4 Monotone Boolean functions, complexes, graphs, and inequality systems

The multi-indices of the subsystems of infeasible systems with the monotonicity property and, in particular, the multi-indices of the subsystems of infeasible systems of linear inequalities determine a partition of the Boolean lattice of multi-indices into two subposets that correspond to the feasible and infeasible subsystems. This partition is uniquely determined by the so-called *border* that is the common collection of the multi-indices of maximal feasible and minimal infeasible subsystems; the family of the multi-indices of MFSs is naturally regarded as the facet family of an abstract simplicial complex. In terms of monotone Boolean functions, the multi-indices of MFSs and IISs correspond to the upper zeros and lower units of some Boolean function which is assigned to the inequality system under consideration, see Chapter 1.

In this chapter, we investigate the relationship of the problems of searching for the maximal feasible subsystems of an inequality system with the problem of optimal inference of monotone Boolean functions. Inference lies on the basis of numerous applications; for this reason, we will explain in detail a specific approach to its efficient realization.

4.1 Optimal inference of monotone Boolean functions

Let us recall several constructions and notation which we used earlier in Section 1.2.

For binary tuples $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} := (\beta_1, \beta_2, \ldots, \beta_m)$ from the unit discrete m-dimensional cube $\mathbf{B}^m := \{0, 1\}^m$, the ordering $\boldsymbol{\alpha} \le \boldsymbol{\beta}$, by definition holds if and only if $\alpha_i \le \beta_i$, for all $i \in [m]$. If $\mathcal{A} \subseteq \mathbf{B}^m$, then $\mathbf{max}\,\mathcal{A}$ and $\mathbf{min}\,\mathcal{A}$ denote the sets of all maximal elements and all minimal elements of the set $\mathbf{max}\,\mathcal{A}$, with respect to that ordering, respectively.

The number of units in a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ will be denoted by $|\boldsymbol{\alpha}|$.

We will denote by $\boldsymbol{\alpha} \oplus \boldsymbol{\beta}$ the coordinate-wise summation of the tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ over the set $\mathbf{B}$ equipped with the properties of the finite field $\mathbb{F}_2$ with two elements.

Any monotone Boolean function (MBF) $\mathfrak{f}\colon\ \mathbf{B}^m \to \mathbf{B}$, which is a map for which the implications

$$\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{B}^m,\ \boldsymbol{\alpha} \le \boldsymbol{\beta} \quad \Longrightarrow \quad \mathfrak{f}(\boldsymbol{\alpha}) \le \mathfrak{f}(\boldsymbol{\beta}) \tag{4.1}$$

hold, induces the partition $\mathbf{B}^m = \mathfrak{f}^{-1}(0) \,\dot\cup\, \mathfrak{f}^{-1}(1)$ of the cube $\mathbf{B}^m$ into the preimages of the elements from the set $\mathbf{B}$. Under such a partitioning, the family

$$\mathfrak{I}(\mathbf{J}) := \left\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)\right\}, \tag{4.2}$$

interpreted as a subset of the Boolean lattice $\mathbb{B}(m)$ of subsets of the index set $[m]$, represents its order ideal generated in $\mathbb{B}(m)$ by the family $\mathbf{J}$ of the inclusion-maximal

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sets from the family $\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)\}$. The ideal $\mathfrak{I}(\mathbf{J})$ is the face poset of the abstract simplicial complex $\Delta(\mathbf{J})$ with the facet family $\mathbf{J}$.

Similarly, it follows from monotonicity property (4.1) that the family

$$\mathfrak{F}(\mathbf{I}) := \{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)\}$$

can be regarded as an order filter of the lattice $\mathbb{B}(m)$ generated by the family $\mathbf{I}$ of inclusion-minimal sets from the family $\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)\}$.

Recall that the set $\mathfrak{f}^{-1}(0)$ consists of the *zeros* of the function $\mathfrak{f}$, and the set $\mathfrak{f}^{-1}(1)$ consists of the *units* of this function. The subset $\mathfrak{Q}(\mathfrak{f}) := \mathbf{max}\, \mathfrak{f}^{-1}(0)$ of maximal elements of the poset $\mathfrak{f}^{-1}(0)$ is the set of *upper zeros* of the function $\mathfrak{f}$; the subset $\mathfrak{P}(\mathfrak{f}) := \mathbf{min}\, \mathfrak{f}^{-1}(1)$ of minimal elements of the poset $\mathfrak{f}^{-1}(1)$ is the set of *lower units* of the function $\mathfrak{f}$.

An upper zero $\boldsymbol{\alpha} \in \mathfrak{Q}(\mathfrak{f})$ of the function $\mathfrak{f}$ is called *maximal* if $|\boldsymbol{\alpha}| = \max_{\boldsymbol{\beta} \in \mathfrak{Q}(\mathfrak{f})} |\boldsymbol{\beta}|$. Dually, a lower unit $\boldsymbol{\alpha} \in \mathfrak{P}(\mathfrak{f})$ of the function $\mathfrak{f}$ is called *minimal* if $|\boldsymbol{\alpha}| = \min_{\boldsymbol{\beta} \in \mathfrak{P}(\mathfrak{f})} |\boldsymbol{\beta}|$.

Let us denote the class of all monotone Boolean functions of m variables by $\mathcal{M}_m$.

Let us assign to the function $\mathfrak{f} \in \mathcal{M}_m$ an *oracle* $\mathcal{O}_\mathfrak{f}$ that is an operator making it possible to compute for an arbitrary point $\boldsymbol{\alpha} \in \mathbf{B}^m$ the value of the function $\mathfrak{f}$ at this point. *Inference* of an *a priori* unknown monotone Boolean function means its reconstruction with the use of the oracle $\mathcal{O}_\mathfrak{f}$. The problem of constructing the algorithms of MBF inference which require the least, in a sense, number of invocations of the oracle, is fundamental.

Given some algorithm G, let $\varphi(G, \mathfrak{f})$ denote the number of its calls of the operator $\mathcal{O}_\mathfrak{f}$ when inferring the function $\mathfrak{f} \in \mathcal{M}_m$. Optimality of the algorithm G, in the sense of the number of invocations of the operator $\mathcal{O}_\mathfrak{f}$, can be ranked, for example, by the following functionals:

$$\varphi(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \varphi(G, \mathfrak{f}) \,, \tag{4.3}$$

$$\eta(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \frac{\varphi(G, \mathfrak{f})}{|\mathfrak{Q}(\mathfrak{f}) \dot{\cup} \mathfrak{P}(\mathfrak{f})|} \,, \tag{4.4}$$

$$\eta_1(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \big(\varphi(G, \mathfrak{f}) - |\mathfrak{Q}(\mathfrak{f}) \dot{\cup} \mathfrak{P}(\mathfrak{f})|\big) \,, \tag{4.5}$$

$$\eta_2(G, m) = \sum_{\mathfrak{f} \in \mathcal{M}_m} \varphi(G, \mathfrak{f}) \,. \tag{4.6}$$

Let us consider the quantity $\varphi(m) := \min_G \varphi(G, m)$, where the minimum is found over all algorithms G of inference of the MBFs of m variables. Let us use the analogous notation $\eta(m)$, $\eta_1(m)$, and $\eta_2(m)$ for criteria (4.4)–(4.6).

The classical, well-known and well-examined criterion $\varphi(G, m)$, defined in (4.3), is called *Shannon's criterion*. For an optimal, with respect to the criterion $\varphi(G, m)$, algorithm of MBF inference, the relation $\varphi(m) = \binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1}$ holds.

At the same time, an algorithm which is optimal in the Shannon formulation is inadequate for miscellaneous practical purposes.

Let us analyze the complexity of inference algorithms with respect to the criterion $\eta(G, m)$.

During the inference process, such algorithms must invoke the values of the function on the set $\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})$. This means that $\varphi(G, \mathfrak{f}) \geq |\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$, for any algorithm G and for any monotone Boolean function $\mathfrak{f} \in \mathcal{M}_m$. Thus, the criterion $\eta(G, m)$ formalizes the natural requirement according to which the computational effort of the algorithm G when inferring the function $\mathfrak{f}$, measured by the quantity $\varphi(G, \mathfrak{f})$, should be proportional to the objective complexity of the inference problem for the function $\mathfrak{f}$, measured by the quantity $|\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$.

The inference process for the function $\mathfrak{f} \in \mathcal{M}_m$, which is realized by Algorithm G, can be described by the sequence

$$G(\mathfrak{f}) = \big(\, g_1(\mathfrak{f}), \mathfrak{f}(g_1(\mathfrak{f})),\ g_2(\mathfrak{f}), \mathfrak{f}(g_2(\mathfrak{f})),\ \ldots,\ g_k(\mathfrak{f}), \mathfrak{f}(g_k(\mathfrak{f}))\,\big)\,,$$

of tuples $g_i(\mathfrak{f}) \in \mathbf{B}^m$, chosen by the algorithm, and of the corresponding values $\mathfrak{f}(g_i(\mathfrak{f})) \in \mathbf{B}$ of the function $\mathfrak{f}$, for $i \in [\varphi(G, \mathfrak{f})]$. In other words, $g_i(\mathfrak{f})$ can be interpreted as sequential calls, by the algorithm G, of the operator $\mathcal{O}_\mathfrak{f}$ during the inference process for the function $\mathfrak{f}$, and $\mathfrak{f}(g_i(\mathfrak{f}))$ are the responses of the operator $\mathcal{O}_\mathfrak{f}$.

We will consider below those algorithms G of MBF inference only, for which the sequence $G(\mathfrak{f})$ is determined for each function $\mathfrak{f} \in \mathcal{M}_m$ uniquely.

Suppose $\mathcal{A} \subseteq \mathbf{B}^m$. Let us denote by $\mathfrak{M}_\mathfrak{f}(\mathcal{A})$ the set of all those points $\boldsymbol{\alpha}$ of the unit cube $\mathbf{B}^m$, at which the values of the function $\mathfrak{f}$ are determined, by the monotonicity property, uniquely by its values at the set $\mathcal{A}$, that is, $\boldsymbol{\alpha} \in \mathfrak{M}_\mathfrak{f}(\mathcal{A})$ if there exists a tuple $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$ such that $\boldsymbol{\beta} \in \mathcal{A} \cap \mathfrak{f}^{-1}(0)$, or a tuple $\boldsymbol{\beta}' \leq \boldsymbol{\alpha}$ such that $\boldsymbol{\beta}' \in \mathcal{A} \cap \mathfrak{f}^{-1}(1)$.

In what follows, we will need, as standard procedures, a routine UZ($\mathfrak{f}, \boldsymbol{\alpha}$) of extracting, on the basis of a point $\boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)$, an upper zero $\boldsymbol{\alpha}'$ of the function $\mathfrak{f}$, such that $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$, and a routine LU($\mathfrak{f}, \boldsymbol{\alpha}$) of extracting, on the basis of a point $\boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)$, a lower unit $\boldsymbol{\alpha}'$, such that $\boldsymbol{\alpha} \geq \boldsymbol{\alpha}'$. The routines UZ($\mathfrak{f}, \boldsymbol{\alpha}$) and LU($\mathfrak{f}, \boldsymbol{\alpha}$) work in accordance to the standard scheme, computing the values of the function $\mathfrak{f}$ at some tuples with the help of the operator $\mathcal{O}_\mathfrak{f}$.

The routine UZ($\mathfrak{f}, \boldsymbol{\alpha}$)

Let a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ contain $k < m$ units $\alpha_i = 1$. Let us re-index the zeros in $\boldsymbol{\alpha}$ from left to right. Let us denote by $\boldsymbol{\beta}^i$ the binary tuple containing $m-1$ zeros and a single unit at the position of the ith zero of the tuple $\boldsymbol{\alpha}$. Then the sequence

$$\big(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1),\ \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2),\ \ldots,\ \boldsymbol{\alpha}^{m-k}, \mathfrak{f}(\boldsymbol{\alpha}^{m-k})\big)\,, \tag{4.7}$$

where $\boldsymbol{\alpha}^1 := \boldsymbol{\alpha} \oplus \boldsymbol{\beta}^1$ and $\boldsymbol{\alpha}^i := \boldsymbol{\alpha} \oplus \boldsymbol{\beta}^i \oplus (1 - \mathfrak{f}(\boldsymbol{\alpha}^1))\boldsymbol{\beta}^1 \oplus (1 - \mathfrak{f}(\boldsymbol{\alpha}^2))\boldsymbol{\beta}^2 \oplus \cdots \oplus (1 - \mathfrak{f}(\boldsymbol{\alpha}^{i-1}))\boldsymbol{\beta}^{i-1}$, determines the upper zero $\boldsymbol{\alpha}' \in \mathfrak{Q}(\mathfrak{f})$ such that $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$, namely

$$\boldsymbol{\alpha}' = \mathbf{max}\,\{\{\boldsymbol{\alpha}\} \cup \{\boldsymbol{\alpha}^i :\ i \in [m-k],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\}\}\,.$$

The routine LU($\mathfrak{f}, \boldsymbol{\alpha}$)
Let a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ contain $k > 0$ units $\alpha_i = 1$. Let us re-index the units in $\boldsymbol{\alpha}$ from left to right. Let us denote by $\boldsymbol{y}^i$ the binary tuple containing $m - 1$ zeros and a single unit at the position of the ith unit of the tuple $\boldsymbol{\alpha}$. Then the sequence

$$(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1),\ \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2),\ \dots,\ \boldsymbol{\alpha}^k, \mathfrak{f}(\boldsymbol{\alpha}^k)) \,, \tag{4.8}$$

where $\boldsymbol{\alpha}^1 := \boldsymbol{\alpha} \oplus \boldsymbol{y}^1$ and $\boldsymbol{\alpha}^i := \boldsymbol{\alpha} \oplus \boldsymbol{y}^i \oplus \mathfrak{f}(\boldsymbol{\alpha}^1)\boldsymbol{y}^1 \oplus \mathfrak{f}(\boldsymbol{\alpha}^2)\boldsymbol{y}^2 \oplus \cdots \oplus \mathfrak{f}(\boldsymbol{\alpha}^{i-1})\boldsymbol{y}^{i-1}$, determines the lower unit $\boldsymbol{\alpha}' \in \mathfrak{P}(\mathfrak{f})$ such that $\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}$, namely

$$\boldsymbol{\alpha}' = \mathbf{min}\,\{\{\boldsymbol{\alpha}\} \cup \{\boldsymbol{\alpha}^i :\ i \in [k],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\}\} \,.$$

Sequences (4.7) and (4.8) themselves will be denoted below by UZ($\mathfrak{f}, \boldsymbol{\alpha}$) and LU($\mathfrak{f}, \boldsymbol{\alpha}$). Note that the sequence UZ($\mathfrak{f}, \boldsymbol{\alpha}$) contains $m - |\boldsymbol{\alpha}|$ invocations of the operator $\mathcal{O}_{\mathfrak{f}}$, and the sequence LU($\mathfrak{f}, \boldsymbol{\alpha}$) contains $|\boldsymbol{\alpha}|$ invocations of the operator $\mathcal{O}_{\mathfrak{f}}$. By definition we will suppose that the sequence UZ($\mathfrak{f}, \boldsymbol{\alpha}$) is empty when $|\boldsymbol{\alpha}| = m$, and the sequence LU($\mathfrak{f}, \boldsymbol{\alpha}$) is empty when $|\boldsymbol{\alpha}| = 0$.

Let us denote by $G(\mathfrak{f}, \boldsymbol{\alpha})$ the sequence of the form

$$G(\mathfrak{f}, \boldsymbol{\alpha}) := \begin{cases} (\boldsymbol{\alpha}, 0,\ \mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha})), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 0\,, \\ (\boldsymbol{\alpha}, 1,\ \mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha})), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 1\,. \end{cases}$$

It follows from the definition that the sequence $G(\mathfrak{f}, \boldsymbol{\alpha})$ contains an upper zero $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$ of the function $\mathfrak{f}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 0$, and $G(\mathfrak{f}, \boldsymbol{\alpha})$ contains a lower unit $\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}$ of the function $\mathfrak{f}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 1$. Let us use the common notation $\arg G(\mathfrak{f}, \boldsymbol{\alpha})$ for these elements.

Let us denote by $\mathcal{B}(\mathbf{B}^m)$ the family of all subsets of the unit cube $\mathbf{B}^m$. Given a fixed choice function $\psi\colon \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$, define an algorithm of MBF inference, as follows:

$$G_\psi(\mathfrak{f}) := \big(G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \dots, G(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)\,, \tag{4.9}$$

where $\boldsymbol{\alpha}^1 := \psi(\mathbf{B}^1)$ and $\boldsymbol{\alpha}^i := \psi(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$.

The inference process for the function $\mathfrak{f}$ is completed by the algorithm G_ψ when we have $\mathfrak{M}_{\mathfrak{f}}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [k]\}) = \mathbf{B}^m$. Analyzing definition (4.9) of the sequence $G_\psi(\mathfrak{f})$, we can conclude that

$$\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [\varphi(G_\psi, \mathfrak{f})]\} = \mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})\,, \tag{4.10}$$

$$|\{\boldsymbol{\alpha}^i\colon\ i \in [\varphi(G_\psi, \mathfrak{f})],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\}| = |\mathfrak{Q}(\mathfrak{f})|\,, \tag{4.11}$$

$$|\{\boldsymbol{\alpha}^i\colon\ i \in [\varphi(G_\psi, \mathfrak{f})],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\}| = |\mathfrak{P}(\mathfrak{f})|\,. \tag{4.12}$$

Proposition 4.1. *For any choice function $\psi\colon\ \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$, the inequality $\eta(G_\psi, m) \leq m + 1$ holds.*

Proof. The sequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ contains at most $m + 1$ invocations of the operator $\mathcal{O}_\mathfrak{f}$ when $|\boldsymbol{\alpha}^i| = 0$ or $|\boldsymbol{\alpha}^i| = m$, and at most m invocations otherwise. It follows from the above argument, taking into account (4.10), that

$$\varphi(G_\psi, \mathfrak{f}) \leq m|\mathfrak{Q}(\mathfrak{f})| + m|\mathfrak{P}(\mathfrak{f})| + 2\,. \tag{4.13}$$

For the functions *identically zero* $\mathfrak{f}_0(\boldsymbol{\alpha}) \equiv 0$ and *identically unit* $\mathfrak{f}_1(\boldsymbol{\alpha}) \equiv 1$, we have $\eta(G_\psi, \mathfrak{f}_0) \leq m+1$ and $\eta(G_\psi, \mathfrak{f}_1) \leq m+1$, respectively. If $\mathfrak{f} \notin \{\mathfrak{f}_0, \mathfrak{f}_1\}$ then $|\mathfrak{P}(\mathfrak{f}) \dot\cup \mathfrak{Q}(\mathfrak{f})| \geq 2$, and the proposition follows from (4.13). □

Proposition 4.2. *Let* $\psi\colon \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$ *be an arbitrary choice function,* $\mathfrak{f} \in \mathcal{M}_m$*, and let* $\boldsymbol{\alpha}^i$ *be a fixed item of the sequence* $G_\psi(\mathfrak{f}) = (G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k))$.

(i) *If* $\boldsymbol{\alpha}^i \in \mathbf{min}\,(\mathbf{B}^m - \mathcal{M}_m(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$, $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$*, then* $\boldsymbol{\alpha}^i$ *is a minimal lower unit of the function* $\mathfrak{f}$*, and thus* $\boldsymbol{\alpha}^i = \arg G(\mathfrak{f}, \boldsymbol{\alpha}^i)$.

(ii) *If* $\boldsymbol{\alpha}^i \in \mathbf{min}\,(\mathbf{B}^m - \mathcal{M}_m(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$, $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$*, then* $\boldsymbol{\alpha}^i$ *is a maximal upper zero of the function* $\mathfrak{f}$*, and thus* $\boldsymbol{\alpha}^i = \arg G(\mathfrak{f}, \boldsymbol{\alpha}^i)$.

Proof. Let us prove assertion (i). Let $\{\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^l\}$ be the set of elements from $\mathbf{B}^m$ which are covered by the element $\boldsymbol{\alpha}^i$ in the poset $\mathcal{B}(\mathbf{B}^m)$. Then $\boldsymbol{\beta}^j \in \mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon s \in [i-1]\})$, $j \in [l]$, because of the minimality of $\boldsymbol{\alpha}^i = \mathbf{min}\,\{\mathbf{B}^m - \mathfrak{M}_\mathfrak{f}\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon s \in [i-1]\}\}$, and $\mathfrak{f}(\boldsymbol{\beta}^j) = 0$, $j \in [l]$, because otherwise the element $\boldsymbol{\alpha}^i$ would belong to the set $\mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\})$, which contradicts the choice of the element $\boldsymbol{\alpha}^i$ of the sequence $G_\psi(\mathfrak{f})$ in (4.9). It follows from the above argument that $\boldsymbol{\alpha}^i$ is a minimal lower unit of the function $\mathfrak{f}$.

Assertion (ii) is proved similarly. □

Proposition 4.2 is of applied significance because the proposition makes it possible, for some choice functions, to simplify the corresponding algorithms of inference G_ψ. Let us consider two interesting examples of the choice functions. Let ψ be an arbitrary choice function. Suppose, for any $\mathcal{A} \subseteq \mathbf{B}^m$,

$$\psi_0(\mathcal{A}) := \psi(\mathbf{min}\,\mathcal{A})\,, \tag{4.14}$$

$$\psi_1(\mathcal{A}) := \psi(\mathbf{max}\,\mathcal{A})\,. \tag{4.15}$$

Let us consider the sequence $G_{\psi_0}(\mathfrak{f}) := (G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k))$. It follows from Proposition 4.2 that any element $\boldsymbol{\alpha}^i$ from $G_{\psi_0}(\mathfrak{f})$, such that $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$, is a minimal lower unit of the function $\mathfrak{f}$. This means that the subsequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ of the sequence $G_{\psi_0}(\mathfrak{f})$, in the case when $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$, can be substituted by $(\boldsymbol{\alpha}^i, 1)$, without affecting the result of inference. The following algorithm is thus defined:

Algorithm G'_{ψ_0}**,**

with the inference sequence $G'_{\psi_0}(\mathfrak{f}) := \big(G'(\mathfrak{f}, \boldsymbol{\alpha}^1), G'(\mathfrak{f}, \boldsymbol{\alpha}^2), \dots, G'(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$, where

$$G'(\mathfrak{f}, \boldsymbol{\alpha}^i) := \begin{cases} (\boldsymbol{\alpha}^i, 0, \mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha}^i)), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\,, \\ (\boldsymbol{\alpha}^i, 1), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\,, \end{cases}$$

$$\boldsymbol{\alpha}^1 := \psi_0(\mathbf{B}^m) = (0, 0, \dots, 0)\,,$$

$$\boldsymbol{\alpha}^i := \psi_0(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [i-1]\}))\,,$$

$$\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) := \begin{cases} \arg G(\mathfrak{f}, \boldsymbol{\alpha}^s), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 0\,, \\ \boldsymbol{\alpha}^s, & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 1\,. \end{cases}$$

The inference process for the function $\mathfrak{f}$ is completed by Algorithm G' when we have $\mathfrak{M}_{\mathfrak{f}}\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [k]\} = \mathbf{B}^m$.

▷ In the description of Algorithm G_{ψ_0}, let us substitute 1 by 0, and 0 by 1 in all positions, except for $\boldsymbol{\alpha}^1$, and $\mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha}^i)$ by $\mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha}^i)$; we obtain *Algorithm* G'_{ψ_1} that is a modification of Algorithm G_{ψ_1} on the basis of Proposition 4.2.

Proposition 4.3. *Let* $\psi \colon \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$ *be an arbitrary choice function, and let the quantities* ψ_0 *and* ψ_1 *be defined by relations* (4.14) *and* (4.15). *Then*

$$\varphi(G'_{\psi_0}, \mathfrak{f}) \le m|\mathfrak{Q}(\mathfrak{f})| + |\mathfrak{P}(\mathfrak{f})| + 1\,, \tag{4.16}$$

$$\varphi(G'_{\psi_1}, \mathfrak{f}) \le |\mathfrak{Q}(\mathfrak{f})| + m|\mathfrak{P}(\mathfrak{f})| + 1\,. \tag{4.17}$$

Proof. Let us prove inequality (4.16). It follows from the definition of Algorithm G'_{ψ_0} and from Proposition 4.2 that relations (4.10) also hold for the sequence $G'_{\psi_0}(\mathfrak{f})$, from where, taking into account the definition of the sequence $G'(\mathfrak{f}, \boldsymbol{\alpha}^i)$, we obtain (4.16).

Relation (4.17) is proved similarly. □

Thus, the algorithm G'_{ψ_0} is efficient for inferring monotone Boolean functions with a relatively small number of maximal upper zeros.

According to the following proposition, the function $\eta(m)$ is not bounded by a constant, uniformly for all m.

Proposition 4.4. *For the function* $\eta(m)$*, it holds*

$$\max\{2, \log_2 m^{1/2}\} \le \eta(m) \le \lfloor \tfrac{m}{2} \rfloor + 2\,.$$

Proof. Let us prove the lower bound. Let us first show that $\eta(m) \ge \max\{2, \log_2 m^{1/2}\}$. Let G be an arbitrary inference algorithm. Suppose $H(k) := \{\mathfrak{f} \in \mathcal{M}_m \colon\ |\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})| \le k\}$. For $i \in [m]$, define the function $\mathfrak{f}_i \in \mathcal{M}_m$ with the single maximal upper zero $\mathfrak{Q}(\mathfrak{f}_i) = \{(1, 1, \dots, 1, 0, 1, \dots, 1)\}$, where the zero is situated at the ith position, and with the single minimal lower unit $\mathfrak{P}(\mathfrak{f}_i) = \{(0, 0, \dots, 0, 1, 0, \dots, 0)\}$; the unit is situated at the ith position. Thus, $|H(2)| \ge m$. Suppose $l := \max\{\varphi(G, \mathfrak{f}) \colon\ \mathfrak{f} \in H(2)\}$. It follows from the requirement of unambiguous determining of the subsequence $G(\mathfrak{f})$ that the number of MBFs $\mathfrak{f} \in \mathcal{M}_m$, for which $\varphi(G, \varphi) \leq l$, does not exceed the number of the binary tuples of length l, that is 2^l. As a consequence, the definition of the quantity l implies that $|H(2)| \geq m$, and we get $l \geq \log_2 m$. Then for some function $\mathfrak{f} \in H(2)$, we have $\varphi(G, \mathfrak{f}) \geq \log_2 m$, and thus $\eta(G, m) \geq \log_2 m^{1/2}$. The inequality $\eta(m) \geq 2$ is proved similarly. The lower bound is verified.

In order to prove the inequality $\eta(m) \leq \lfloor \frac{m}{2} \rfloor + 2$, let us present a specific algorithm G of MBF inference, with $\eta(G, m) \leq \lfloor \frac{m}{2} \rfloor + 2$. Let us introduce for the cube $\mathbf{B}^m$ another relation $\preceq$ of partial ordering: we set $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ if and only if $|\boldsymbol{\alpha}| \leq |\boldsymbol{\beta}|$. Denote by $\mathbf{B}^{m,k}$ the set of all binary tuples $\boldsymbol{\alpha} \in \mathbf{B}^m$ that contain precisely k units. Let us define the choice function $\psi_2 : \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$:

$$\psi_2(\mathcal{A}) := \begin{cases} \boldsymbol{\beta}^1 := \psi(\mathbf{max}_{\preceq}\mathbf{min}_{\preceq}\mathcal{A}), & \text{if } |\boldsymbol{\beta}^1| > \lfloor \frac{m}{2} \rfloor \,, \\ \boldsymbol{\beta}^2 := \psi(\mathbf{min}_{\preceq}\mathbf{max}_{\preceq}\mathcal{A}), & \text{if } |\boldsymbol{\beta}^1| \leq \lfloor \frac{m}{2} \rfloor \text{ and } |\boldsymbol{\beta}^2| \leq \lfloor \frac{m}{2} \rfloor \,, \\ \boldsymbol{\beta}^3 := \psi(\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A}) \,, & \text{if } |\boldsymbol{\beta}^1| \leq \lfloor \frac{m}{2} \rfloor \leq |\boldsymbol{\beta}^2| \,, \end{cases}$$

where ψ means an arbitrary and fixed choice function for $\mathcal{A} \subseteq \mathbf{B}^m$.

Let us prove that this choice function is well defined, that is, the function $\psi_2(\mathcal{A})$ is defined for any tuple set $\mathcal{A} \subseteq \mathbf{B}^m$. For this, it suffices to show that $|\boldsymbol{\beta}^1| \leq \lfloor \frac{m}{2} \rfloor \leq |\boldsymbol{\beta}^2|$ implies the relation $\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A} \neq \emptyset$. Let $(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \ldots, \boldsymbol{\gamma}^k)$ be an inclusion-maximal chain of the poset $(\mathcal{A}, \leq)$. It is evident that $\boldsymbol{\gamma}^1 \in \mathbf{min}\,\mathcal{A}$ and $\boldsymbol{\gamma}^k \in \mathbf{max}\,\mathcal{A}$, therefore, it follows from $|\boldsymbol{\beta}^1| \leq \lfloor \frac{m}{2} \rfloor \leq |\boldsymbol{\beta}^2|$ and from the definition of the tuples $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ that $|\boldsymbol{\gamma}^1| \leq |\boldsymbol{\beta}^1| \leq \lfloor \frac{m}{2} \rfloor \leq |\boldsymbol{\beta}^2| \leq |\boldsymbol{\gamma}^k|$; this means that among the elements of the chain $(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \ldots, \boldsymbol{\gamma}^k)$ there exists a tuple with $\lfloor \frac{m}{2} \rfloor$ units, that is, $\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A} \neq \emptyset$.

▷ Let us consider the sequence $G_{\psi_2}(\mathfrak{f}) := \big(G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$. It follows from Proposition 4.2 and from the definition of the choice function ψ_2 that if $|\boldsymbol{\alpha}^i| > \lfloor \frac{m}{2} \rfloor$ and $f(\boldsymbol{\alpha}^i) = 1$, then $\boldsymbol{\alpha}^i$ is a minimal lower unit, and if $|\boldsymbol{\alpha}^i| < \lfloor \frac{m}{2} \rfloor$ and $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$, then $\boldsymbol{\alpha}^i$ is a maximal upper zero of the function $\mathfrak{f}$. This means that in these cases we can, without affecting the result of the inference process, substitute in the sequence $G_{\psi_2}(\mathfrak{f})$ its subsequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ by $(\boldsymbol{\alpha}^i, 1)$ or by $(\boldsymbol{\alpha}^i, 0)$, respectively. An *Algorithm* G'_{ψ_2} is thus defined, with the inference sequence $G'_{\psi_2}(\mathfrak{f}) := (G'(\mathfrak{f}, \boldsymbol{\alpha}^1), G'(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G'(\mathfrak{f}, \boldsymbol{\alpha}^k))$, where

$$G'(\mathfrak{f}, \boldsymbol{\alpha}) := \begin{cases} \big(\boldsymbol{\alpha}, \mathfrak{f}(\boldsymbol{\alpha})\big), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 1 \text{ and } |\boldsymbol{\alpha}| > \lfloor \frac{m}{2} \rfloor \\ & \text{or } \mathfrak{f}(\boldsymbol{\alpha}) = 0 \text{ and } |\boldsymbol{\alpha}| < \lfloor \frac{m}{2} \rfloor \,, \\ G(\mathfrak{f}, \boldsymbol{\alpha}), & \text{otherwise} \,, \end{cases}$$

$$\boldsymbol{\alpha}^1 := \psi_2(\mathbf{B}^m) \,,$$

$$\boldsymbol{\alpha}^i := \psi_2(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\; s \in [i-1]\})) \,,$$

$$\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) := \begin{cases} \boldsymbol{\alpha}^s, & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 1 \text{ and } |\boldsymbol{\alpha}^s| > \lfloor \frac{m}{2} \rfloor \\ & \text{or } \mathfrak{f}(\boldsymbol{\alpha}^s) = 0 \text{ and } |\boldsymbol{\alpha}^s| < \lfloor \frac{m}{2} \rfloor \,, \\ \arg G(\mathfrak{f}, \boldsymbol{\alpha}^s), & \text{otherwise} \,. \end{cases}$$

The Algorithm $G'_{\psi_2}(\mathfrak{f})$ completes inference of the function $\mathfrak{f} \in \mathcal{M}_m$ if and only if $\mathfrak{M}_{\mathfrak{f}}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon s \in [k]\}) = \mathbf{B}^m$. Comparing the sequence G_{ψ_2} to $G'_{\psi_2}(\mathfrak{f})$, we verify that G'_{ψ_2} also satisfies relations (4.10). On the other hand, it follows from the definition of the sequence $G'(\mathfrak{f}, \boldsymbol{\alpha})$ that $G'(\mathfrak{f}, \boldsymbol{\alpha})$ contains at most $\lfloor \frac{m}{2} \rfloor + 2$ invocations of the operator $\mathcal{O}_{\mathfrak{f}}$, for any $i \in [k]$. It follows from the above argument that $\varphi(G'_{\psi_2}, \mathfrak{f}) = (\lfloor \frac{m}{2} \rfloor + 2)|\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$ and, thus, $\eta(G'_{\psi_2}, \mathfrak{f}) \le \lfloor \frac{m}{2} \rfloor + 2$. □

4.2 An inference algorithm for monotone Boolean functions associated with graphs

The study of infeasible systems, whose constraints correspond to the vertices of undirected graphs, and the subsystems with two constraints are feasible (or, on the contrary, infeasible) if and only if the corresponding vertex pairs are edges of the graphs, is of special applied interest.

In this section, with a graph is associated a monotone Boolean function whose zeros correspond to the feasible subsystems of the initial infeasible system of constraints.

Let a simple undirected graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ be given, with the vertex set $V(\mathbf{G}) := \{v_1, \dots, v_n\}$ and the edge family $\mathcal{E}(\mathbf{G}) := \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_p\}$. If $U \subset V(\mathbf{G})$, then $\mathbf{G}\langle U \rangle$ denotes the induced subgraph of the graph $\mathbf{G}$, on the vertex set U. For a vertex $v \in V(\mathbf{G})$, as earlier, $\mathcal{N}(v) \subset V(\mathbf{G})$ denotes the neighborhood of the vertex v in the graph $\mathbf{G}$. For a subset of vertices $U \subseteq V(\mathbf{G})$, we let $\binom{U}{2}$ denote the family of all unordered two subsets of the set U. If $\boldsymbol{x} := (x_1, \dots, x_n) \in \mathbf{B}^n := \{0,1\}^n$, then $\operatorname{supp}(\boldsymbol{x}) := \{i \in [n] \colon x_i = 1\}$.

Consider the monotone Boolean function $\mathfrak{f}_{\mathbf{G}} \colon \mathbf{B}^n \to \mathbf{B}$ whose set of *units* $\mathfrak{f}_{\mathbf{G}}^{-1}(1)$ is defined as follows:

$$\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) := 1 \iff \#\Big(\mathcal{E}(\mathbf{G}) \cap \Big(\tbinom{\{v_i \in V(\mathbf{G}) \colon i \in \operatorname{supp}(\boldsymbol{x})\}}{2}\Big)\Big) \ge 1 \; ; \tag{4.18}$$

in other words, we suppose $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) := 1$ if and only if the induced subgraph $\mathbf{G}\langle\{v_i \in V(\mathbf{G}) \colon i \in \operatorname{supp}(\boldsymbol{x})\}\rangle$ has at least one edge.

Another monotone Boolean function $\mathfrak{g}_{\mathbf{G}} \colon \mathbf{B}^n \to \mathbf{B}$, which is naturally associated with the graph $\mathbf{G}$, is defined by the set of its *zeros* $\mathfrak{g}_{\mathbf{G}}^{-1}(0)$ as follows:

$$\mathfrak{g}_{\mathbf{G}}(\boldsymbol{x}) := 0 \iff \text{subgraph } \mathbf{G}\langle\{v_i \in V(\mathbf{G}) \colon i \in \operatorname{supp}(\boldsymbol{x})\}\rangle \text{ is complete} \; ; \tag{4.19}$$

we relate to the complete graphs the empty graph $\mathbf{G}\langle\emptyset\rangle$, and the isolated vertices $\mathbf{G}\langle\{v_i\}\rangle$, $v_i \in V(\mathbf{G})$.

The graph-theoretic construction that establishes the connection between MBFs from (4.18) and (4.19) is the complement of the graph. The *complement* $\overline{\mathbf{G}}$ *of the graph* $\mathbf{G}$ by definition has the vertex set $V(\mathbf{G})$ and the edge family $\binom{V(\mathbf{G})}{2} - \mathcal{E}(\mathbf{G})$. Definitions (4.18) and (4.19) imply the following useful identities:

$$\mathfrak{f}_{\mathbf{G}} = \mathfrak{g}_{\overline{\mathbf{G}}}\,, \quad \mathfrak{f}_{\overline{\mathbf{G}}} = \mathfrak{g}_{\mathbf{G}}\,.$$

If $\mathcal{X} \subseteq \mathbf{B}^n$ is a set of binary tuples of length n, then, as earlier, $\mathbf{max}\,\mathcal{X}$ denotes the subset of maximal elements of $\mathcal{X}$ with respect to the partial order on $\mathbf{B}^n$, and $\mathbf{max}_{\preceq}\mathcal{X}$ denotes the subset of all tuples from $\mathcal{X}$ that have the maximal number of unit components.

Problem 4.5. *For the function* $\mathfrak{f}_{\mathbf{G}}$ *defined in* (4.18), *to find the set*

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}}) := \mathbf{max}\,\mathfrak{f}_{\mathbf{G}}^{-1}(0)$$

of its upper zeros.

Problem 4.6. *For the function* $\mathfrak{f}_{\mathbf{G}}$, *to find the set*

$$\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$$

of its maximal upper zeros.

An algorithm for finding a maximal upper zero of a monotone Boolean function associated with an undirected graph

Let us consider Problem 4.6, for graphs from a certain class, in more detail.

Proposition 4.7. *Let* $v_i \in V(\mathbf{G})$ *be a vertex of a graph* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$, *such that for its neighborhood* $\mathcal{N}(v_i)$ *the induced subgraph* $\mathbf{G}\langle\mathcal{N}(v_i)\rangle$ *of the graph* $\mathbf{G}$ *is complete. Then there exists a maximal upper zero* $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ *of the function* $\mathfrak{f}_{\mathbf{G}}$ *such that* $x'_i = 1$.

Proof. Let us consider an arbitrary maximal upper zero $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ of the function $\mathfrak{f}_{\mathbf{G}}$, and associate with this zero the index set $I := \{s \in [n]\colon\ v_s \in \mathcal{N}(v_i)\}$. It is easy to see that among the elements of the set $I \dot\cup \{i\}$ there is at least one index j such that $x_j = 1$, because otherwise we could find a tuple $\boldsymbol{x}' \in \mathbf{B}^n$ such that $x'_i = 1$ and $x'_s = x_s$ for all indices $s \in [n] - \{i\}$. Thus, because of $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) = 0$, and by the assumption that $x_s = 0$ for all $s \in I$, the definition of the function $\mathfrak{f}_{\mathbf{G}}$ implies that $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}') = 0$. This contradicts the maximality of the upper zero $\boldsymbol{x}$, because we obtain the strict inclusion $\mathrm{supp}(\boldsymbol{x}') \supsetneqq \mathrm{supp}(\boldsymbol{x})$ and $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}') = \mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) = 0$. Now let us consider the two possible cases. If $x_i = 1$, then we are done. If $x_i = 0$ and $x_s = 1$ for some index $s \in I$, then for the tuple $\boldsymbol{x}$, one can find the tuple $\boldsymbol{x}' \in \mathbf{B}^n$ (by the rule: $x'_j := x_j$ for all $j \in [n] - \{i, s\}$, $x'_i := 1$, and $x'_s := 0$), which is an upper zero of the function $\mathfrak{f}_{\mathbf{G}}$, in view of the completeness of the induced subgraph $\mathbf{G}\langle\mathcal{N}(v_i)\rangle$, and $|\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})|$; we thus obtained a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}}$ such that $x'_i = 1$, as was to be proved. □

For an integer $k \in [n-1]$, *we call a vertex* $v \in V(\mathbf{G})$ *of the graph* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ *a* k-vertex, *if* $|\mathcal{N}(v)| = k$ *and the induced subgraph* $\mathbf{G}\langle\mathcal{N}(v)\rangle$ *of the graph* $\mathbf{G}$ *is complete.*

For integers $k, m \in [n-1]$, we call a vertex $v \in V(\mathbf{G})$ of the graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ a (k, m)-vertex, *if $k = |\mathcal{N}(v)|$ and $m = \binom{k}{2} - \#(\mathcal{E}(\mathbf{G}) \cap \binom{\mathcal{N}(v)}{2})$.*

A (k, m)-vertex $v \in V(\mathbf{G})$ of the graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ is its k-vertex when $m = 0$.

On the basis of Proposition 4.7, one can propose an efficient recursive algorithm for solving Problem 4.6, which finishes its work either by the construction of a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$, or by the reduction of Problem 4.6 for the function $\mathfrak{f}_{\mathbf{G}}$ to the new Problem 4.6 for a function $\mathfrak{f}_{\mathbf{G}'}$, where $\mathbf{G}' \subset \mathbf{G}$, that is, by the decrease of the dimension of the problem to be solved.

Given a vertex $v \in V_0 \subseteq V(\mathbf{G})$, denote by $\mathcal{N}(v, V_0) \subset V_0$ the neighborhood of the vertex v in the induced subgraph $\mathbf{G}\langle V_0\rangle$.

– *Algorithm* 1: Algorithm $A(\mathbf{G}, V_0)$ for finding a maximal upper zero $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbf{B}^n$ of the function $\mathfrak{f}_{\mathbf{G}}$.

```
Input data: G, V_0
Output data: V_0, x
  x_i = 0, i ∈ [n], v_i ∈ V_0
  for each v_i ∈ V_0 do
    if v_i is an |N(v_i, V_0)|-vertex in the subgraph G⟨V_0⟩ then
      x_i ← 1
      V_0 ← V_0 − ({v_i} ∪̇ N(v_i, V_0))
      A(G, V_0)
    end of condition
  end of loop
```

If at the finish of the work of *Algorithm* 1, we get $V_0 = \emptyset$, then, according to Proposition 4.7, the resulting tuple $\boldsymbol{x} \in \mathbf{B}^n$ is a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

However, if at the finish of the work of *Algorithm* 1, we have $V_0 \neq \emptyset$, then for all vertices of the graph $\mathbf{G}\langle V - V_0\rangle$, we determined the values of some components x_i such that there exists a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}}$ with precisely the same values for these components, that is, $x'_i = x_i$; and yet we achieve the decrease of the dimension of the problem from $|V|$ to $|V_0|$.

Lemma 4.8. *Let two graphs $\mathbf{G}_1 := (V, \mathcal{E}(\mathbf{G}_1))$ and $\mathbf{G}_2 := (V, \mathcal{E}(\mathbf{G}_2))$ be given, with the same vertex set V, and*

$$\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2) .$$

Then

$$\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}^{-1}_{\mathbf{G}_2}(0) \subseteq \mathfrak{f}^{-1}_{\mathbf{G}_1}(0) .$$

Proof. It is clear that $\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}^{-1}_{\mathbf{G}_2}(0)$.

Consider an arbitrary tuple $\boldsymbol{x} \in \mathbf{B}^n$ such that $\boldsymbol{x} \in \mathfrak{f}^{-1}_{\mathbf{G}_2}(0)$. By the definition of the set of zeros $\mathfrak{f}^{-1}_{\mathbf{G}_2}(0)$ of the MBF $\mathfrak{f}_{\mathbf{G}_2}$, we have

$$\#\left(\mathcal{E}(\mathbf{G}_2) \cap \binom{\{v_i:\ i \in \mathrm{supp}(\boldsymbol{x})\}}{2}\right) = 0 .$$

By the hypothesis of the lemma, we have $\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2)$ and $V(\mathbf{G}_1) = V(\mathbf{G}_2)$; as a consequence,

$$\#\left(\mathcal{E}(\mathbf{G}_1) \cap \binom{\{v_i:\ i \in \operatorname{supp}(\boldsymbol{x})\}}{2}\right) = 0\ , \quad \forall \boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)\ ,$$

and

$$\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ . \tag{4.20}$$

Then for any tuples $\boldsymbol{x} \in \mathbf{B}^n$ such that $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)$, inclusion (4.20) holds, that is,

$$\mathfrak{f}_{\mathbf{G}_2}^{-1}(0) \subseteq \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ ,$$

as was to be proved. □

It should be mentioned that

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \not\subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})\ . \tag{4.21}$$

Indeed, consider the graphs

$$\begin{aligned} \mathbf{G}_1 :&= (V(\mathbf{G}_1),\ \mathcal{E}(\mathbf{G}_1)) = (V, \emptyset)\ , \\ \mathbf{G}_2 :&= (V(\mathbf{G}_2),\ \mathcal{E}(\mathbf{G}_2)) = \left(V, \tbinom{V}{2}\right)\ , \end{aligned}$$

for which we have $V(\mathbf{G}_1) = V(\mathbf{G}_2)$ and $\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2)$. The graph $\mathbf{G}_1$ has no edges, therefore, the set of upper zeros of the function $\mathfrak{f}_{\mathbf{G}_1}$ consists of the unique tuple

$$\boldsymbol{x} := (1, 1, \ldots, 1)\ .$$

The graph $\mathbf{G}_2$ is complete; thus, the set of upper zeros of the function $\mathfrak{f}_{\mathbf{G}_2}$ has the form

$$\begin{aligned} \boldsymbol{x}^1 :&= (1, 0, \ldots, 0)\ , \\ \boldsymbol{x}^2 :&= (0, 1, \ldots, 0)\ , \\ &\ \ \vdots \\ \boldsymbol{x}^n :&= (0, 0, \ldots, 1)\ . \end{aligned}$$

Any tuple $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2})$ is a zero of the function $\mathfrak{f}_{\mathbf{G}_1}$, that is,

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ , \quad \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \not\subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})\ ,$$

as Lemma 4.8 asserts; this justifies (4.21).

Let us define the quantity $\max_0 \mathfrak{f}_{\mathbf{G}} := |\operatorname{supp}(\boldsymbol{x})|$, where $\boldsymbol{x} \in \mathbf{max}_{\preceq} \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$, which is the number of unit components in a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Corollary 4.9. *Let* $\mathbf{G}_1 := (V, \mathcal{E}_1)$ *and* $\mathbf{G}_2 := (V, \mathcal{E}_2)$ *be graphs such that* $\mathcal{E}_1 \subseteq \mathcal{E}_2$. *Then*

$$\max_0 \mathfrak{f}_{\mathbf{G}_1} \geq \max_0 \mathfrak{f}_{\mathbf{G}_2}\ .$$

Proof. Let $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2})$. According to Lemma 4.8, we have $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)$.

By the definition of the maximal upper zeros of the function, for any tuple $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)$ there exists a tuple $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})$ such that $\boldsymbol{x}' \geq \boldsymbol{x}$. Then

$$\max{}_0\mathfrak{f}_{\mathbf{G}_1} = |\mathrm{supp}(\boldsymbol{x}')| \geq |\mathrm{supp}(\boldsymbol{x})| = \max{}_0\mathfrak{f}_{\mathbf{G}_2} ,$$

as was to be proved. □

Proposition 4.10. *Let* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ *be a graph in which vertices* v_i *and* v_j *are not adjacent. Then*

$$\max{}_0\mathfrak{f}_{\mathbf{G}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 . \tag{4.22}$$

Proof. The inequality $\max_0\mathfrak{f}_{\mathbf{G}} \geq \max_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ follows from Corollary 4.9.

Let us prove the inequality $\max_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max_0\mathfrak{f}_{\mathbf{G}} - 1$. Let $\boldsymbol{x} := (x_1, \dots, x_n)$ be a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Case 1
Suppose that $x_i = 0$ and $x_j = 0$. Then $\boldsymbol{x}$ is clearly a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$, and it is a maximal upper zero, because otherwise we would obtain, by definition, that there exists a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ such that $\boldsymbol{x}' > \boldsymbol{x}$ and $|\mathrm{supp}(\boldsymbol{x}')| > |\mathrm{supp}(\boldsymbol{x})|$. According to Lemma 4.8, we obtain that $\boldsymbol{x}'$ is a zero of the function $\mathfrak{f}_{\mathbf{G}}$, but this contradicts the maximality of $\boldsymbol{x}$.

Thus, in this case, we have

$$\max{}_0\mathfrak{f}_{\mathbf{G}} = \max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 .$$

Case 2
Suppose that $x_i = 1$ and $x_j = 0$.

If the edge (v_i, v_j) is added, then the tuple $\boldsymbol{x}$ is again a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ and, as shown earlier, it is also a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$.

Case 3
Suppose that $x_i = 1$ and $x_j = 1$.

If the edge (v_i, v_j) is added, then we obtain that $\boldsymbol{x}$ is not a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$. In this case, we can find a tuple $\boldsymbol{x}'$ for which $x'_s = x_s$ for all $s \in [n] - \{i\}$, and $x'_i = 0$. The tuple $\boldsymbol{x}'$ will be a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$. Besides, by construction,

$$|\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})| - 1 .$$

By the definition of the maximal upper zeros of the function, we have

$$\max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq |\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})| - 1 = \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 ,$$

as was to be proved. □

Corollary 4.11. *For a graph* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$, *let* $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_t\} \subset \binom{V(\mathbf{G})}{2} - \mathcal{E}(\mathbf{G})$ *be a subfamily of t vertex pairs that are not edges of the graph* $\mathbf{G}$.

Then

$$\max_0 \mathfrak{f}_{\mathbf{G} \cup \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_t\}} \geq \max_0 \mathfrak{f}_{\mathbf{G}} - t\,.$$

Proof. It suffices to apply Proposition 4.10, t times, to the graph $\mathbf{G}$. □

On the basis of Proposition 4.10, one can modify *Algorithm* 1 in such a way that the work of the algorithm will continue until the set of remaining vertices V_0 becomes empty and, besides, a zero $\boldsymbol{x}$ of the function $\mathfrak{f}_{\mathbf{G}}$ will be found, for which, at the same time, we will calculate the estimate $(\max_0 \mathfrak{f}_{\mathbf{G}} - |\mathrm{supp}(\boldsymbol{x})|)$ of the deviation of the number of unit components in the resulting tuple $\boldsymbol{x}$ from the number of unit components in a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

– *Algorithm* 2: Algorithm $A_m(\mathbf{G}, V_0)$.

```
Input data: G, V_0, m ∈ [n]
Output data: V_0, Ind, x
  Ind = 0
  for each v_i ∈ V_0 do
    if v_i is an (|N(v_i, V_0)|, m)-vertex in the subgraph G⟨V_0⟩ then
      x_i ← 1
      V_0 ← V_0 − ({v_i} ∪̇ N(v_i, V_0))
      Ind ← 1
  end of loop
```

Algorithm 2 sequentially checks, for the given value of m and for each vertex of the initial set V_0, whether it is an $(|\mathcal{N}(v_i, V_0)|, m)$-vertex. If there are no such vertices, then no operations are performed, and the resulting set V_0 at the finish of the work of the algorithm coincides with the input set V_0, the flag Ind $= 0$, a binary tuple $\boldsymbol{x}$ is not determined. In the case where such a vertex v_i is found, the output set V_0 will be obtained from the input set V_0 by means of the "removal" of the vertex v_i and its neighborhood, Ind $= 1$, and the corresponding component x_i of the tuple $\boldsymbol{x}$ takes the value of 1.

– *Algorithm* 3: Algorithm $B(\mathbf{G}, V_0)$.

```
Input data: G, V_0
Output data: x ∈ f_G^{-1}(0)
  while V_0 ≠ ∅
    m = 0
    Ind = 1
    while (Ind = 1) & V_0 ≠ ∅ do
```

```
      A_m(G, V_0)
      Ind ← Ind(A_m(G, V_0))
    end of loop

    while (Ind = 0) & V_0 ≠ ∅ do
      m ← m + 1
      A_m(G, V_0)
      Ind ← Ind(A_m(G, V_0))
    end of loop
  end of loop
```

In the course of the work of *Algorithm* 3, as the result of repeated calls of *Algorithm* 2, the tuple $\boldsymbol{x}$ is formed, which is a zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Proposition 4.12. *Let v_i be a (k, m)-vertex in a graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$. Then there exists a tuple $\boldsymbol{x}' \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ such that $x_i' = 1$ and*

$$|\mathrm{supp}(\boldsymbol{x}')| \geq \max\nolimits_0 \mathfrak{f}_{\mathbf{G}} - m\ .$$

Proof. Suppose, according to the definition of the (k, m)-vertices, that for $v_i \in V(\mathbf{G})$ we have

$$\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} := \binom{\mathcal{N}(v_i)}{2} - \left(\mathcal{E}(\mathbf{G}) \cap \binom{\mathcal{N}(v_i)}{2}\right).$$

Then the vertex v_i is a k-vertex in the graph $\mathbf{G}_1$, which is obtained from the graph $\mathbf{G}$ by the addition of the m edges $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ into the neighborhood of the vertex v_i of the graph $\mathbf{G}$ up to the complete induced subgraph $\mathbf{G}_1\langle\mathcal{N}(v_i)\rangle$.

According to Proposition 4.7, there exists a tuple $\boldsymbol{x}$ such that $x_i = 1$ and $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})$.

According to Corollary 4.11, for the graph $\mathbf{G}_1$ we have

$$|\mathrm{supp}(\boldsymbol{x})| = \max\nolimits_0 \mathfrak{f}_{\mathbf{G}_1} \geq \max\nolimits_0 \mathfrak{f}_{\mathbf{G}} - m\ .$$

It follows from Lemma 4.8 that $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}}^{-1}(0)$. By the definition of the upper zeros, there exists a tuple $\boldsymbol{x}' \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ such that $\boldsymbol{x}' \geq \boldsymbol{x}$ and, as a consequence,

$$|\mathrm{supp}(\boldsymbol{x}')| \geq |\mathrm{supp}(\boldsymbol{x})| \geq \max\nolimits_0 \mathfrak{f}_{\mathbf{G}} - m\ ,$$

as was to be proved. □

In every next loop of *Algorithm* 1, the search is terminated when some k-vertex is found. Such an approach minimizes the number of operations in the working loop of the algorithm, but it does not necessarily lead to the best solution in the case where $V_0 \neq \emptyset$.

Let us present an *Algorithm* 4, in each next working loop of which the parameters k and m are calculated for every vertex from the current set V_0. *Algorithm* 4 admits a larger number of operations in each working loop, but it can provide a more precise approximation to a maximal upper zero.

– *Algorithm* 4:

```
Input data: G, V_0, m = 0
Output data: x ∈ Ω(f_G), and m which is the estimate of deviation from max_0 f_G
  while V_0 ≠ ∅
    for all vertices v_i ∈ V_0 ≠ ∅, to calculate the parameters k_i and m_i such that v_i is
    a (k_i, m_i)-vertex in the graph G⟨V_0⟩; in the set V_0, to extract the subset V'_0 ⊆ V_0
    of vertices with the minimal values of the parameter m_i. Among the extracted
    vertices in the set V'_0, to find a vertex v_{i_0} ∈ V'_0 with the maximal value of the
    parameter k_{i_0}
    x_{i_0} ← 1
    m ← m + m_{i_0}
    V_0 ← V_0 − ({v_{i_0}} ∪̇ N(v_{i_0}, V_0))
  end of loop
```

Algorithm 4 finds a tuple $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$, for which the precision estimate $\max_0 \mathfrak{f}_{\mathbf{G}} - |\mathrm{supp}(\boldsymbol{x})| \le m$ of the solution is true.

Let us estimate the complexity of *Algorithm* 4.

For each vertex v_i from the current set V_0, it is necessary to find the number of vertices in the neighborhood $\mathcal{N}(v_i, V_0)$ and the number of new edges that should be added into the neighborhood $\mathcal{N}(v_i, V_0)$ for turning the induced subgraph $\mathbf{G}\langle\mathcal{N}(v_i, V_0)\rangle$ into a complete graph. We remove the vertices $v_i \dot\cup \mathcal{N}(v_i, V_0)$ and the edges $\boldsymbol{e}_i \in \mathbf{G}\langle\{v_i\} \dot\cup \mathcal{N}(v_i, V_0)\rangle$ until the current set of vertices V_0 becomes empty. Given the input data $V(\mathbf{G}) = \{v_1, \dots, v_n\}$ and $\mathcal{E}(\mathbf{G}) = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_p\}$, we obtain the following estimate. The common number of iterations undertaken during the work of *Algorithm* 4 is less than or equal to n; every iteration demands no more than $O(np)$ actions for the computation of the parameters k and m; and no more than $O(p)$ actions are needed for the removal of a vertex and its neighborhood from the current graph. Thus, *Algorithm* 4 has the complexity of $O(n \cdot np + np) = O(n^2 p)$.

Solving the problem of searching for a maximal upper zero

For some applied problems that are reduced to Problem 4.6, either exact results were obtained, or the significant decrease of the dimension of Problem 4.6 was achieved.

Example 4.13. *The graph* $\mathbf{G} := (V := \{v_1, \dots, v_{22}\}, \mathcal{E})$ *is specified by the incidence lists* $\mathcal{N}(v_i)$ *of its vertices,* $i \in [22]$, $V_0 = V$:

$$\mathcal{N}(v_1) := \{v_2, v_3\}\,,\ \mathcal{N}(v_2) := \{v_1, v_3\}\,,\ \mathcal{N}(v_3) := \{v_1, v_2, v_4, v_9\}\,,$$

$$\mathcal{N}(v_4) := \{v_3, v_5, v_6, v_{11}\}\,,\ \mathcal{N}(v_5) := \{v_4, v_6\}\,,\ \mathcal{N}(v_6) := \{v_4, v_5, v_7, v_{10}, v_{12}\}\,,$$

$$\mathcal{N}(v_7) := \{v_6, v_8\}\,,\ \mathcal{N}(v_8) := \{v_7, v_{12}, v_{16}, v_{17}\}\,,\ \mathcal{N}(v_9) := \{v_3, v_{11}, v_{13}\}\,,$$
$$\mathcal{N}(v_{10}) := \{v_6, v_{11}, v_{12}, v_{14}, v_{15}\}\,,\ \mathcal{N}(v_{11}) := \{v_4, v_9, v_{10}, v_{14}\}\,,$$
$$\mathcal{N}(v_{12}) := \{v_6, v_8, v_{10}, v_{16}\}\,,\ \mathcal{N}(v_{13}) := \{v_9, v_{14}\}\,,\ \mathcal{N}(v_{14}) := \{v_{10}, v_{11}, v_{13}, v_{15}\}\,,$$
$$\mathcal{N}(v_{15}) := \{v_{10}, v_{14}, v_{16}, v_{20}, v_{21}\}\,,\ \mathcal{N}(v_{16}) := \{v_8, v_{12}, v_{15}, v_{17}, v_{19}\}\,,$$
$$\mathcal{N}(v_{17}) := \{v_8, v_{16}, v_{18}, v_{19}\}\,,\ \mathcal{N}(v_{18}) := \{v_{17}, v_{19}\}\,,$$
$$\mathcal{N}(v_{19}) := \{v_{16}, v_{17}, v_{18}, v_{20}, v_{21}, v_{22}\}\,,\ \mathcal{N}(v_{20}) := \{v_{15}, v_{19}, v_{21}, v_{22}\}\,,$$
$$\mathcal{N}(v_{21}) := \{v_{15}, v_{19}, v_{20}, v_{22}\}\,,\ \mathcal{N}(v_{22}) := \{v_{19}, v_{20}, v_{21}\}\,.$$

Acting in accordance with Algorithm 1*, for each vertex* $v_i \in V_0$ *we check whether it is a* k*-vertex in the graph* $\mathbf{G}$*.*

$A(\mathbf{G}, V_0)$*:*

1. v_1 *is a* 2*-vertex* $\Rightarrow\ x_1 \leftarrow 1;\ V_0 \leftarrow V_0 - \{v_1, v_2, v_3\}$.
2. v_4 *is not a* 3*-vertex.*
3. v_5 *is a* 2*-vertex* $\Rightarrow\ x_5 \leftarrow 1;\ V_0 \leftarrow V_0 - \{v_4, v_5, v_6\}$.
4. v_7 *is a* 1*-vertex* $\Rightarrow\ x_7 \leftarrow 1;\ V_0 \leftarrow V_0 - \{v_7, v_8\}$.
5. $v_{9\,(10,11,12,13,14,15,16,17)}$ *is not a* 2 (4, 3, 2, 2, 4, 5, 4, 3)*-vertex.*
6. v_{18} *is a* 2*-vertex* $\Rightarrow\ x_{18} \leftarrow 1;\ V_0 \leftarrow V_0 - \{v_{17}, v_{18}, v_{19}\}$.
7. $v_{9\,(10,11,12,13,14,15,16,20,21)}$ *is not a* 2 (4, 3, 2, 2, 4, 5, 2, 3, 3)*-vertex.*
8. v_{22} *is a* 2*-vertex* $\Rightarrow\ x_{22} \leftarrow 1;\ V_0 \leftarrow V_0 - \{v_{20}, v_{21}, v_{22}\}$.

$\boldsymbol{x} = (1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1)$ *is a zero of the function* $\mathfrak{f}_{\mathbf{G}}$, $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}}^{-1}(0)$*; besides, a maximal upper zero* $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ *of the function* $\mathfrak{f}_{\mathbf{G}}$ *has the form*

$$\boldsymbol{x}' = (1, 0, 0, 0, 1, 0, 1, 0, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, 0, 1, 0, 0, 0, 1)\,.$$

Thus, the dimension of the problem was decreased from $|V_0| = 22$ to $|V_0| = |\{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}| = 8$.

For exhausting the vertex set V_0, we follow *Algorithm* 3; among the vertices from the set V_0 we search for (k, m)-vertices (the case of $m = 0$ corresponds to the search for k-vertices, which was undertaken by *Algorithm* 1).

Example 4.14.

$V_0 \neq \emptyset$*,* $m = 0$*:*

$\mathrm{Ind} = 0 \Rightarrow m \leftarrow m + 1 = 1,\ A_1(\mathbf{G}, V_0)$*:*

v_9 *is a* (2, 1)*-vertex,* then*:* $x_9 \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_9, v_{11}, v_{13}\} \Rightarrow$

$\mathrm{Ind} = 1 \Rightarrow m = 0,\ A_0(\mathbf{G}, V_0)$*:*

$v_{10\,(12)}$ *is not a* 3(2)*-vertex;*

v_{14} *is a* 2*-vertex,* then*:* $x_{14} \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_{10}, v_{14}, v_{15}\}$.

$\mathrm{Ind} = 1 \Rightarrow m = 0,\ A_0(\mathbf{G}, V_0)$*:*

v_{12} *is a* 1*-vertex,* then*:* $x_{12} \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_{12}, v_{16}\}$.

$V_0 = \emptyset \Rightarrow$

$\boldsymbol{x}' = (1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1)$ *is a zero of the function* $\mathfrak{f}_{\mathbf{G}}$*, and it is a maximal upper zero of the function* $\mathfrak{f}_{\mathbf{G}\cup\{(v_{11}, v_{13})\}}$*; then, according* *to Proposition* 4.10*, the number of unit components in a maximal upper zero of the function* $\mathfrak{f}_{\mathbf{G}}$ *is restricted by the inequality:*

$$\max\nolimits_0 \mathfrak{f}_{\mathbf{G}} \leq \max\nolimits_0 \mathfrak{f}_{\mathbf{G} \cup \{(v_{11}, v_{13})\}} + 1 = |\text{supp}(\boldsymbol{x}')| + 1 = 9\ .$$

It is convenient to describe the result of the work of *Algorithm* 3 in the form of Table 4.1. The columns of the table correspond to the current state of the set V_0. We sequentially remove k-vertices and their neighborhoods from the set V_0, associating to the corresponding components x_i the value of 1 in the case where v_i is a k-vertex, and the value of 0 otherwise.

Table 4.2 describes the work of *Algorithm* 4. Every column of the table represents an iteration of *Algorithm* 4; the nonzero elements of a column correspond to the set V_0, and in an ith row the values of k and m are related to the vertex v_i in the current subgraph $\mathbf{G}\langle V_0 \rangle$.

For the resulting tuple $\boldsymbol{x} = (1,0,0,0,1,0,1,0,0,0,1,1,1,0,1,0,0,1,0,0,0,1)$ it holds that $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ and $|\text{supp}(\boldsymbol{x})| = 9 \geq \max_0 \mathfrak{f}_{\mathbf{G}} - 1$.

Table 4.1. The result of the work of *Algorithm* 3

m	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	***x***
Ind	1	1	1	1	1	1	0	1	1	
v_1	1	0	0	0	0	0	0	0	0	1
v_2	1	0	0	0	0	0	0	0	0	0
v_3	1	0	0	0	0	0	0	0	0	0
v_4	1	1	0	0	0	0	0	0	0	0
v_5	1	1	0	0	0	0	0	0	0	1
v_6	1	1	0	0	0	0	0	0	0	0
v_7	1	1	1	0	0	0	0	0	0	1
v_8	1	1	1	0	0	0	0	0	0	0
v_9	1	1	1	1	1	1	0	0	0	1
v_{10}	1	1	1	1	1	1	1	0	0	0
v_{11}	1	1	1	1	1	1	0	0	0	0
v_{12}	1	1	1	1	1	1	1	1	0	1
v_{13}	1	1	1	1	1	1	0	0	0	0
v_{14}	1	1	1	1	1	1	1	0	0	1
v_{15}	1	1	1	1	1	1	1	0	0	0
v_{16}	1	1	1	1	1	1	1	1	0	0
v_{17}	1	1	1	1	0	0	0	0	0	0
v_{18}	1	1	1	1	0	0	0	0	0	1
v_{19}	1	1	1	1	0	0	0	0	0	0
v_{20}	1	1	1	1	1	0	0	0	0	0
v_{21}	1	1	1	1	1	0	0	0	0	0
v_{22}	1	1	1	1	1	0	0	0	0	1

Table 4.2. The work of *Algorithm* 4

	k/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*k*/*m*	*x*
v_1	2/0	2/0								1
v_2	2/0	2/0								0
v_3	4/5	4/5								0
v_4	4/5	4/5	3/2							0
v_5	2/0	2/0	2/0							1
v_6	5/8	5/8	5/8							0
v_7	2/1	2/1	2/1	1/0						1
v_8	4/4	4/4	4/4	4/4						0
v_9	3/3	3/3	2/1	2/1	2/1	2/1	2/1	2/1		0
v_{10}	5/7	5/7	5/7	4/4	4/4	4/4				0
v_{11}	4/5	4/5	4/5	3/2	3/2	3/2	2/1	1/0		1
v_{12}	4/4	4/4	4/4	3/2	2/1	2/1				1
v_{13}	2/1	2/1	2/1	2/1	2/1	2/1	2/1	1/0	0/0	1
v_{14}	4/4	4/4	4/4	4/4	4/4	4/4	3/3			0
v_{15}	5/8	3/2	3/2	3/2	3/2	3/2	1/0			1
v_{16}	5/7	4/4	4/4	4/4	3/3	2/1				0
v_{17}	4/3	3/2	3/2	3/2	2/1					0
v_{18}	2/0	1/0	1/0	1/0	1/0					1
v_{19}	6/10									0
v_{20}	4/2									0
v_{21}	4/2									0
v_{22}	3/0									1

Earlier, for the tuple $\boldsymbol{x}'$ obtained with the help of *Algorithm* 3, we also obtained that

$$|\mathrm{supp}(\boldsymbol{x}')| = 8 \geq \max_0 \mathfrak{f}_{\mathbf{G}} - 1$$

or, in other words, $\max_0 \mathfrak{f}_{\mathbf{G}} \leq 9$. Since $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ and $|\mathrm{supp}(\boldsymbol{x})| = 9$, we see that $\max_0 \mathfrak{f}_{\mathbf{G}} = 9$.

4.3 Monotone Boolean functions and inequality systems

The problem of extracting inclusion-maximal feasible subsystems of an infeasible monotone system of constraints is naturally reduced to the problem of inference of monotone Boolean functions.

We will consider here the problem of extracting all the MFSs of an infeasible system of linear inequalities of the form (3.20), described on page 75, that is a rank n system

$$\mathsf{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 :\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\}$$

of homogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$.

The reduction to the problem of MBF inference consists in the following. Let $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a binary tuple. Let us pick in $\boldsymbol{\alpha}$ all its unit components. Suppose that their indices are $i_1, i_2, \ldots, i_k$. Consider the subsystem, with the multi-index $\{i_1, i_2, \ldots, i_k\}$, of the system S, and denote this subsystem by $\mathsf{S}(\boldsymbol{\alpha})$. Let us set

$$\mathfrak{f}(\boldsymbol{\alpha}) := \begin{cases} 0, & \text{if } \mathsf{S}(\boldsymbol{\alpha}) \text{ is feasible ,} \\ 1, & \text{if } \mathsf{S}(\boldsymbol{\alpha}) \text{ is infeasible .} \end{cases}$$

The function $\mathfrak{f}$ is monotone and the set of its upper zeros is in one-to-one correspondence with the family of maximal feasible subsystems of the system S.

It turns out that after some modification of the operator $\mathcal{O}_{\mathfrak{f}}$, natural for the class of monotone Boolean functions under consideration, it is possible to present an algorithm of MBF inference which is optimal with respect to all criteria (4.3)–(4.6).

(1) The new operator $\mathcal{O}'_{\mathfrak{f}}$ is to determine the value of the function $\mathfrak{f}(\boldsymbol{\alpha})$ at a given tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$;
(2) if $\mathfrak{f}(\boldsymbol{\alpha}) = 1$, then the operator $\mathcal{O}'_{\mathfrak{f}}$ is to extract one lower unit $\boldsymbol{\alpha}'$ of the function $\mathfrak{f}$, such that $\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}$.

This modification is indeed reasonable, because for the class of MBFs under consideration, which are associated with the infeasible systems S, some variants of the well-developed technique of linear programming can be chosen as such an operator.

Let us denote by $\varphi(\mathcal{O}'_{\mathfrak{f}}, G, \mathfrak{f})$ the number of calls of the operator $\mathcal{O}'_{\mathfrak{f}}$ by an algorithm G when inferring a function $\mathfrak{f} \in \mathcal{M}_m$. For any algorithm G of MBF inference, and for any function $\mathfrak{f} \in \mathcal{M}_m$, the inequality $\varphi(\mathcal{O}'_{\mathfrak{f}}, G, \mathfrak{f}) \geq |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|$ holds.

Proposition 4.15. *There exists an algorithm G^* of MBF inference, such that $\varphi(\mathcal{O}'_{\mathfrak{f}}, G^*, \mathfrak{f}) = |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|$ for any function $\mathfrak{f} \in \mathcal{M}_m$.*

Proof. Let us construct such an algorithm G^*. Let us suppose $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 0$, and $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}'$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 1$, where $\boldsymbol{\alpha}'$ is a lower unit of the function $\mathfrak{f}$, determined by the operator $\mathcal{O}'_{\mathfrak{f}}$. By definition, the inference sequence $G^*(\mathfrak{f})$ for Algorithm G^* and for the function $\mathfrak{f} \in \mathcal{M}_m$ is as follows:

$$\begin{aligned} G^*(\mathfrak{f}) &:= \big(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1), \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2), \ldots, \boldsymbol{\alpha}^k, \mathfrak{f}(\boldsymbol{\alpha}^k)\big) , \\ \boldsymbol{\alpha}^1 &:= (1, 1, \ldots, 1) \in \mathbf{B}^m , \\ \boldsymbol{\alpha}^i &:= \psi\big(\mathbf{max}\,(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon \; s \in [i-1]\}))\big) , \end{aligned}$$

where ψ is an arbitrary choice function. The inference process is completed when we have $\mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon \; s \in [k]\}) = \mathbf{B}^m$, that is,

$$\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon \; s \in [k]\} \supseteq \mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f}) . \tag{4.23}$$

Further, if $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^i)$ is a maximal upper zero of the function $\mathfrak{f}$ because of the maximality of

$$\boldsymbol{\alpha}^i = \psi(\mathbf{max}\,(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [i-1]\}))) \,,$$

in analogy to the argument of Proposition 4.2; if $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha})$ is by definition a minimal lower unit of the function $\mathfrak{f}$. Thus, the inclusion

$$\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [k]\} \subseteq \mathfrak{Q}(\mathfrak{f}) \,\dot{\cup}\, \mathfrak{P}(\mathfrak{f}) \tag{4.24}$$

holds.

Let us now prove the implication

$$t, p \in [k],\ \ t < p \quad \Longrightarrow \quad \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)\,. \tag{4.25}$$

Indeed, if $\mathfrak{f}(\boldsymbol{\alpha}^t) \neq \mathfrak{f}(\boldsymbol{\alpha}^p)$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$, by definition. If $\mathfrak{f}(\boldsymbol{\alpha}^t) = \mathfrak{f}(\boldsymbol{\alpha}^p) = 0$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$ because $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$. If $\mathfrak{f}(\boldsymbol{\alpha}^t) = \mathfrak{f}(\boldsymbol{\alpha}^p) = 1$ then $\boldsymbol{\alpha}^p \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$, and $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \leq \boldsymbol{\alpha}^p$ implies $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$, that is, $A'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq A'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$. It follows from relations (4.23)–(4.25) that $k = \varphi(\mathcal{O}'_{\mathfrak{f}}, G^*, \mathfrak{f}) = |\mathfrak{Q}(\mathfrak{f}) \,\dot{\cup}\, \mathfrak{P}(\mathfrak{f})|$. □

Note that during the inference process for the function $\mathfrak{f}$, it suffices to store in the memory of a computer system just the set $\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^1), \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^2), \ldots, \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^k)\}$, that is, at most $|\mathfrak{Q}(\mathfrak{f}) \,\dot{\cup}\, \mathfrak{P}(\mathfrak{f})|$ binary tuples of length m.

The algorithm G^* of MBF inference is optimal with respect to criteria (4.3)–(4.6).

Notes

A thorough presentation of the Boolean function theory and of its various applications can be found, for example, in books [30, 86, 90, 110, 153].

Among numerous works devoted to the study of monotone Boolean functions, we point out at review [79], and at book [124] which is devoted to Dedekind's problem on the number of MBFs.

A thorough survey of the state-of-art theory and practice in inference of monotone Boolean functions is given in book [144] and in concise note [143].

In this chapter, we follow in our presentation work [52].

In the typical case, inference of monotone Boolean functions requires asymptotically the twice as less number of invocations of the oracle than in the worst case, see [124, 126].

The algorithm $\varphi(G, m)$ of inference of monotone Boolean functions which is optimal with respect to classical Shannon's criterion is presented in work [66], where it was proved that $\varphi(m) = \binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1}$.

For various applied problems, the algorithms that are optimal with respect to Shannon's criterion are inadequate. For example, such inference algorithms from works [66] and [131] require at least $\binom{m}{\lfloor m/2 \rfloor}$ calls of the operator $\mathcal{O}_{\mathfrak{f}}$ during the inference process for such simple functions as *identically zero* $\mathfrak{f}_0 \equiv 0$ and *identically unit* $\mathfrak{f}_1 \equiv 1$.

Comparing the algorithms described in Section 4.1 to other known algorithms, let us note that for the algorithms "A_1" from [66] and "A_2" from [131], optimal with respect to Shannon's criterion $\varphi(G, m)$, we have $\eta(A_1, m), \eta(A_2, m) \geq \binom{m}{\lfloor m/2 \rfloor}$. This observation follows from the fact that during the inference process for the function *identically zero* $\mathfrak{f}_0 \equiv 0$, the algorithms "A_1" and "A_2" call the operator $\mathcal{O}_{\mathfrak{f}}$ at least $\binom{m}{\lfloor m/2 \rfloor}$ times, where $\binom{m}{\lfloor m/2 \rfloor}$ is the number of chains in the chain partition, considered in [66, 131], of the unit cube $\mathbf{B}^m$.

The close relationship between the problem of inference of monotone Boolean functions and central problems of combinatorial optimization is well known; for example, it was shown in [78] how the knapsack problem is reduced to inference of some MBF.

The adaptive algorithm of solving the multidimensional knapsack problem, presented in [145], can also be efficiently applied to inference of MBFs with a small number of maximal upper zeros. The efficiency of this algorithm is justified by illustrative inference of the specific MBF, of ten binary variables, with the following five maximal upper zeros:

$$\begin{aligned}
\boldsymbol{\alpha}^1 &= (1, 1, 0, 1, 0, 1, 1, 0, 0, 1)\ , \\
\boldsymbol{\alpha}^2 &= (1, 0, 1, 0, 0, 1, 0, 1, 0, 0)\ , \\
\boldsymbol{\alpha}^3 &= (1, 0, 0, 1, 1, 1, 0, 0, 1, 1)\ , \\
\boldsymbol{\alpha}^4 &= (0, 1, 1, 1, 0, 0, 0, 0, 0, 1)\ , \\
\boldsymbol{\alpha}^5 &= (0, 0, 0, 0, 1, 0, 0, 1, 0, 0)\ .
\end{aligned}$$

The algorithm from [145] requires 150 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$ during the inference process for this MBF, while the algorithms from [66, 131] that are optimal with respect to Shannon's criterion, require at least 252 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$. Let us estimate the number of invocations of the operator $\mathcal{O}_{\mathfrak{f}}$ by Algorithm G'_{ψ_0} during the inference process for this specific function $\mathfrak{f}$. It is easily checked that the function has 20 minimal lower units. Then, according to Proposition 4.3, we obtain that Algorithm G'_{ψ_0} requires at most 71 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$. Algorithm G'_{ψ_0} admits realization when it suffices to store in the memory of a computer system the upper zeros of the function $\mathfrak{f} \in \mathcal{M}_m$ only. Algorithm G'_{ψ_1} is efficient when inferring MBFs with a relatively small number of minimal lower units. It admits realization when it suffices to store in the memory of a computer system the minimal lower units of the function $\mathfrak{f} \in \mathcal{M}_m$ only.

Algorithms of finding the upper zeros and lower units of monotone Boolean functions, similar to those which we considered in Section 4.1, are used, for example, in [21, 89, 149]. Such algorithms known under the common name *algorithms Find-Border* gained widespread acceptance. They are also discussed in [18, 34, 79, 88, 124, 142, 144].

Advantages and disadvantages of different algorithms of inferring MBFs, as well as perspectives on further investigations of this problem, are thoroughly discussed in book [144]: a new criterion function, minimized over all inference algorithms G that changes seriously our point of view on an assessment of the efficiency of approaches to inference, is as follows: $\min_G \frac{\sum_{\mathfrak{f}\in\mathcal{M}_m} \varphi(G,\mathfrak{f})}{\Psi(m)}$, where $\Psi(m)$ is the number of all monotone Boolean functions over $\mathbf{B}^m$, the quantity which is central for Dedekind's problem; see [141] and further works in this direction.

In Section 4.2, we propose an inference algorithm for monotone Boolean functions associated with graphs, and discuss the related problem of searching for their maximal upper zeros, following article [59].

Justification of reducibility of the problem of extracting maximal feasible subsystems of an infeasible system of linear inequalities S, considered in Section 4.3, to the problem of inference of the corresponding MBF, is presented in seminal work [159].