes, a very useful method for investigating the combinatorial structure of polytopes is to consider their Gale transforms. The Gale transform establishes a relationship between the face structure of an r-dimensional polytope, which has m vertices, with the positive dependence of a certain arrangement of m vectors that lie in the space $\mathbb{R}^{m-r-1}$. In this section, we use the Gale transform to establish a link between the combinatorial properties of infeasible systems (3.20) and those of the facets and diagonals of polytopes; from now on, when considering the diagonals of polytopes, we always mean G-diagonals defined earlier on page 59. For a finite nonempty tuple of points $\boldsymbol{X} \subset \mathbb{R}^r$, we introduce the notion of diagonal as follows:

An inclusion-minimal subtuple $\boldsymbol{D} \subseteq \boldsymbol{X}$, with the property

$$\operatorname{conv} \boldsymbol{D} \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} \neq \emptyset ,$$

is called a *diagonal* of the tuple $\boldsymbol{X}$.

Let us recall relevant definitions. Let a finite sequence of points $\boldsymbol{X} := (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m) \subset \mathbb{R}^r$, such that $\operatorname{aff} \boldsymbol{X} \simeq \mathbb{R}^r$, be given. Consider the $(m-r-1)$-dimensional space $\mathbf{K}(\boldsymbol{X})$ of solutions $(\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m$ to the following system of homogeneous linear equations:

$$\sum_{i \in [m]} \beta_i \boldsymbol{x}_i = \mathbf{0} , \quad \sum_{i \in [m]} \beta_i = 0 .$$

In the space $\mathbf{K}(\boldsymbol{X})$, fix its arbitrary ordered basis $(\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{m-r-1})$. Let $\boldsymbol{B}(\boldsymbol{X})$ be the $(m-r-1) \times m$ matrix whose rows are the vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{m-r-1}$ of this basis. For each index $i \in [m]$, let us denote by $\boldsymbol{x}_i^*$ the ith column of the matrix $\boldsymbol{B}(\boldsymbol{X})$ regarded as a vector from the space $\mathbb{R}^{m-r-1}$.

The sequence $\boldsymbol{X}^* = (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_m^*)$ is called a *Gale transform* of the sequence $\boldsymbol{X}$.

A Gale transform is not unique. Since for any two distinct transforms there exists a linear isomorphism of $\mathbb{R}^{m-r-1}$ onto itself that sends the first Gale transform to the second transform, one usually takes an arbitrary basis of $\mathbf{K}(\boldsymbol{X})$ as a Gale transform.

In the general case, the Gale transform can contain coinciding points. Therefore, each of the pairwise distinct points of the Gale transform is assigned its multiplicity, which is the number of its preimages.

Let $L \subseteq [m]$. If $\boldsymbol{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m\} \subset \mathbb{R}^r$ then the sequence $\boldsymbol{X}(L) := \{\boldsymbol{x}_i \in \boldsymbol{X} \colon i \in L\}$ is called a *cofacet* of the sequence $\boldsymbol{X}$, if $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{aff} \boldsymbol{X}([m] - L) = \emptyset$.

Recall some basic properties of the Gale transform.

Proposition 3.37. *Let a sequence of points $\boldsymbol{X} := (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m) \subset \mathbb{R}^r$ be given, $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$. Let $\boldsymbol{X}^* := (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_m^*) \subset \mathbb{R}^{m-r-1}$ be a Gale transform of the tuple $\boldsymbol{X}$.*

(i) *If $\sum_{i \in [m]} \boldsymbol{x}_i = \mathbf{0}$ then the tuple $\boldsymbol{X}$ is a Gale transform of the sequence $\boldsymbol{X}^*$.*

(ii) $\sum_{i \in [m]} \boldsymbol{x}_i^* = \mathbf{0}$, $\operatorname{lin} \boldsymbol{X}^* = \mathbb{R}^{m-r-1}$, $\operatorname{pos} \boldsymbol{X}^* = \mathbb{R}^{m-r-1}$.

(iii) $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*$.

(iv) *A subtuple $\boldsymbol{X}(L)$ is a coface of the tuple $\boldsymbol{X}$ if and only if $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*(L)$.*

It is customary to formulate some properties of the Gale transform in terms of Gale diagrams.

A *Gale diagram* $\mathfrak{G}(\mathcal{P})$ of a bounded convex r-polytope $\mathcal{P} \subset \mathbb{R}^r$, with m vertices $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$, is the sequence of points $(\mathfrak{g}(\boldsymbol{x}_1), \mathfrak{g}(\boldsymbol{x}_2), \ldots, \mathfrak{g}(\boldsymbol{x}_m)) \in \mathbb{R}^{m-r-1}$ defined as follows: $\mathfrak{g}(\boldsymbol{x}_i) := \mathbf{0}$ for $\boldsymbol{x}_i^* = \mathbf{0}$, and $\mathfrak{g}(\boldsymbol{x}_i) := \frac{\boldsymbol{x}_i^*}{\|\mathbf{x}_i^*\|}$ for $\boldsymbol{x}_i^* \neq \mathbf{0}$.

Thus, a Gale diagram consists of a finite sequence of points from the set $\mathbb{S}^{m-r-2} \cup \{\mathbf{0}\}$, where $\mathbb{S}^{m-r-2}$ is the $(m - r - 2)$-dimensional unit sphere centered at the origin $\mathbf{0}$.

Given a subsequence $\boldsymbol{V} \subseteq \operatorname{vert} \mathcal{P}$, $\mathfrak{G}(\boldsymbol{V})$ denotes the subset of the Gale diagram $\mathfrak{G}(\mathcal{P})$ that corresponds to the tuple $\boldsymbol{V}$.

Corollary 3.38. (i) *A set $\boldsymbol{X} \subset \operatorname{vert} \mathcal{P}$ is a coface of the vertex tuple of a polytope $\mathcal{P}$ if and only if $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \mathfrak{G}(\boldsymbol{X})$.*

(ii) *A set of points $\boldsymbol{X} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ represents the vertex set of some r-polytope $\mathcal{P}$ if and only if*
  (a) *either $\mathfrak{g}(\boldsymbol{x}_i) = \mathbf{0}$ for all $\boldsymbol{x}_i \in \boldsymbol{X}$, that is, when $\mathcal{P}$ is an r-simplex, or*
  (b) *for any open half-space $\mathbf{C}_>$ of $\mathbb{R}^{m-r-1}$, such that $\overline{\mathbf{C}_>} \ni \mathbf{0}$, the condition $|\{i \in [m] \colon \mathfrak{g}(\boldsymbol{x}_i) \in \mathbf{C}_>\}| \geq 2$ is satisfied.*

(iii) *If $\boldsymbol{F}$ is a face of the vertex tuple of a polytope $\mathcal{P}$, and $\boldsymbol{Z} := \operatorname{vert} \mathcal{P} - \operatorname{vert} \boldsymbol{F}$ is the corresponding coface, then $\operatorname{ri} \operatorname{conv} \mathfrak{G}(\boldsymbol{Z}) \ni \mathbf{0}$.*

(iv) *A polytope $\mathcal{P}$ is simplicial if and only if for each hyperplane $\mathbf{H}$ containing the origin $\mathbf{0}$, it holds $\mathbf{0} \notin \operatorname{ri} \operatorname{conv} (\mathfrak{G}(\mathcal{P}) \cap \mathbf{H})$.*

(v) *A polytope $\mathcal{P}$ is an r-faced pyramid if and only if in its Gale diagram the origin $\mathbf{0}$ has multiplicity r.*

We will need the following statement:

Lemma 3.39. *Suppose $\boldsymbol{X} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\} \subset \mathbb{R}^r$. The inclusion $\mathbf{0} \in \operatorname{ri} \operatorname{conv} X$ holds if and only if there exist coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$ such that $\sum_{i \in [m]} \lambda_i \boldsymbol{x}_i = \mathbf{0}$.*

Lemma 3.40. *Let $\boldsymbol{X} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ be a sequence of points in the space $\mathbb{R}^r$, $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$, and $\boldsymbol{C} := (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$ the sequence of points in the space $\mathbb{R}^{m-r-1}$ such that $\boldsymbol{C} = \boldsymbol{X}^*$. The subsystem, with a multi-index L, of the system $\{\langle \boldsymbol{c}_i^*, \mathbf{x}\rangle > 0 \colon \mathbf{x} \in \mathbb{R}^{m-r-1};\ i \in [m]\}$ is infeasible if and only if $\boldsymbol{X}(L)$ contains a nonempty coface of $\boldsymbol{X}$.*

Proof. By Lemma 3.34, the subsystem $\{\langle \boldsymbol{c}_i^*, \mathbf{x}\rangle > 0 \colon \mathbf{x} \in \mathbb{R}^{m-r-1},\ i \in L\}$ is infeasible if and only if there exist nonnegative factors λ_k, $k \in L$, at least one of which is nonzero, such that $\sum_{k \in L} \lambda_k \boldsymbol{a}_k = \mathbf{0}$. By Lemma 3.39, the latter is possible if and only if $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*(L')$, where $\emptyset \neq L' \subseteq L$; this proves the statement, in view of Proposition 3.37 (iv). ☐

The following auxiliary assertion that concerns arbitrary point tuples was the basis of the definition of G-diagonals in the case of the vertex tuples of convex polytopes, see page 59.

Lemma 3.41. *Consider a tuple $\boldsymbol{X} := (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$ of points in $\mathbb{R}^r$ such that $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$. A subtuple $\boldsymbol{X}(L)$ is a diagonal of the tuple $\boldsymbol{X}$ if and only if $\boldsymbol{X}(L)$ is contained in none of the faces of the tuple $\boldsymbol{X}$, and every subtuple $\boldsymbol{X}(L')$, where $\emptyset \neq L' \subset L$, is contained in at least one face of the tuple $\boldsymbol{X}$.*

Proof. It suffices to show that the subtuple $\boldsymbol{X}(L)$ is included in some face of the tuple $\boldsymbol{X}$ if and only if $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$.

Suppose that $\boldsymbol{X}(L) \subset \boldsymbol{X}(M)$, for some face $\boldsymbol{X}(M)$ of the tuple $\boldsymbol{X}$. For the face $\boldsymbol{X}(M)$, we by definition have $\operatorname{aff} \boldsymbol{X}(M) \cap \operatorname{conv} \boldsymbol{X}([m] - M) = \emptyset$. Using Lemma 3.35, one can verify that $\operatorname{conv} \boldsymbol{X}(M) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$, as a consequence, for the tuple $\boldsymbol{X}(L)$ the equality $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$ also holds.

Now let the relation $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$ hold for some subtuple $\boldsymbol{X}(L)$. Then, by Lemma 3.35, there exists a hyperplane $\mathbf{H}$ that separates the sets $\operatorname{conv} \boldsymbol{X}(L)$ and $\operatorname{conv} \boldsymbol{X}$. Let us suppose $M := \{i \in [m] : \ \boldsymbol{x}_i \in \mathbf{H}\}$. Note that $\operatorname{aff} \boldsymbol{X}(M) \cap \operatorname{conv} \boldsymbol{X}([m] - M) \subseteq \mathbf{H} \cap \operatorname{conv}([m] - M) = \emptyset$, that is, the subtuple is a face of the tuple $\boldsymbol{X}$. Further, $\operatorname{conv} \boldsymbol{X}(L) \subset \mathbf{H}$ and, thus, $\boldsymbol{X}(L) \subset \boldsymbol{X}(M)$. □

Another auxiliary assertion that we will use is as follows:

Lemma 3.42. *Let $\boldsymbol{X}(L)$ be a diagonal of a tuple $\boldsymbol{X} := (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$ of points in the space $\mathbb{R}^r$. Then the convex hull $\operatorname{conv} \boldsymbol{X}(L)$ is a simplex.*

Proof. Let us consider an arbitrary point $\boldsymbol{x}^* \in \operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X}$. It follows from Carathéodory's theorem on the representability of points in the convex hull of a subset from $\mathbb{R}^r$ that $\boldsymbol{x}^* \in \operatorname{conv} \boldsymbol{X}(L')$, for some subset $L' \subseteq L$ of cardinality $|L'| = \dim \boldsymbol{X}(L) + 1$. The assumption that the hull $\operatorname{conv} \boldsymbol{X}(L)$ is not a simplex contradicts the minimality of the subtuple $\boldsymbol{X}(L)$ because $|L'| = \dim \boldsymbol{X}(L) + 1 < |L|$. □

Theorem 3.43. *Let an irreducible infeasible system of linear inequalities S, and a sequence $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m)$ of points, of affine dimension $d := m - r - 1$, in $\mathbb{R}^d$, be given, such that $\boldsymbol{B}^* := (\boldsymbol{b}_1^*, \boldsymbol{b}_2^*, \dots, \boldsymbol{b}_m^*) = (\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \dots, \lambda_m \boldsymbol{a}_m)$ for some factors $\lambda_1, \lambda_2, \dots, \lambda_m > 0$. The following statements are true:*

(i) *A set $I \subset [m]$ is the multi-index of some IIS of the system S if and only if its complement $[m] - I$ is the multi-index of a facet of the tuple $\boldsymbol{B}$.*
(ii) *A set $J \subset [m]$ is the multi-index of some MFS of the system S if and only if its complement $[m] - J$ is the multi-index of a diagonal of the tuple $\boldsymbol{B}$.*

Proof. Since all factors λ_k are positive, it suffices to consider, when proving the theorem, the system

$$\{\langle \boldsymbol{b}_i^*, \mathbf{x} \rangle > 0 : \ \mathbf{x} \in \mathbb{R}^r, \ i \in [m]\} \tag{3.21}$$

instead of system (3.20). Let I be the multi-index of an IIS of system (3.21). It follows from Lemma 3.40 that I is the multi-index of an inclusion-minimal nonempty coface of the tuple $\boldsymbol{B}$, that is, the complement $[m] - I$ is the multi-index of a facet of the tuple $\boldsymbol{B}$. The converse assertion is proved by applying Lemma 3.40 in the opposite direction.

Let us denote by $\mathbf{I}$ the family of the multi-indices of IISs of system (3.21), and by $\mathbf{F}$ the family of the multi-indices of facets of the tuple $\boldsymbol{B}$. A subset $J \subset [m]$ is the multi-index of a MFS of system (3.21) if and only if the complement $[m] - J$ is an inclusion-minimal subset of the set $[m]$, such that $([m] - J) \cap I \neq \emptyset$, for the multi-index $I \in \mathbf{I}$ of any IIS of system (3.21).

On the other hand, by Lemma 3.41, the complement $[m] - J$ is the multi-index of a diagonal of the tuple $\boldsymbol{B}$ if and only if the set $[m] - J$ is an inclusion-minimal subset of the set $[m]$, such that $([m] - J) \cap ([m] - F) \neq \emptyset$, for the multi-index F of any facet of the tuple $\boldsymbol{B}$.

Since $\mathbf{I} = \{[m] - F\colon\ F \in \mathbf{F}\}$, it follows from the above argument that J is the multi-index of a MFS of system (3.20) if and only if the complement $[m] - J$ is the multi-index of a diagonal of the tuple $\boldsymbol{B}$. □

Corollary 3.44. (i) *The family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of all IISs of some irreducible infeasible system* (3.20)*, of rank* r*, over* $\mathbb{R}^r$ *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of all facets of some tuple of* m *points, of affine dimension* $d := m - r - 1$*, in the space* $\mathbb{R}^d$.

(ii) *The family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of all MFSs of some irreducible infeasible system* (3.20)*, of rank* r*, over* $\mathbb{R}^r$ *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some tuple of* m *points, of affine dimension* $d := m - r - 1$*, in the space* $\mathbb{R}^d$.

We now turn to a study of the properties of infeasible systems of linear inequalities

$$\mathsf{S}_2 := \left\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\right\}, \tag{3.22}$$

of the form (3.20), whose set of determining vectors $\boldsymbol{A}(\mathsf{S}_2) := \{\boldsymbol{a}_i\colon\ i \in [m]\}$ satisfies the following structural condition: *for any open half-space* $\mathbf{C}_> \subset \mathbb{R}^r$*, bounded by a codimension one linear subspace, the condition*

$$|\{\boldsymbol{a} \in \boldsymbol{A}(\mathsf{S}_2)\colon\ \boldsymbol{a} \in \mathbf{C}_>\}| \geq 2 \tag{3.23}$$

is satisfied.

Proposition 3.45. *For system* (3.22, 3.23)*, the inclusion* $\mathbf{0} \in \operatorname{ri}\operatorname{conv}\boldsymbol{A}(\mathsf{S}_2)$ *holds.*

Proof. Suppose to the contrary that $\mathbf{0} \notin \operatorname{ri}\operatorname{conv}\boldsymbol{A}(\mathsf{S}_2)$. Then there exists a subset $\boldsymbol{A}' \subset \boldsymbol{A}(\mathsf{S}_2)$ such that $\dim\operatorname{aff}\boldsymbol{A}' = r - 1$, and $\boldsymbol{A}(\mathsf{S}_2)$ is contained in a closed half-space bounded by the hyperplane $\operatorname{aff}\boldsymbol{A}'$ because the impossibility of the mentioned inclusion would mean that the tuple $\boldsymbol{A}(\mathsf{S}_2)$ is contained in some open hemisphere of the unit sphere $\mathbb{S}^{r-1}$ and, as a consequence, the feasibility of the system S_2. But the inclusion $\boldsymbol{A}(\mathsf{S}_2)$ into a closed hemisphere of the sphere $\mathbb{S}^{r-1}$ contradicts condition (3.23). □

Proposition 3.46. *System of linear inequalities* (3.22, 3.23) *is irreducible.*

Proof. Suppose to the contrary that, according to Proposition 3.36, the set pos $\boldsymbol{A}(\mathsf{S}_2)$ is not a linear subspace. But this is possible (see the proof of Proposition 3.45) if and only if pos $\boldsymbol{A}(\mathsf{S}_2)$ is some closed half-space $\mathbf{C}_{\geq}$ bounded by a codimension one linear subspace $\mathbf{H}$ of $\mathbb{R}^n$; but this contradicts condition (3.23). □

Propositions 3.39 and 3.45 imply the following statement:

Corollary 3.47. *For a system* S_2 *of the form* (3.22, 3.23), *there exists a tuple of positive factors* λ_i, *such that the origin* $\mathbf{0}$ *is a convex combination of the vector tuple* $\boldsymbol{A}(\mathsf{S}_2)$ *with these coefficients, that is,* $\mathbf{0} = \sum_{i\in[m]} \lambda_i \boldsymbol{a}_i$, $\lambda_i > 0$, $\sum_{i\in[m]} \lambda_i = 1$.

The following proposition clarifies the link between the properties of systems (3.22, 3.23) and those of convex polytopes.

Proposition 3.48. *Let* S_2 *be a system of the form* (3.22, 3.23).
(i) *A family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of IISs of the system* S_2 *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of facets of some bounded convex* $(m - r - 1)$*-polytope with m vertices.*
(ii) *A family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of MFSs of the system* S_2 *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some bounded convex* $(m - r - 1)$*-polytope with m vertices.*
The mentioned polytope is not a pyramid; in particular, it is not a simplex.

Proof. Let us associate with the system S_2 a modified system $\mathsf{S}_2' := \{\langle \lambda_i \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i \in \boldsymbol{A}(\mathsf{S}_2)\}$ such that the coefficients λ_i satisfy the conditions from Corollary 3.47. Since all coefficients λ_i are positive, the sets of solutions to feasible subsystems, with the same multi-indices, for the systems S_2 and S_2' coincide. Since $\sum_{i\in[m]} \lambda_i \boldsymbol{a}_i = \mathbf{0}$, in accordance with Proposition 3.37 (i), the tuple $\{\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \ldots, \lambda_m \boldsymbol{a}_m\}$ is a Gale transform of the tuple $\{\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \ldots, \lambda_m \boldsymbol{a}_m\}^*$. By Corollary 3.38 (ii)(b), and according to condition (3.23), we obtain $\{\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \ldots, \lambda_m \boldsymbol{a}_m\}^*$ is the vertex tuple of a bounded convex $(m - r - 1)$-polytope in $\mathbb{R}^{m-r-1}$. The proof of assertions (i) and (ii) is completed by applying Corollary 3.44.

Recall that the vector tuple $\boldsymbol{A}(\mathsf{S}_2)$, by convention, does not contain the origin $\mathbf{0}$; therefore, in accordance with Corollaries 3.38 (ii)(a) and 3.38 (v), the tuple $\{\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \ldots, \lambda_m \boldsymbol{a}_m\}^*$ cannot be the vertex tuple of a pyramid. □

Combinatorially dual systems of linear inequalities

Proposition 3.49. *Let* S_2 *be a rank r infeasible system* (3.22) *of m homogeneous strict linear inequalities that has* $\mathfrak{p}$ *minimal infeasible subsystems. Let* $\mathbf{J}$ *and* $\mathbf{I}$ *be the families of the multi-indices of all its MFSs and IISs, respectively.*

The system S_2 *satisfies condition* (3.23) *if and only if there exists a rank* $r + \mathfrak{p} - m$ *infeasible system* S_2^0 (*whose families of the multi-indices of all MFSs and IISs are denoted by* $\mathbf{J}^0$ *and* $\mathbf{I}^0$, *respectively*) *of* $\mathfrak{p}$ *homogeneous strict linear inequalities, that has m minimal infeasible subsystems, such that*

(i) *for each multi-index* $I \in \mathbf{I}$ *of an IIS, there exists the index* $t \in [\mathfrak{p}]$ *of an inequality of the system* S_2^0 *such that* $\#\{I^0 \in \mathbf{I}^0 : \ t \in I^0\} = |I|$;
(ii) *for each multi-index* $I^0 \in \mathbf{I}^0$ *of an IIS, there exists the index* $t \in [m]$ *of an inequality of the system* S_2 *such that* $\#\{I \in \mathbf{I} : \ t \in I\} = |I|$;
(iii) *for each multi-index* $J \in \mathbf{J}$ *of a MFS, there exists a family* $\mathcal{M} \subset \mathbf{I}^0$ *of the multi-indices of IISs,* $\#\mathcal{M} = m - |J|$, *such that* $\bigcup_{I \in \mathcal{M}} I = [\mathfrak{p}]$;
(iv) *for each multi-index* $J^0 \in \mathbf{J}^0$ *of a MFS, there exists a family* $\mathcal{M} \subset \mathbf{I}$ *of the multi-indices of IISs,* $\#\mathcal{M} = \mathfrak{p} - |J|$, *such that* $\bigcup_{I \in \mathcal{M}} I = [m]$.

Proof. Proposition 3.48 puts in correspondence with the system S_2 a convex $(m-r-1)$-polytope $\mathcal{P}$ with m vertices and $\mathfrak{p}$ facets. In turn, for this polytope there exists a *dual* $(m-r-1)$-polytope $\mathcal{P}^0$ with $\mathfrak{p}$ vertices and m facets; the face lattices of the polytopes $\mathcal{P}$ and $\mathcal{P}^0$ are anti-isomorphic. The proof is completed by reapplying of Proposition 3.48, to the polytope $\mathcal{P}^0$; and, besides, we need to show that the set of vectors determining the system S_2^0 does not contain the origin $\mathbf{0}$. Suppose to the contrary that it is not the case. Then, in accordance with Corollary 3.38 (v), the polytope $\mathcal{P}^0$ is a pyramid. But the pyramid $\mathcal{P}^0$ is a polytope which is dual to the pyramid $\mathcal{P}$; thus, the set $\boldsymbol{A}(\mathsf{S}_2)$ of vectors determining the system S_2 contains the origin $\mathbf{0}$, a contradiction with the hypothesis of the proposition. □

Systems of linear inequalities and simplicial/simple polytopes. The Dehn–Sommerville relations. Bounds for the number of subsystems

In this section, in addition to the linear inequality systems of the form (2.26), (3.20), and (3.22, 3.23), we consider the inequality system S_2, whose description is given in (3.22, 3.23), which satisfies one of the following new conditions:

$$\textit{every subsystem of rank at most } r-1 \textit{ is feasible}; \tag{3.24}$$

$$\textit{each inequality belongs to } \mathfrak{p} + r - m + 1 \textit{ IISs}. \tag{3.25}$$

We will show below, in Proposition 3.52, that a system of the form (3.22, 3.23, 3.24) can equivalently be defined as a system of the form (3.22, 3.23), such that every of its minimal infeasible subsystem is composed of $r + 1$ inequalities.

The next statement follows immediately from definitions:

Proposition 3.50. *If in the set* $A(\mathsf{S}_2)$ *of vectors, which define a rank* 2 *system* S_2 *of the form* (3.22, 3.23), *there are no antipodal pairs then the system* S_2 *satisfies condition* (3.24).

Proposition 3.51. *If* S_2 *is a system of the form* (3.22, 3.23, 3.24) *then the following assertions are true:*

(i) *A family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of IISs of the system* S_2 *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of facets of some bounded convex simplicial* $(m - r - 1)$*-polytope with* m *vertices.*

(ii) *A family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of MFSs of the system* S_2 *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some bounded convex simplicial* $(m - r - 1)$*-polytope with* m *vertices.*

The mentioned polytope is not a simplex.

Proof. First, it is necessary to repeat the argument that we used when proving Proposition 3.48. Taking into account that, for any hyperplane $\mathbf{H}$ that contains the origin $\mathbf{0}$, we have $\mathbf{0} \notin \operatorname{conv}(\boldsymbol{A}(S_2) \cap \mathbf{H})$, the proof is completed by applying Corollary 3.38 (iv). □

Proposition 3.52. *A system of the form* (3.22, 3.23) *satisfies condition* (3.24) *if and only if all its minimal infeasible subsystems have the same cardinality* $r + 1$.

Given a system S_2 *of the form* (3.22, 3.23, 3.24)*, the number of all its infeasible subsystems, of cardinality* k*, which contain a fixed IIS, is equal to* $\binom{m-r-1}{m-k}$.

Proof. Since in accordance with Proposition 3.51, all index sets $[m] - I$, where $I \in \mathbf{I}$, are the multi-indices of the vertex tuples of facets of a simplicial $(m - r - 1)$-polytope, they all have the same cardinality $m - r - 1$. As a consequence, any multi-index I of a minimal infeasible subsystem has cardinality $r + 1$. □

The next two statements show that the number of feasible and infeasible subsystems, of different cardinalities, of a system S_2 of the form (3.22, 3.23, 3.24) obey special relations.

Proposition 3.53. *Let* S_2 *be a system of the form* (2.26,3.23,3.24)*; let* ν_i *and* τ_i *be the numbers of its feasible and infeasible subsystems, of cardinality* i*, respectively.*

The following relations (*where* x *is a formal variable*) *hold:*

$$\begin{cases} \nu_j = \binom{m}{j}, \text{ if } 0 \le j \le r\,, \\ \nu_{m-1} = \nu_m = 0\,, \\ \sum_{j=r+1}^{m} \left(\binom{m}{j} - \nu_j\right)(\mathrm{x} - 1)^{m-j} = \sum_{j=r+1}^{m} (-1)^{j-r-1} \left(\binom{m}{j} - \nu_j\right) \mathrm{x}^{m-j}\,. \end{cases}$$

We will call these relations the Dehn–Sommerville equations for the feasible subsystems of the system S_2. *The substitution in these relations of* $\binom{m}{j} - \nu_j$ *by* τ_j *leads to the* Dehn–Sommerville equations for the infeasible subsystems of the system S_2.

Proof. In accordance with Proposition 3.51, the index sets $[m] - J$, where $J \in \mathbf{J}$, are the multi-indices of diagonals of the vertex tuple of a simplicial $(m - r - 1)$-polytope. We will regard the family of its diagonals as a family of subsets of the atom set of its face lattice $\mathcal{L}$; this lattice is of rank $\rho(\mathcal{L}) = m - r$, and its atoms are the vertices of the polytope under consideration. Let n_t be the number of the t-subsets of the atom set of the lattice $\mathcal{L}$ that contain, as a subset, at least one diagonal. Let W_j be the number of rank j elements of the lattice $\mathcal{L}$; here $W_0 = 1$ and $W_1 = m$. In other words, W_j denotes the number of faces with j vertices. We have

$$n_t = \begin{cases} \binom{W_1}{t} - W_t = \binom{m}{t} - W_t, & \text{if } 0 \le t \le m - r - 1\,, \\ \binom{W_1}{t} = \binom{m}{t}, & \text{if } m - r \le t \le m\,. \end{cases}$$

It is clear that $n_0 = n_1 = 0$.

Further, $v_k = n_{m-k}$, therefore,

$$v_k = \begin{cases} \binom{m}{k}, & \text{if } 0 \le k \le r\,, \\ \binom{m}{k} - W_{m-k}, & \text{if } r + 1 \le k \le m\,. \end{cases}$$

Let us consider the case $r+1 \le k \le m$ in more detail; in this situation, $v_k = \binom{m}{k} - W_{m-k}$. The Dehn–Sommerville equations for the Whitney numbers of the second kind W_i of the lattice $\mathcal{L}$ are as follows:

$$\sum_{i=0}^{\rho(\mathcal{L})-1} W_i(\mathrm{x} - 1)^i = \sum_{i=0}^{\rho(\mathcal{L})-1} (-1)^{\rho(\mathcal{L})-i-1} W_i \mathrm{x}^i\,,$$

or, in our case,

$$\sum_{i=0}^{m-r-1} W_i(\mathrm{x} - 1)^i = \sum_{i=0}^{m-r-1} (-1)^{m-r-i-1} W_i \mathrm{x}^i, \quad W_0 = 1, \quad W_1 = m\,.$$

Let us equivalently rewrite the latter expression in the form

$$\sum_{j=r+1}^{m} W_{m-j}(\mathrm{x} - 1)^{m-j} = \sum_{j=r+1}^{m} (-1)^{j-r-1} W_{m-j} \mathrm{x}^{m-j}, \quad W_0 = 1, \quad W_1 = m\,.$$

By substituting W_{m-j} by $\binom{m}{j} - v_j$, we complete the proof. □

As an illustration, we present the solutions to several initial Dehn–Sommerville equations.

Corollary 3.54. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24), *and* v_i *the number of its feasible subsystems of cardinality* i. *Then*

(i) *if* $m = r + 3$, *then* $v_{r+1} = \binom{m}{2} - m$;

(ii) *if* $m = r + 4$, *then*

$$v_{r+1} = \binom{m}{3} - 2m + 4\,,$$
$$v_{r+2} = \binom{m}{2} - 3m + 6\,;$$

(iii) *if* $m = r + 5$, *then*

$$\begin{aligned} v_{r+2} &= 2v_{r+1} - 2\binom{m}{4} + \binom{m}{3}\,, \\ v_{r+3} &= -v_{r+1} + \binom{m}{4} - \binom{m}{2} + m\,. \end{aligned}$$

Corollary 3.55. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24), *and* τ_i *the number of its infeasible subsystems of cardinality* i. *Then the following relations hold:*
(i) *if* $r + 1 \le i \le m$, *then*

$$\tau_i = \sum_{j=r+1}^{m} (-1)^{j-r-1}\binom{m-j}{m-i}\tau_j\,;$$

(ii) *if* $k \in [\lfloor \frac{m-r}{2} \rfloor]$, *then*

$$\sum_{j=r+k}^{m} (-1)^{m-j+1}\binom{j-r-1}{k-1}\tau_j = \sum_{j=m-k+1}^{m} (-1)^{r+j}\binom{j-r-1}{m-r-k}\tau_j\,.$$

Proof. The above relations follow immediately from Proposition 3.51 and from the assertions: if $\mathcal{L}$ is the face lattice of a simplicial polytope, then
(i) the Dehn–Sommerville equations for $\mathcal{L}$ are equivalent to

$$W_i = \sum_{j=0}^{\rho(\mathcal{L})-1} (-1)^{\rho(\mathcal{L})-j-1}\binom{j}{i}W_j\,,$$

for $i \in [\rho(\mathcal{L}) - 1]$;
(ii) the Dehn–Sommerville equations for $\mathcal{L}$ are equivalent to

$$\sum_{j=0}^{k-1}(-1)^{\rho(\mathcal{L})+j}\binom{\rho(\mathcal{L})-j-1}{\rho(\mathcal{L})-k}W_j = \sum_{j=0}^{\rho(\mathcal{L})-k} (-1)^{j+1}\binom{\rho(\mathcal{L})-j-1}{k-1}W_j\,,$$

for $k \in [\lfloor \frac{\rho(\mathcal{L})}{2} \rfloor]$. □

Proposition 3.56. *A system* S_2 *of the form* (3.22, 3.23) *satisfies condition* (3.24) *if and only if its combinatorially dual system* S_2^0 *satisfies condition* (3.25).

Proof. The proof is analogous to that of Proposition 3.49. Since the polytope $\mathcal{P}$, which is put in correspondence with the system S_2 by the Gale transform, is simplicial, then its dual polytope $\mathcal{P}^0$ is simple. Therefore, the system S_2^0 satisfies (3.25). □

Proposition 3.57. *Let a system* S_2*of the form* (3.22, 3.23) *satisfy condition* (3.25), *and let* $I_1, I_2, \ldots, I_{m-r-k-1} \in \mathbf{I}$, *where* $0 \le k \le m - r - 2$, *be the multi-indices of some of its IISs. Let us suppose* $I := \bigcup_{j\in[m-r-k-1]} I_j$. *If* $I \neq [m]$, *then the chosen multi-indices* $I_1, I_2, \ldots, I_{m-r-k-1} \in \mathbf{I}$ *only are precisely those families of the multi-indices of IISs of the system* S_2 *whose union is* I.

Proof. A system S_2 of the form (3.22, 3.23) satisifes condition (3.25) if and only if the multi-index of each of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of a facet of some simple $(m-r-1)$-polytope $\mathcal{P}$ with m vertices. Now the proposition follows from the next observation: let $\mathcal{P}$ be a simple d-polytope, and $F_1, F_2, \dots, F_{d-k}$ its facets, where $0 \le k \le d-1$. Let us suppose $F := \bigcap_{i\in[d-k]} F_i$, and assume that $F \neq \emptyset$. Then F represents a k-dimensional face of the polytope $\mathcal{P}$, and the facets $F_1, F_2, \dots, F_{d-k}$ are precisely those faces of $\mathcal{P}$ that contain F. ☐

Proposition 3.58. *A system* S_2 *of the form* (3.22, 3.23) *satisifes condition* (3.25) *if and only if for any of its IIS, with a multi-index* $I \in \mathbf{I}$*, there exist precisely* $m-r-1$ *multi-indices of IISs* $I_1, I_2, \dots, I_{m-r-1}$ *such that for each* $j \in [m-r-1]$ *it holds* $|I \cup I_j| = r+2$.

Proof. According to Propositions 3.51 and 3.56, the system S_2 satisfies condition (3.24) if and only if the multi-index of any of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of some facet of a simplicial $(m-r-1)$-polytope $\mathcal{P}$ with m vertices, which is dual to a simple $(m-r-1)$-polytope $\mathcal{P}^0$. But $\mathcal{P}^0$ is simple if and only if each of its vertex is incident to precisely $m-r-1$ one-dimensional faces. The one-dimensional faces of the polytope $\mathcal{P}^0$ are anti-isomorphic to the $(m-r-2)$-dimensional simplices of $\mathcal{P}$, from where the proof follows. ☐

The Dehn–Sommerville relations presented in Proposition 3.53 can be reformulated, according to combinatorial duality described in Propositions 3.49 and 3.56, for systems of the form (3.22, 3.23, 3.25) as follows:

Proposition 3.59. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.25)*, let* $\mathbf{I}$ *be the family of the multi-indices of its IISs, and* $\mathfrak{n}_i$ *the number of those subfamilies* $\{I_1, I_2, \dots, I_i\} \subseteq \mathbf{I}$*, for which* $\bigcup_{j\in[i]} I_j = [m]$*. Let us suppose* $\mathfrak{n}_0 := 0$*. Then the following relations hold* x *is a formal variable:*

$$\begin{cases} \mathfrak{n}_0 = \mathfrak{n}_1 = 0\,, \\ \mathfrak{n}_k = \binom{\mathfrak{p}}{k},\ \textit{if}\ m-r \le k \le \mathfrak{p}\ , \\ \sum_{j=r+\mathfrak{p}-m+1}^{\mathfrak{p}} \left(\binom{\mathfrak{p}}{j} - \mathfrak{n}_{\mathfrak{p}-j}\right)(\mathrm{x}-1)^{\mathfrak{p}-j} \\ \qquad = \sum_{j=r+\mathfrak{p}-m+1}^{\mathfrak{p}} (-1)^{j-r-\mathfrak{p}+m-1} \left(\binom{\mathfrak{p}}{j} - \mathfrak{n}_{\mathfrak{p}-j}\right)\mathrm{x}^{\mathfrak{p}-j}\,. \end{cases}$$

Since any subset of vertices of a face of a simplicial polytope is also the vertex set of some of its face, Proposition 3.51 makes it possible to estimate the number of subsystems of different cardinalities in systems of the form (3.22, 3.23, 3.24):

Proposition 3.60. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24)*; let* ν_k *and* τ_k *denote the numbers of its feasible and infeasible subsystems, of cardinality k, respectively. Suppose*

$$\Phi_j(m-r-1, m) := \sum_{i=0}^{\lfloor (m-r-1)/2 \rfloor} \binom{i}{j}\binom{m+r+i}{i} + \sum_{i=0}^{\lfloor (m-r-1)/2 \rfloor -1} \binom{m-r-i-1}{j}\binom{m+r+i}{i}\,.$$

Then

$$\tau_k \le \Phi_{k-r-1}(m-r-1, m)\,,$$
$$\nu_k \ge \binom{m}{k} - \Phi_{k-r-1}(m-r-1, m)\,,$$

for $r+1 \le k \le m-2$.

Proof. According to Proposition 3.51 (i), a system S_2 of the form (3.22, 3.23) satisifies condition (3.24) if and only if the multi-index of any of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of some facet of a simplicial $(m-r-1)$-polytope $\mathcal{P}$ with m vertices. The proposition follows from the *upper bound theorem* proved by McMullen: if we set, for a simplicial d-dimensional polytope with m vertices,

$$\Phi_j(d, m) := \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{i}{j}\binom{m-d+i-1}{i} + \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{d-i}{j}\binom{m-d+i-1}{i}\,,$$

then, for $1 \le j \le d-1$, the number of its j-dimensional faces is at most $\Phi_{d-j-1}(d, m)$. □

Proposition 3.61. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24)*; let* ν_k *and* τ_k *denote the numbers of its feasible and infeasible subsystems, of cardinality* k*, respectively. Suppose*

$$\varphi(m-r-1, m) := \begin{cases} (m-r-2)m - (m-r)(m-r-3), & \text{if } j = 0\,, \\ \binom{m-r-1}{j+1}m - \binom{m-r}{j+1}(m-r-j-2), & \text{if } j \in [m-r-3]\,. \end{cases}$$

Then

$$\tau_k \ge \varphi_{k-r-1}(m-r-1, m)\,,$$
$$\nu_k \le \binom{m}{k} - \varphi_{k-r-1}(m-r-1, m)\,,$$

for $r+1 \le k \le m-2$.

Proof. The argument is similar to that presented in the proof of Proposition 3.60. The proposition follows from the *lower bound theorem* proved by Barnette: if we set, for a simplicial d-dimensional polytope with m vertices,

$$\varphi_j(d, m) := \begin{cases} (d-1)m - (d+1)(d-2), & \text{if } j = 0\,, \\ \binom{d}{j+1}m - \binom{d+1}{j+1}(d-j-1), & \text{if } j \in [d-2]\,, \end{cases}$$

then, for $1 \le j \le d-1$, the number of its j-dimensional faces is at least $\varphi_{d-j-1}(d, m)$. □

Diagonals of cyclic polytopes and MFSs of inequality systems

As noted earlier, one fundamental extremal construction in the problems of combinatorial polytope theory is the *cyclic polytope*. Recall that it is defined as the convex hull of m distinct points on the moment curve $\boldsymbol{x}(t) := (t, t^2, \dots, t^d) \in \mathbb{R}^d$, and it is denoted by $\mathfrak{C}(d, m)$. We will count the number of diagonals of the polytope $\mathfrak{C}(d, m)$ and, as a consequence, we will estimate the number of MFSs of inequality systems. The question on the number of diagonals of the cyclic polytopes is one of the questions of most interest to the combinatorial theory of this important class of polytopes.

Let $\boldsymbol{V} := \{\boldsymbol{x}_i := \boldsymbol{x}(t_i) : i \in [m]\}$ be the vertex tuple of a cyclic polytope $\mathfrak{C}(d, m)$, where $i < j \Rightarrow t_i < t_j$. The polytope $\mathfrak{C}(d, m)$ is simplicial (i.e., any proper face of such a polytope is a simplex) and $\lfloor \frac{d}{2} \rfloor$-neighborly (i.e., the convex hull of any of its $\lfloor \frac{d}{2} \rfloor$ vertices is a face of the polytope). We will call a subtuple $\boldsymbol{X} \subseteq \boldsymbol{V}$ *connected*, if it is of the form $\boldsymbol{X} = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_j)$, for some indices $i \le j$.

Let $\boldsymbol{Y} \subseteq \boldsymbol{V}$. We will call the inclusion-maximal connected subtuples of the tuple $\boldsymbol{Y}$ *components* of the tuple $\boldsymbol{Y}$. A component $\boldsymbol{Y}' \subseteq \boldsymbol{Y}$ with an odd number of elements will be called *odd*, otherwise it will be called *even*. A component $\boldsymbol{Y}' \subseteq \boldsymbol{Y}$ will be called *end* when $\boldsymbol{x}_1 \in \boldsymbol{Y}'$ or $\boldsymbol{x}_m \in \boldsymbol{Y}'$. Any subtuple $\boldsymbol{Y} \subseteq \boldsymbol{V}$ is partitioned into disjoint components. A subtuple $\boldsymbol{Y} \subseteq \boldsymbol{V}$ will be called an (r, s)-tuple if $|\boldsymbol{Y}| = r$ and $\boldsymbol{Y}$ has precisely s odd nonend components. We will denote by $\mathrm{s}(\boldsymbol{Y})$ the number of odd nonend components of the subtuple $\boldsymbol{Y}$. We will use the following characterization of the proper faces of the tuple $\boldsymbol{V}$:

Lemma 3.62. *A tuple $\boldsymbol{X} \subset \boldsymbol{V}$ is a proper face of the tuple $\boldsymbol{V}$ if and only if* $\mathrm{s}(\boldsymbol{X}) \le d - |\boldsymbol{X}|$.

Let us denote by $\boldsymbol{X} \backslash \boldsymbol{x}$ the tuple obtained from the tuple $\boldsymbol{X}$ by removing an element $\boldsymbol{x}$. Since the polytope $\mathfrak{C}(d, m)$ is simplicial, for any proper face $\boldsymbol{X}$ of the tuple $\boldsymbol{V}$ the inclusion $\boldsymbol{X}' \subset \boldsymbol{X}$ implies that the tuple $\boldsymbol{X}'$ is also a (proper) face of the tuple $\boldsymbol{V}$. Thus, the next assertion follows from Lemma 3.41:

Lemma 3.63. *A tuple $\boldsymbol{X} \subset \boldsymbol{V}$ is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{X}$ is not a proper face of the tuple $\boldsymbol{V}$, but $\boldsymbol{X} \backslash \boldsymbol{x}$ is its proper face, for any $\boldsymbol{x} \in \boldsymbol{X}$.*

Lemma 3.64. *Suppose $m \ge d + 2$. Then any diagonal $\boldsymbol{X}$ of the tuple $\boldsymbol{V}$ contains only one-element components.*

Proof. By Lemma 3.63, the tuple $\boldsymbol{X}$ cannot be a proper face of the tuple $\boldsymbol{V}$, therefore, taking into account Lemma 3.62, we have $\mathrm{s}(\boldsymbol{X}) > d - |\boldsymbol{X}|$. Assume that the tuple $\boldsymbol{X}$ has a component $\boldsymbol{X}_0$ that contains at least two elements. Since, by Lemma 3.42, the convex hull $\operatorname{conv} \boldsymbol{X}$ is a simplex then $|\boldsymbol{X}| \le d + 1$. By the hypothesis of the Lemma, we have $m \ge d + 2$; thus, the component $\boldsymbol{X}_0$ does not contain $\boldsymbol{x}_1$ and $\boldsymbol{x}_m$ simultaneously. Specifically, suppose $\boldsymbol{X}_0 = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_j)$, where $i \ne 1$. Let us set $\boldsymbol{x} := \boldsymbol{x}_j$ when the cardinality $|X|$ is even, and $\boldsymbol{x} := \boldsymbol{x}_{i+1}$ when the cardinality $|\boldsymbol{X}|$ is odd. Then it follows from the inequality $\mathrm{s}(\boldsymbol{X}) > d - |\boldsymbol{X}|$ that $\mathrm{s}(\boldsymbol{X}\backslash\boldsymbol{x}) = \mathrm{s}(\boldsymbol{X}) + 1 > d - |\boldsymbol{X}| + 1 = d - |\boldsymbol{X}\backslash\boldsymbol{x}|$, that is, $\boldsymbol{X}\backslash\boldsymbol{x}$ is not a proper face of the tuple $\boldsymbol{X}$, a contradiction with Lemma 3.63. □

Let us denote by $\mathrm{D}(d, m)$ the number of all diagonals, and by $\mathrm{D}_s(d, m)$ the number of diagonals, with s elements, of the tuple $\boldsymbol{V}$ of vertices of a cyclic polytope $\mathfrak{C}(d, m)$.

Proposition 3.65.

$$\mathrm{D}(d, m) = \begin{cases} 1, & \text{if } m \le d + 1\,, \\ 2\binom{m-k-2}{k} + \binom{m-k-2}{k+1}, & \text{if } m \ge d + 2 \text{ and } d = 2k\,, \\ \binom{m-k-2}{k+1} + \binom{m-k-3}{k}, & \text{if } m \ge d + 2 \text{ and } d = 2k + 1\,. \end{cases}$$

Proof. If $m \le d + 1$, then $\mathfrak{C}(d, m)$ is a simplex and, as a consequence, the tuple $\boldsymbol{V}$ has the unique diagonal that coincides with $\boldsymbol{V}$.

Suppose $m \ge d + 2$. Since the polytope $\mathfrak{C}(d, m)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly, it follows from Lemma 3.63 that

$$\mathrm{D}_s(d, m) = 0\,, \quad 1 \le s \le \lfloor \tfrac{d}{2} \rfloor\,. \tag{3.26}$$

1. Suppose $d = 2k$. Let us find the number $\mathrm{D}_{k+1}(d, m)$. Since the polytope $\mathfrak{C}(d, m)$ is k-neighborly, it follows from Lemma 3.63 that a subtuple $\boldsymbol{Y} \subset \boldsymbol{V}$ with $k + 1$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{Y}$ is not a proper face of the tuple $\boldsymbol{V}$. The latter is possible, according to Lemma 3.62, if and only if $\boldsymbol{Y}$ is a $(k + 1, s)$-tuple and $s > d - (k + 1) = k - 1$, that is, $\boldsymbol{Y}$ consists of $k + 1$ one-element components and, besides, $\boldsymbol{x}_1$ and $\boldsymbol{x}_m$ do not belong to $\boldsymbol{Y}$ simultaneously.

The enumeration of such tuples is reduced to the following problem: on a line, $m - (k + 1)$ black points and $m - k$ white points are chosen; besides, every black point is situated between two white points. It is necessary to enumerate all the tuples with $k + 1$ white points that do not contain two white end points simultaneously. We have

$$\mathrm{D}_{k+1}(d, m) = 2\binom{m-k-2}{k} + \binom{m-k-2}{k+1}\,. \tag{3.27}$$

Let us show that $\mathrm{D}_s(d, m) = 0$, when $s > k + 1$. Suppose to the contrary that there exists a diagonal $\boldsymbol{Y}$ of the tuple $\boldsymbol{V}$ that contains $k + 1 + p$ elements, where $p \ge 1$. By Lemma 3.64, the tuple $\boldsymbol{Y}$ consists of $k + 1 + p$ one-element components. By removing from this tuple the first p elements, we get the tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element components, besides, $\boldsymbol{x}_1 \notin \boldsymbol{Y}'$. As shown earlier, such a tuple $\boldsymbol{Y}'$ is a diagonal of the tuple $\boldsymbol{V}$; this contradicts the minimality of $\boldsymbol{Y}$. Thus, $\mathrm{D}_s(d, m) = 0$, when $s > k + 1$, from where, taking into account (3.26) and (3.27), we obtain Proposition 3.65 in the case of $d = 2k$.

2. Suppose $d = 2k + 1$. Let us find the number $\mathrm{D}_{k+1}(2k + 1, m)$. Arguing the same way as in the case of $d = 2k$, we verify that a tuple $\boldsymbol{Y}$ with $k + 1$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{Y}$ consists of $k + 1$ one-element nonend components. Such tuples can be enumerated as was done earlier, that yields

$$\mathrm{D}_{k+1}(2k + 1, m) = \binom{m-k-2}{k+1}\,. \tag{3.28}$$

Let us find the number $\mathrm{D}_{k+2}(2k+1, m)$. Any diagonal $\boldsymbol{Y}$ with $k+2$ elements of the tuple $\boldsymbol{V}$ consists, by Lemma 3.62, of $k+2$ one-element components. Let us show that, besides, $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. Suppose to the contrary that, for example, $\boldsymbol{x}_1 \notin \boldsymbol{Y}$. Then, by removing the last element of the tuple $\boldsymbol{Y}$, we get the tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element nonend components, that is a diagonal of the tuple $\boldsymbol{V}$, a contradiction with the minimality of $\boldsymbol{Y}$.

Now let $\boldsymbol{Y}$ be a tuple consisting of $k+2$ one-element components, and $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. Then $\boldsymbol{Y}$ is a $(k+2, k)$-tuple and, by Lemma 3.62, it is not a proper face of the tuple $\boldsymbol{V}$. By removing any element from $\boldsymbol{Y}$, we obtain a $(k+1, s')$-tuple $\boldsymbol{Y}'$, where $s' \le k \le d-(k+1)$, which is a proper face of the tuple $\boldsymbol{V}$. It follows from Lemma 3.63 that the tuple $\boldsymbol{Y}$ is a diagonal of the tuple $\boldsymbol{V}$. Thus, a tuple $\boldsymbol{Y}$ with $k+2$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if it consists of $k+2$ one-element components, and $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. The number of such tuples is

$$\mathrm{D}_{k+2}(2k+1, m) = \binom{m-k-3}{k} . \tag{3.29}$$

Let us show that $\mathrm{D}_s(2k+1, m) = 0$ when $s > k+2$. Suppose to the contrary that there exists a diagonal $\boldsymbol{Y}$ of the tuple $\boldsymbol{V}$ that consists of $k+1+p$ elements, where $p \ge 2$. By removing from this tuple the first $p-1$ elements and the last element, we obtain a tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element nonend components. As we saw earlier, such a tuple is a diagonal of the tuple $\boldsymbol{V}$ that contradicts the minimality of $\boldsymbol{Y}$. Thus, $\mathrm{D}_s(d, m) = 0$ when $s > k+2$, from where, taking into account relations (3.26), (3.28) and (3.29), we obtain Proposition 3.65 in the case of $d = 2k+1$. □

Corollary 3.66. *The maximal number of MFSs of irreducible systems of the form* (3.22, 3.23)*, of rank $r \ge 1$, of $m \ge 2$ inequalities is at least*

$$\begin{cases} 2\binom{k-1}{n-1} + \binom{k-1}{n-2} , & \text{if } m+n = 2k+1 , \\ \binom{k-1}{n-1} + \binom{k-2}{n} , & \text{if } m+n = 2k . \end{cases}$$

Notes

The notion for which we use in Section 3.1 the term A-*diagonal* was introduced in [8] under the name *missing face* (or *missing simplex*); the construction called G-*diagonal* was considered in [19] as *minimal diagonal* and, independently, in work [50] under the name *diagonal*; F-*diagonals* were introduced in [45] under the name *diagonals*. See also works [69–71, 108] on the missing faces of polytopes.

The notion of missing face is also standard in the theory of abstract simplicial complexes: let a complex Δ with vertex set V be given. A subset $N \subseteq V$ is called a *missing face* of the complex Δ if $N \notin \Delta$, but any proper subset of the set N is a face of the complex Δ, see, for example, in [25, §2.2].

We say on page 59 that a polytope $\mathcal{P}$ is obtained by the operation of *cross*, borrowing terminology from [35].

Propositions 3.4 and 3.7 are actually proved in [19], in the language of cones.

The Baire category theorem mentioned in Example 3.12 is presented, for example, in [73, Ch. 6].

The assertion that the face structure of a simplicial polytope is determined by the structure of the family of its A-diagonals, mentioned on page 66, is proved in [8, Th. 2.4].

The theory of positive bases of a finite dimensional space is a construction that is well known in combinatorial geometry [31, 118, 119, 129]; see also [29, Ch. 2], [128, Ch. 1].

The statement, presented on page 67, that for a positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ the inequalities $n + 1 \leq |\boldsymbol{B}| \leq 2n$ hold, is discussed, for example, in work [129].

A set $\boldsymbol{X}$ which is a minimal basis of the space lin $\boldsymbol{X}$ is called in work [20] a *minimally dependent set*. In our study, we use the term *minimal sub-basis* for emphasizing the origin of the minimally dependent sets under consideration. On the basis of well-known facts – see, for example, in [117, Lemma 2.4] – note that the minimal sub-bases of a positive basis $\boldsymbol{B}$, defined in such a manner, are precisely all its inclusion-minimal sub-bases.

We recall on page 68 that a positive basis $\boldsymbol{B}$, with $n + r$ points, of $\mathbb{R}^n$ is a SPB if and only if there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_r$, where $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_r$ are pairwise disjoint minimal sub-bases of the positive basis $\boldsymbol{B}$; see on this subject in works [117, 118, 129].

Proposition 3.18 is proved in [117, 118].

On page 68, we for brevity say that a set $\boldsymbol{X} \subset \mathbb{R}^n$ is *one-sided* if it is contained entirely in an open half-space bounded by a linear hyperplane; note that such a set $\boldsymbol{X}$ is called in work [20] *strict one-sided*.

The notion of *diagram of a positive basis* useful for investigating positive bases was introduced in work [129].

We consider on page 68 a tuple $\boldsymbol{B}$ of vectors that span positively the space $\mathbb{R}^r$, and the corresponding *linear representation* $\boldsymbol{E}$. The fact that the point set from the tuple $\boldsymbol{E}$ is one-sided is mentioned in [129].

Assertions (i)–(iv) of Proposition 3.19 are also proved in work [129].

In the proof of Proposition 3.25, we discuss the impossibility of the separation of the sets $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ in a hyperplane $\mathbf{H}$ by a plane $\boldsymbol{E}$; we are supported in this argument by the theorem on the separability of convex sets from [64, Ch. 2], see Lemma 3.35; see also, for example, in [14, §8], [26, §III.28], [84, §1.3], [150, §4.5]. See [41, §1.12], [42, §1.12], [146, §II.7] on the separability of polyhedral sets.

The proof of Proposition 3.27 is supported by Carathéodory's theorem; see, for example, in [14, §3.7], [24, §1.2], [64, §2.3], [84, §1.2], [117], [163, Lect. 1] on this classical result of convex analysis.

Another classical statement on convex sets, Helly's theorem, is used in the proof of the implication (vi)⇒(iv) of Proposition 3.28; Helly-type assertions can be found, for example, in [14, §3.7], [24, §1.2], [64, Ch. 2], and [120, §IV.21].

The proof of Proposition 3.33 is completed by making reference to a known fact that can be found, for example, in [117, Th. 2.1].

Lemma 3.34 is presented in [146] and, as noted earlier, Lemma 3.35 can be found in [64, Ch. 2].

The relation $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) + \dim \boldsymbol{K}^* = r$ and the related equality $(\boldsymbol{K} \cap -\boldsymbol{K}) + \operatorname{lin} \boldsymbol{K}^* = \mathbb{R}^r$, mentioned in the proof of Proposition 3.36, are given in [120].

Gale transforms and diagrams of a point tuple are powerful tools of studying in combinatorial geometry and in the theory of polytopes, see, for example, in [15, 60, 65, 91, 100, 156], [92, §5.6], [140, Ch. 5], [152, §3.6]. In the mentioned works, in particular, one can find the properties of Gale transforms and diagrams that are presented in Proposition 3.37 and Corollary 3.38.

Lemma 3.39 is proved in work [60].

We consider combinatorially dual systems of linear inequalities and, in particular, their relationship with simplicial and simple polytopes, and we present the Dehn–Sommerville relations, and estimate the number of subsystems, following [93]; see also [56, 57].

The assertion, used in the proof of Proposition 3.49, on the existence for any convex polytope of its dual polytope (the face lattices of such polytopes are by definition anti-isomorphic) is a fundamental observation of convex analysis; see, for example, in [24, §2.10], [64, §3.4], [163, §2.3]. A polytope dual to a pyramid is also a pyramid of the same dimension [24, §2.10].

The Dehn–Sommerville relations for the Whitney numbers of the second kind, taken as the basis of the proof of Proposition 3.53, are formulated in [134, §3.14].

The equivalent reformulations of the Dehn–Sommerville relations, given in the proof of Corollary 3.55, can be found in [156, §1.5].

The observation, mentioned in the proof of Proposition 3.57, is discussed in [24, Th. 12.14].

In the proof of Proposition 3.58, we recall that each vertex of a simple polytope is incident to the same number of its one-dimensional faces; see on this in [24, Th. 12.12].

The upper bound theorem, proved by P. McMullen, which is taken as the basis of the proof of Proposition 3.60, is reproduced in [24, Corollary 18.3]. The lower bound theorem – see the proof of Proposition 3.61 – was proved by D. Barnette; it is presented in [24, Corollary 19.6].

We recall on page 88 that the cyclic polytopes are simplicial and highly neighborly; see on this, for example, in [64, §4.7], [156, §1.2].

We say on the connected subtuples of the vertex set of a cyclic polytope, following [101]. Lemma 3.62 is also proved in [101].