5 Inequality systems, committees, (hyper)graphs, and alternative covers

A *committee* of a rank n infeasible system (2.26),

$$\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m];\ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2}\}\ ,$$

of homogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$, introduced on page 33, is defined as a finite set of vectors $\mathcal{K} \subset \mathbb{R}^n - \{\mathbf{0}\}$ satisfying the relation

$$|\{\boldsymbol{x} \in \mathcal{K}\colon\ \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > 0\}| > \tfrac{1}{2}|\mathcal{K}|\ ,$$

for every vector $\boldsymbol{a}_i$, $i \in [m]$.

As earlier, we will denote by $\mathbf{J} := \{J_s \subset [m]\colon s \in [\mathfrak{q}]\}$ the family of the multi-indices of all MFSs of the system $\mathfrak{S}$.

Among the techniques that are used in the committee method is the extracting minimal infeasible and maximal feasible subsystems of the inequality system $\mathfrak{S}$. Solutions to feasible subsystems are combined into committee constructions that generalize the notion of *solution* to feasible systems.

The committee method is efficiently applied to the synthesis of decision-making procedures and, in particular, to contradictory problems of pattern recognition: the problem of committee discrimination of the so-called training sample, for the purpose of forming the decision rules of recognition, can be reduced to the following basic *two-class* setting:

Let $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ be finite sets of vectors of the *feature space* $\mathbb{R}^{n-1}$ that compose the above-mentioned *training sample*. By augmenting artificially each vector from the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ by a new nth component, equal to 1, we obtain two sets $\boldsymbol{B}, \boldsymbol{C} \subset \mathbb{R}^n$ of *extended vectors* of the training sample.

It is necessary to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \boldsymbol{a}, \mathbf{x}\rangle > 0, & \boldsymbol{a} \in \boldsymbol{B}\ , \\ \langle \boldsymbol{a}, \mathbf{x}\rangle < 0, & \boldsymbol{a} \in \boldsymbol{C}\ . \end{cases} \tag{5.1}$$

Strict inequalities are used here, because from the applied point of view the use of nonstrict inequalities would be too risky and lead to the synthesis of unstable decision rules.

If $\boldsymbol{x}$ is a solution to system (5.1), then classification of a new extended vector $\boldsymbol{g} \in \mathbb{R}^n$ (that is the making reference of $\boldsymbol{g}$ to one of the classes represented partially by the sets $\boldsymbol{B}$ and $\boldsymbol{C}$) is performed on the basis of the sign of the scalar product $\langle \boldsymbol{x}, \boldsymbol{g}\rangle$. However, the system under consideration can turn out to be infeasible, and this most frequent case requires the development of special methods of problem solving.

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Unification of two subsystems that compose system (5.1),

$$\begin{cases} \langle \boldsymbol{a}, \mathbf{x} \rangle > 0, & \boldsymbol{a} \in \boldsymbol{B}\,, \\ -\langle \boldsymbol{a}, \mathbf{x} \rangle > 0, & \boldsymbol{a} \in \boldsymbol{C}\,, \end{cases}$$

leads us (after normalizing the determining vectors) to constructions $\mathfrak{S}$ of the form (2.26).

5.1 The graph of MFSs of an infeasible system of linear inequalities and committees

Let L be the multi-index of a feasible subsystem of the system $\mathfrak{S}$, and $\{J_i\colon\ i \in T \subseteq [q]\} \subseteq \mathbf{J}$ the family of some (not necessarily all) multi-indices of MFSs that contain the multi-index L as a subset; in other words, $\{J_i\colon\ i \in T\} \subseteq \{J \in \mathbf{J}\colon\ J \supseteq L\}$. The algorithmic problem of extracting the multi-indices of all MFSs of the system $\mathfrak{S}$ that contain the multi-index L, provided their subfamily $\{J_i\colon\ i \in T\}$ is known, will be called the $(L, \{J_i\colon\ i \in T\})$*-problem for the system* $\mathfrak{S}$ or, for brevity, the $(L, \{J_i\colon\ i \in T\})$*-problem*, when it is clear what system of inequalities is meant. An important specific case of the above problem is the $(\emptyset, \emptyset)$*-problem of extracting the multi-indices of all MFSs* of the system $\mathfrak{S}$.

Let us first consider a combinatorial algorithm of solving this problem which will serve in what follows as the basic mechanism of a graph-combinatorial algorithm.

Combinatorial algorithm of extracting MFSs of an infeasible system of linear inequalities

We denote the algorithm that we describe here by CMB$(L, \{J_i \in \mathbf{J}\colon\ i \in T\})$, and by $\{J_i\colon\ i \in T_{\mathrm{pr}}\}$ the family of the multi-indices of MFSs of the system $\mathfrak{S}$ which contain the subfamily $\{J_i\colon\ i \in T\}$, as well as of the multi-indices of all MFSs found by algorithm CMB$(L, \{J_i \in \mathbf{J}\colon\ i \in T\})$ to the present moment. Before launching the algorithm, we have $\{J_i \in \mathbf{J}\colon\ i \in T_{\mathrm{pr}}\} = \{J_i\colon\ i \in T\}$.

We will use the following statement:

Proposition 5.1. *For a subfamily $\{J_i\colon\ i \in T\} \subseteq \mathbf{J}$ and for the multi-index L of a feasible subsystem of the system $\mathfrak{S}$, there exists its maximal feasible subsystem with a multi-index $J_s \supseteq L$, $\mathbf{J} \ni J_s \notin \{J_i\colon\ i \in T\}$, if and only if in the blocker $\mathfrak{B}(\{[m] - J_i\colon\ i \in T\})$ there exists a minimal system of representatives M of the family $\{[m] - J_i\colon\ i \in T\}$ such that the subsystem, with the multi-index $M \cup L$, of the system $\mathfrak{S}$ is feasible.*

Proof. The sufficiency is evident. Let us prove the necessity. Since $J_s \notin \{J_i\colon\ i \in T\}$, we have $J_s \cap ([m] - J_i) \neq \emptyset$ for all indices $i \in T$; as a consequence, there exists a minimal system of representatives $M \subseteq J_s$ of the family $\{[m] - J_i\colon\ i \in T\}$, and the subsystem with the multi-index $M \cup L$ is feasible. □

Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$

1. To find all the minimal systems of representatives of the family $\{[m] - J_i:\ i \in T_{\text{pr}}\}$, that is to form the blocker $\mathfrak{B}(\{[m] - J_i:\ i \in T_{\text{pr}}\})$.
2. To check feasibility of the subsystems of the system $\mathfrak{S}$ with the multi-indices $M \cup L$, for every multi-index $M \in \mathfrak{B}(\{[m] - J_i:\ i \in T_{\text{pr}}\})$.
3. If all these systems are infeasible then the algorithm finishes because, according to Propositon 5.1, the family $\{J_i:\ i \in T_{\text{pr}}\}$ coincides with $\mathbf{J}$.
4. If there is a multi-index $M \in \mathfrak{B}(\{[m] - J_i:\ i \in T_{\text{pr}}\})$ such that the subsystem with the multi-index $M \cup L$ is feasible, then to augment this subsystem up to a MFS. To add the multi-index of the obtained MFS to the current family $\{J_i:\ i \in T_{\text{pr}}\}$ of the multi-indices of MFSs; go to step 1.

In Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$, it is necessary to repeatedly solve the problem of forming the blocker of some family $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ of subsets of the set $[m]$, that is of extracting all the minimal systems of representatives of the family $\mathcal{A}$. This is a well-known combinatorial problem; various algorithms for solving this problem are proposed. We present here one more algorithm; it takes into account its specific use in Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$.

An algorithm of forming the blocker of a set family

Let $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ be a family of subsets of the set $[m]$. The blocker $\mathfrak{B}(\{A_1, A_2, \dots, A_k\})$ of a subfamily $\{A_1, A_2, \dots, A_k\} \subseteq \mathcal{A}$, $k \in [\alpha]$, will be denoted by $\{M_j^{(k)}:\ j \in [\beta_k]\}$. We will suppose by definition that the empty set is the unique system of representatives for the empty set family: $\mathfrak{B}(\emptyset) := \{M_1^{(0)}\} := \{\emptyset\}$. The following assertion is true:

Proposition 5.2. *Let $k \in [\alpha]$. For any index $s \in [\beta_k]$, there exists precisely one index $i_s \in [\beta_{k-1}]$ such that $M_{i_s}^{(k-1)} \subseteq M_s^{(k)}$.*

Proof. The existence of at least one index $i_s \in [\beta_{k-1}]$, such that $M_{i_s}^{(k-1)} \subseteq M_s^{(k)}$, is evident. Let us prove its uniqueness. Assume the converse: let $i_1, i_2 \in [\beta_{k-1}]$ be two different indices such that $M_{i_1}^{(k-1)} \subseteq M_s^{(k)}$ and $M_{i_2}^{(k-1)} \subseteq M_s^{(k)}$. Let us first show that

$$(M_{i_1}^{(k-1)} \cup M_{i_2}^{(k-1)}) \cap A_k = \emptyset\,. \tag{5.2}$$

Assume the converse: let, for example, $a \in M_{i_1}^{(k-1)} \cap A_k \neq \emptyset$. Then, because of the minimality of $M_s^{(k)}$, we have $M_{i_1}^{(k-1)} = M_s^{(k)}$ and, because of the inclusion $M_{i_2}^{(k-1)} \subseteq M_s^{(k)} = M_{i_1}^{(k-1)}$ and of the minimality of $M_{i_1}^{(k-1)}$, we obtain that $M_{i_1}^{(k-1)} = M_{i_2}^{(k-1)}$. By the definition of $M_s^{(k)}$, there exists $c \in M_s^{(k)} \cap A_k$, and $c \in M_s^{(k)} - M_{i_1}^{(k-1)}$, in view of (5.2). Since the sets $M_{i_1}^{(k-1)}$ and $M_{i_2}^{(k-1)}$ are inclusion-incomparable, there exists an element $b \in M_{i_2}^{(k-1)} - M_{i_1}^{(k-1)} \subset M_s^{(k)} - M_{i_1}^{(k-1)}$. Since $c \in A_k$ and $b \in M_{i_2}^{(k-1)}$, in view of (5.2), $c \neq b$. But then $M_s^{(k)} - \{b\} \supseteq M_{i_1}^{(k-1)} - \{c\}$, where $(M_{i_1}^{(k-1)} \cup \{c\}) \cap A_i \neq \emptyset$ for all $i \in [k]$, a contradiction with the minimality of $M_s^{(k)}$. This contradiction proves the proposition. □

Let us put in correspondence with the family $\mathcal{A}$ the directed graph $\mathbf{G}(\mathcal{A})$ with the vertex set $\{M_i^{(k)}: \ i \in [\beta_k], \ k \in \{0\} \dot\cup [\alpha]\}$, for which an arc from $M_{i_1}^{k_1}$ to $M_{i_2}^{k_2}$ exists if and only if $k_2 = k_1 + 1$ and $M_{i_1}^{k_1} \subseteq M_{i_2}^{k_2}$. Let us denote by $\Gamma(M_i^{(k)})$ the *out-neighborhood* of $M_i^{(k)}$, which is the set of the final vertices of all the arcs whose initial vertex is $M_i^{(k)}$; similarly, $\Gamma^{-1}(M_i^{(k)})$ will denote the *in-neighborhood* of $M_i^{(k)}$, which is the set of the initial vertices of all the arcs whose final vertex is $M_i^{(k)}$.

A graph is called a *rooted tree* with *root* y_0, if

- any vertex except y_0 has *in-degree one*, that is, such a vertex is the final vertex of precisely one arc;
- the vertex y_0 has in-degree zero;
- the vertex y_0 has nonzero *out-degree*, that is, y_0 is the initial vertex of at least one arc.

It follows immediately from Proposition 5.2 and from the definition of the graph $\mathbf{G}(\mathcal{A})$ that this is a rooted tree with the root $M_1^{(0)} = \emptyset$.

According to Proposition 5.2, for each vertex $M_i^{(k)}$ except the root, $\Gamma^{-1}(M_i^{(k)}) = 1$. Because of the minimality of $M_i^{(k)}$, we have $|M_i^{(k)} - \Gamma^{-1}(M_i^{(k)})| \leq 1$. We will call the number

$$\nu(M_i^{(k)}) := \begin{cases} 0, & \text{if } M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}) = \emptyset\,, \\ a \in M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}), & \text{if } M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}) \neq \emptyset\,, \end{cases}$$

the *inner number* of the vertex $M_i^{(k)}$ of the rooted tree $\mathbf{G}(\mathcal{A})$. The rooted tree $\mathbf{G}(\mathcal{A})$ is uniquely determined if every vertex of the tree is marked by its inner number, because for each vertex $M_i^{(k)}$ in this rooted tree there exists a unique chain $(M_1^{(0)}, M_{i_1}^{(1)}, \ldots, M_{i_{k-1}}^{(k-1)}, M_i^{(k)})$ connecting $M_i^{(k)}$ with the root $M_1^{(0)}$ and, besides

$$M_i^{(k)} = \{\nu(M_{i_1}^{(1)}) \cup \ldots \cup \nu(M_i^{(k)})\} - \{0\}\,. \tag{5.3}$$

The idea of Algorithm ROOTEDTREE that extracts all the minimal systems of representatives of the family $\mathcal{A}$ with the use of the rooted tree $\mathbf{G}(\mathcal{A})$, consists in the construction of the rooted tree by means of sequential inspection of its vertices. We traverse the rooted tree moving each time along arcs of the rooted tree $\mathbf{G}(\mathcal{A})$ as far as possible, and coming one step back in the direction opposite to that of an arc when further movement in the direction of the arc does not lead us to an uninspected vertex of the rooted tree $\mathbf{G}(\mathcal{A})$. A step in the direction of an arc will be called *forward*, and a step in the opposite direction will be called *backward*. A traversal can be arranged in such a way that it will be necessary to store, at any current moment, a relatively small amount of information.

Let us suppose that we are currently at a vertex $M_i^{(k)}$. We will need the following data:

- LENGTH – equals k for the current vertex $M_i^{(k)}$.
- NUMBERS(i) – a one-dimensional array of the inner numbers of vertices, composing the chain that connects the current vertex with the root of the rooted tree, ordered in accordance with this chain; it uniquely determines the current vertex.
- FORESTEP – a variable taking the value *BACKWARD* if we have arrived at the current vertex by moving in the opposite direction of an arc of the rooted tree, and the value *FORWARD* if we have arrived by moving in the direction of the arc.
- FORENUM – a variable equal to the inner number of the vertex by moving from which we have arrived at the current vertex after performing a backward step.
- TRAVERSAL – a variable taking the value *COMPLETED* if the traversal of the rooted tree $\mathbf{G}(\mathcal{A})$ is completed, and the value *NOTCOMPLETED* otherwise.
- $m, \alpha, \mathcal{A}$ – defined earlier.
- M_{pr} – the current vertex, that is the set consisting of the nonzero elements of the array NUMBERS and of its first LENGTH items.

In the array NUMBERS a stack is organized; the LENGTH variable determines the length of this stack. A forward step corresponds to the pushing of the inner number of the next vertex into the NUMBERS stack; a backward step corresponds to the pulling of the tail element out of the stack.

The above information completely determines the state of the algorithm. In order to justify functionality of Algorithm ROOTEDTREE, it suffices to describe the process of passaging from the current vertex to the next vertex when traversing the rooted tree $\mathbf{G}(\mathcal{A})$:

Algorithm ROOTEDTREE($\mathcal{A}$) *– a passage from the current vertex of the rooted tree* $\mathbf{G}(\mathcal{A})$

0. The start of an elementary traversal of the rooted tree $\mathbf{G}(\mathcal{A})$.
1. If LENGTH $= \alpha$, then to retrieve M_{pr} as the immediate minimal system of representatives of the family $\mathcal{A}$.
2. If LENGTH $\neq \alpha$ or (FORESTEP = *BACKWARD* and FORENUM $\in \{0, m\}$), go to instruction 6 – a backward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
3. If FORESTEP = *FORWARD* and $M_{\text{pr}} \cap A_{\text{LENGTH}+1} \neq \emptyset$, then $i_0 = 0$; go to instruction 7 – a forward move within the rooted tree $\mathbf{G}(\{A_1, A_2, \ldots, A_{\text{LENGTH}}\})$.
4. If FORESTEP = *BACKWARD*, then i := FORENUM; otherwise i := 0.
5. To inspect the integers starting with $i+1$, up to m, in the ascending order, up to the first number i_0 such that $M_{\text{pr}} \cup \{i_0\}$ is a minimal system of representatives of the family $\{A_1, A_2, \ldots, A_{\text{LENGTH}+1}\}$. If there does not exist such a number, go to instruction 6 – a backward move within the rooted tree $\mathbf{G}(\mathcal{A})$; else, go to instruction 7 – a forward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
6. A backward move within the rooted tree $\mathbf{G}(\mathcal{A})$. If LENGTH = 0, then TRAVERSAL = *COMPLETED*; go to instruction 7. Else:
   FORESTEP := *BACKWARD*,
   FORENUM := NUMBERS(LENGTH),
   LENGTH := LENGTH – 1; go to instruction 8.

7. A forward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
   FORESTEP := *FORWARD*,
   LENGTH := LENGTH + 1,
   NUMBERS(LENGTH) = i_0; go to instruction 8.
8. The end of an elementary traversal of the rooted tree $\mathbf{G}(\mathcal{A})$.

By repeating the above described actions until the variable TRAVERSAL takes the value *COMPLETED*, we will inspect all vertices of the rooted tree $\mathbf{G}(\mathcal{A})$ and find all the minimal systems of representatives $\mathcal{A}$; this is guaranteed by Proposition 5.2 and by the arrangement of the traversal.

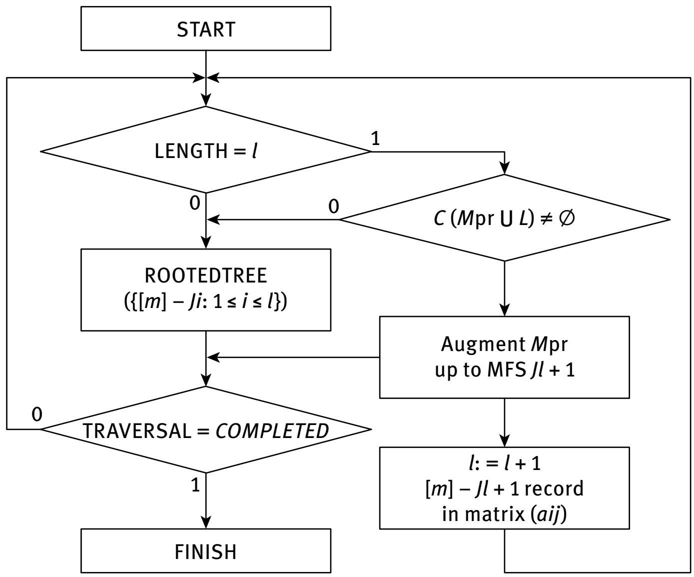
We now show how the above algorithm is used within Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon i \in T\}$), taking into account essential structural properties of the rooted tree of the minimal systems of representatives of the family $\{[m] - J_i\colon\ i \in T\}$. We will propose a realization of Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) more economical than its general scheme.

An economical realization of the combinatorial algorithm

Let us denote by $\{J_i\colon\ i \in T_{\mathrm{pr}}\}$ the family of the multi-indices of MFSs which are already known to the current moment. We emphasize the following observation: in the general scheme of Algorithm CMB, after the multi-index of a new MFS has been found, this multi-index is added to the family of the known multi-indices, and the process is repeated from the very beginning. As a matter of fact, it is possible to arrange Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) in such a way that we will find all multi-indices from the family $\{[m] - J_i\colon\ J_i \in \mathbf{J},\ J_i \supseteq L\}$ at one-pass traversal of the rooted tree of its minimal systems of representatives. Assume for simplicity that $\{J_i \in \mathbf{J}\colon\ i \in T\} = \{J_1, J_2, \dots, J_t\}$. The set T can be empty. Let the multi-indices of new MFSs get their indices starting with $t+1$, in order of their appearance during the work of the algorithm. Let information on the family $\{[m]-J_i\colon\ i \in T\}$ be given by the binary array $(a_{ij})_{m\times|T|}$; for every just found MFS with a multi-index J_j, the jth column of the matrix (a_{ij}) is automatically filled up; let l be the current number of columns of the matrix a_{ij}, that is the number of the already known MFSs. Let us use the algorithm of elementary traversal of the rooted tree $\mathbf{G}(\{[m] - J_i\colon\ i \in [l]\})$.

Let us show that Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) depicted as follows finishes, all those multi-indices of MFSs will be found that contain the multi-index L as a subset, that is, $\{J_1, J_2, \dots, J_{l^*}\} = \{J_i \in \mathbf{J}\colon\ J_i \supseteq L\}$.

For this, it suffices to show, in accordance with Proposition 5.1, that $\mathbf{C}_>(M \cup L) = \emptyset$ for any minimal system of representatives $M \in \mathfrak{B}(\{[m] - J_1, [m] - J_2, \dots, [m] - J_{l^*}\})$. Suppose to the converse that there exists $M_i^{(l^*)} \in \mathfrak{B}(\{[m]-J_1, [m]-J_2, \dots, [m]-J_{l^*}\})$ such that the subsystem with the multi-index $M_i^{l^*} \cup L$ is feasible. Let us consider the chain that connects the vertex $M_i^{(l^*)}$ with the root $M_1^{(0)}$ of the rooted tree $\mathbf{G}(\{[m] - J_i\ :\ i \in [l]\})$, which is the chain $(M_1^{(0)}, M_{i_1}^{(0)}, \dots, M_{i_{l^*-1}}^{(l^*-1)}, M_{i_{l^*}}^{(l^*)})$. Let $k \in [l^*]$ be the max-



imal integer such that Algorithm ROOTEDTREE($\{[m] - J_i:\ i \in [l]\}$) has led us to the vertex $M_{i_k}^{(k)}$.

Let us consider that moment of the execution of Algorithm ROOTEDTREE($\{[m] - J_i:\ i \in [l]\}$) when we left the vertex $M_i^{(k)}$ last time. It is clear that such a passage was backward, that is, we left $M_i^{(k)}$ for the vertex $\Gamma^{-1}\left(M_{i_k}^{(k)}\right) = M_{i_{k-1}}^{(k-1)}$. The latter is possible in the two cases:

1. $k < l \le l^*$ – Algorithm ROOTEDTREE($\{[m] - J_i:\ i \in [l]\}$) had already led us to all the vertices from $\Gamma\left(M_{i_k}^{(k)}\right)$; this contradicts the choice of k;
2. $k = l \le l^*$ and $\mathbf{C}_>(M^{(k)} \cup L) = \emptyset$; this contradicts the assumption $\emptyset \ne \mathbf{C}_>(M_i^{(l)} \cup L) \supseteq \mathbf{C}_>(M_i^{(k)} \cup L)$.

These contradictions prove that after Algorithm CMB($L, \{J_i \in \mathbf{J}:\ i \in T\}$) finishes, the multi-indices of all desirable MFSs will be found, that is, $\{J_1, J_2, \ldots, J_{l^*}\} = \{J_i \in \mathbf{J} : J_i \supseteq L\}$.

The combinatorial algorithm of solving the ($L, \{J_i \in \mathbf{J}:\ J_i \supseteq L\}$)-problem is difficult to use in practice because the computational burden, when extracting every new MFS, increases fast as the number of the already found MFSs increases. By the *combinatorial dimension* of the ($L, \{J_i \in \mathbf{J}:\ J_i \supseteq L\}$)-problem, as well as of Algorithm CMB($L, \{J_i \in \mathbf{J} : i \in T\}$), we will mean below the pair $(m, \#\{J_i \in \mathbf{J}:\ J_i \supseteq L\})$. This notion, being quite coarse, nevertheless reflects some properties of the ($L, \{J_i \in \mathbf{J}:\ i \in T\}$)-problem that affect the rapid growth of the computational burden as the number of the already found MFSs increases.

Graph-combinatorial algorithms of extracting MFSs of an infeasible system of linear inequalities

We consider here a reduction of the $(L, \emptyset)$-problem to a sequence of $(L_i, \{J_s : s \in T_i\})$-problems of lower combinatorial dimension. For this purpose, we will also construct an approximate algorithm of solving the $(L, \emptyset)$-problem, such that the computational burden when finding a new MFS grows much more slower than in the case of Algorithm CMB$(L, \emptyset)$.

Let us first consider the $(\emptyset, \emptyset)$-problem, which is the problem of extracting all the MFSs of the system $\mathfrak{S}$.

We will describe Algorithm GRAPH-CMB of solving the $(\emptyset, \emptyset)$-problem based on the connectedness of the graph of MFSs of the system $\mathfrak{S}$; it involves the above described Algorithm CMB.

Algorithm GRAPH-CMB

1. We find the multi-index J_1 of the first MFS, by augmenting up to this MFS, the feasible subsystem with the multi-index $\{1\}$, that is the subsystem consisting of the first inequality. The multi-index J_1 is assigned the mark 0.
2. Among the multi-indices of the already found MFSs, we pick at random a multi-index with the mark 0 and go to instruction 3. If there are no such multi-indices, then we go to instruction 4.
3. For the chosen multi-index J_s, we distinguish among the found multi-indices the family $\{J_i : i \in T\}$ of those multi-indices that are adjacent with J_s in the graph of MFSs of the system $\mathfrak{S}$. We then solve the $([m]-J_s, \{J_i : i \in T\})$-problem by Algorithm CMB$([m]-J_s, \{J_i : i \in T\})$, that is, we find all the multi-indices of MFSs adjacent with J_s in the graph of MFSs; all the new found multi-indices of MFSs get the mark 0; the multi-index J_s gets the mark 1. We go to instruction 2.
4. The algorithm finishes.

After the algorithm finishes, the multi-indices of all MFSs have the mark 1, that is, for the multi-index of every found MFS we also obtain the multi-indices of all the MFSs that are adjacent with it in the graph of MFSs; because of the connectedness of the graph of MFSs (Proposition 2.20) of the system $\mathfrak{S}$, Algorithm GRAPH-CMB thus finds the multi-indices of all MFSs.

Algorithm GRAPH-CMB can be arranged in such a way that the property mentioned in Proposition 2.35 (iii) will be used: the diameter of the graph $\mathtt{MFSG}(\mathfrak{S})$ of the system $\mathfrak{S}$ does not exceed half the number of inequalities in this system. If, when executing instruction 3 of Algorithm GRAPH-CMB, the distance between the vertices J_1 and J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ turns out to be equal to $\lfloor \frac{m}{2} \rfloor$, then we can immediately assign the mark 1 to the vertex J_s, without solving the $([m]-J_s, \{J_i \in \mathbf{J} : i \in T\})$-problem, and we can go to instruction 2. In view of the mentioned Proposition 2.35 (iii), thus modified Algorithm GRAPH-CMB still finds the multi-indices of all MFSs of the system $\mathfrak{S}$.

Let us now turn to the constructing of a graph-combinatorial algorithm of solving the $(L, \emptyset)$-problem; let us denote it by GRAPH-CMB(L). This algorithm is based on the property: the subgraph $\mathtt{MFSG}^2\langle\{J \in \mathbf{J} : J \supseteq L\}\rangle$ is connected, for any multi-index L of a feasible subsystem of the system $\mathfrak{S}$.

Algorithm GRAPH-CMB(L)

1. By augmenting, up to a MFS, the subsystem with the multi-index L, we find the multi-index J_1 of this MFS; the multi-index J_1 is assigned the mark 0.
2. From the family of the multi-indices of already found MFSs, we pick at random a multi-index with the mark 0, say the multi-index J_s, and choose all those multi-indices of MFSs $\{J_i \in \mathbf{J}: i \in T_s\}$ found that are adjacent to J_s in the graph of MFSs. If there are no multi-indices of MFSs with the mark 0, then we go to instruction 3. If $J_s \supseteq L$, then we set $L' := [m] - J_s$; if $J_s \not\supseteq L$, then we set $L' := ([m] - J_s) \cup L$. By launching Algorithm CMB(L, $\{J_i \in \mathbf{J}: i \in T_s\}$), we find all the multi-indices of MFSs that are adjacent with J_s in the graph of MFSs and contain L as a subset. The multi-indices of the new found MFSs are assigned the mark 0; the multi-index J_s is assigned the mark 1 when $J_s \not\supseteq L$, and it is assigned the mark 2 when $J_s \supseteq L$. We repeat the execution of instruction 2.
3. The algorithm finishes.

Proposition 5.3. *After Algorithm GRAPH-CMB(L) finishes, the multi-indices of the found MFSs, with the mark 2, compose the family of all the multi-indices of MFSs of the system $\mathfrak{S}$ that contain the multi-index L as a subset.*

Proof. Assume the converse. Then the family $\{J_i \in \mathbf{J}: J_i \supseteq L\}$ can be partitioned into two disjoint subfamilies: $\{J_i \in \mathbf{J}: J_i \supseteq L\} = \{J_i \in \mathbf{J}: i \in T\} \dot\cup \{J_i \in \mathbf{J}: i \in T'\}$, where $\{J_i \in \mathbf{J}: i \in T\}$ is the family of all the multi-indices of MFSs found by Algorithm GRAPH-CMB(L) and marked by 2. Since the subgraph $\mathtt{MFSG}^2\langle\{J \in \mathbf{J}: J \supseteq L\}\rangle$ is connected, there exists an edge $\{J_s, J_t\}$ in the square $\mathtt{MFSG}^2(\mathfrak{S})$ of the graph of MFSs of the system $\mathfrak{S}$ such that $J_s \in \{J_i \in \mathbf{J}: i \in T\}$ and $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$. As a consequence, the following two mutually exclusive cases are only possible: (1) the pair $\{J_s, J_t\}$ is an edge of the graph $\mathtt{MFSG}(\mathfrak{S})$, and (2) there exists the multi-index J_r of a MFS such that the pairs $\{J_s, J_r\}$ and $\{J_r, J_t\}$ are edges of the graph $\mathtt{MFSG}(\mathfrak{S})$.

Let us consider the first case. The mark 2 is assigned in Algorithm GRAPH-CMB(L) to the multi-index J_s of some MFS if and only if all the multi-indices of MFSs that are adjacent with J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ have been found; as a consequence, the multi-index J_t has already been found by the algorithm. This contradicts the inclusion $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$ because each found multi-index, after the algorithm finishes, has the mark 2 if it contains the multi-index L as a subset.

Let us consider the second case. As given earlier, the multi-index J_r was found by Algorithm GRAPH-CMB(L). Since at the finish of the algorithm the mark of the multi-index J_r is not 0, and the multi-index J_t is adjacent with J_r and it contains L as a subset, then the multi-index J_t was found by Algorithm GRAPH-CMB($[m] - J_r$, $\{J_i \in \mathbf{J}: i \in T_r\}$), and it was assigned the mark 2 at the finish of the algorithm, a contradiction with the inclusion $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$. □

The difference of Algorithm GRAPH-CMB(L) from Algorithm GRAPH-CMB consists in the following: in the former algorithm, for every multi-index $J_s \not\supseteq L$ of a found MFS, the $(L \cup ([m] - J_s), \{J_i \in \mathbf{J}: i \in T_s\})$-problem is solved which, in the general case, has combinatorial dimension lower than that of $(([m] - J_s), \{J_i: i \in T_s\})$-problem solved by the latter algorithm. Thus, in the general case, Algorithm GRAPH-CMB(L) is more economical, when finding the MFSs that contain the subsystem with the multi-index L, than Algorithm GRAPH-CMB.

Approximate combinatorial and graph-combinatorial algorithms

In practice, it is often suffices to know just a subfamily of the family of the multi-indices of MFSs of the system $\mathfrak{S}$. Therefore, it is natural to discuss an approximate $(L, \{J_s \in \mathbf{J} : s \in T_i\})$-problem.

By $(L, \{J_s \in \mathbf{J}: s \in T_i\})^{(k)}$*-problem* for the system $\mathfrak{S}$, we will mean the problem of extracting arbitrary $\min\{k, \#\{J \in \mathbf{J}: J \supseteq L\}\}$ multi-indices of MFSs that contain L as a subset. The combinatorial algorithm $\mathrm{CMB}(L, \{J_i \in \mathbf{J}: i \in T\})^{(k)}$ of solving the $(L, \{J_s \in \mathbf{J}: s \in T_i\})^{(k)}$-problem is obtained from Algorithm $\mathrm{CMB}(L, \{J_i \in \mathbf{J} : i \in T\})$ when we require that the latter algorithm finishes in the case where k multi-indices of MFSs have already been found. By *combinatorial complexity* of the $(L, \{J_s \in \mathbf{J}: s \in T_i\})^{(k)}$-problem or of Algorithm $\mathrm{CMB}(L, \{J_i \in \mathbf{J}: i \in T\})^{(k)}$, we will mean the pair $(m, \min\{k, \#\{J \in \mathbf{J}: J \supseteq L\}\})$. The combinatorial dimension of the $(\emptyset, \emptyset)^{(k)}$-problem, equal to $(m, \min\{k, \mathrm{q}\})$, will be still high; besides, Algorithm $\mathrm{CMB}(\emptyset, \emptyset)^{(k)}$ naturally faces the same difficulties as earlier.

The graph-combinatorial algorithm GRAPH-CMB$^{(k)}$ differs from Algorithm GRAPH-CMB: in the appropriate place, the approximate algorithm $\mathrm{CMB}(L, \{J_i \in \mathbf{J} : i \in T\})^{(k)}$ is used instead of the exact algorithm $\mathrm{CMB}(L, \{J_i \in \mathbf{J}: i \in T\})$, that is, for every multi-index of a MFS found, one searches, in the general case, for not necessarily all the multi-indices, adjacent with this multi-index in the graph $\mathtt{MFSG}(\mathfrak{S})$, but for just a number of them. It is remarkable that even for quite small k's, say for those close to n, Algorithm GRAPH-CMB$^{(k)}$ finds, thanks to the connectedness of the graph $\mathtt{MFSG}(\mathfrak{S})$, a large number of the multi-indices of MFSs of the system $\mathfrak{S}$; besides, the computational burden when extracting a new MFS, grows slower for small k's.

In conclusion, let us make a few remarks concerning Algorithm GRAPH-CMB$^{(k)}$. Suppose that for some system $\mathfrak{S}$ it was found, with the help of Algorithm GRAPH-CMB$^{(k)}$, a quite large number of its MFSs. The search for new MFSs of the system $\mathfrak{S}$ can be arranged in the following way: to find the multi-indices of MFSs adjacent simultaneously with two (three, and so on) already found multi-indices of MFSs, thus applying Algorithm $\text{GRAPH-CMB}(([m] - J_s) \cup ([m] - J_t), \{J_i \in \mathbf{J}: i \in T_{st}\})^{(k)}$ to those pairs J_s, J_t of already found multi-indices, for which $\mathbf{C}_>(([m] - J_s) \cup ([m] - J_t)) \neq \emptyset$; here $\{J_i \in \mathbf{J}: i \in T_{st}\}$ are those multi-indices of MFSs already found that are adjacent with both J_s and J_t. Since the vertex degrees in the graphs of MFSs are quite high, for many multi-indices of MFSs, not found yet, there are two, three, or a larger number of already found multi-indices of MFSs, that are adjacent with them. For this reason, combinatorial complexity of the $(([m] - J_s) \cup ([m] - J_t), \{J_i \in \mathbf{J}: i \in T_{st}\})$-problem can turn out to be less than combinatorial complexity of the $(([m] - J_s), \{J_i \in \mathbf{J}: i \in T_s\})$- and $(([m] - J_t), \{J_i \in \mathbf{J}: i \in T_t\})$-problems.

Odd cycles in the graph of MFSs, and committees

The following fundamental property of the graph $\mathtt{MFSG}(\mathfrak{S})$ underlies well-known methods of constructing committees of the system $\mathfrak{S}$:

Theorem 5.4. *Let a sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}}, J_{i_1})$ compose an odd cycle in the graph $\mathtt{MFSG}(\mathfrak{S})$ of the system $\mathfrak{S}$. Suppose that pairwise distinct vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$ are solutions to the MFSs with the multi-indices $J_1, J_2, \dots, J_{2k+1}$ respectively. Then the collection of vectors $\mathcal{K} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}\}$ is a committee of the system $\mathfrak{S}$.*

Proof. Assume the converse. Since the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$ are pairwise distinct, the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}\}$ contains $2k+1$ elements. Then there exists an integer $i_0 \in [m]$ such that the inequality with the index i_0 is satisfied by at most k vectors from the set $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$. As a consequence, in the sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}})$ of the multi-indices of MFSs there exist $k+1$ multi-indices of MFSs which do not contain the element i_0 and thus are not adjacent in the graph of MFSs of the system $\mathfrak{S}$. This contradicts the assumption that the sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}}, J_{i_1})$ composes a cycle in the graph of MFSs of the system $\mathfrak{S}$. ☐

5.2 The hypergraph of MFSs of an infeasible system of linear inequalities and committees

Let us consider a rank n infeasible system

$$\mathfrak{S}_{\boldsymbol{b}} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > b_i \colon \ \boldsymbol{b} := (b_1, \dots, b_m) \in \mathbb{R}^m, \ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; \ \|\boldsymbol{a}_i\| > 0, \ i \in [m]\} \quad (5.4)$$

of inhomogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$, with the property: each subsystem with two inequalities is feasible. Such a system has a *committee*, which is a finite set of vectors $\mathcal{K} \subset \mathbb{R}^n$ such that $|\{\boldsymbol{x} \in \mathcal{K} \colon \ \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > b_i\}| > \frac{1}{2}|\mathcal{K}|$, for every vector $\boldsymbol{a}_i$, $i \in [m]$.

By a *multi-committee* of system (5.4) we mean a *sequence* (also considered, if necessary, as an unordered *multiset*) of vectors $\mathcal{K} \subset \mathbb{R}^n$ with the same property: $|\{\boldsymbol{x} \in \mathcal{K} \colon \ \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > b_i\}| > \frac{1}{2}|\mathcal{K}|$, for every vector $\boldsymbol{a}_i$, $i \in [m]$.

A multi-committee $\mathcal{K}$ of a system $\mathfrak{S}_{\boldsymbol{b}}$ of linear inequalities (5.4) is called *minimal* if the system $\mathfrak{S}_{\boldsymbol{b}}$ has no multi-committee of cardinality less than $|\mathcal{K}|$.

Let us consider a linear operator $\Phi\colon \ \mathbb{R}^n \to \mathbb{R}^2$ such that the sequence of images $(\Phi(\boldsymbol{a}_1), \dots, \Phi(\boldsymbol{a}_m))$ of the vectors from the sequence $\boldsymbol{A}(\mathfrak{S}_{\boldsymbol{b}}) := \{\boldsymbol{a}_i \colon \ i \in [m]\}$ that determines the system $\mathfrak{S}_{\boldsymbol{b}}$ does not contain the origin and antipodal pairs. The system

$$\{\langle \Phi(\boldsymbol{a}_i), \mathbf{y}\rangle > 0 \colon \ \mathbf{y} \in \mathbb{R}^2, \ i \in [m]\} \quad (5.5)$$

has a committee, for example, according to Theorem 5.4 and Proposition 2.33 and, thus, the system

$$\{\langle \Phi(\boldsymbol{a}_i), \mathbf{y}\rangle > b_i \colon \ \mathbf{y} \in \mathbb{R}^2, \ i \in [m]\} \quad (5.6)$$

also has a committee.

If a sequence $\mathcal{K}' := (\boldsymbol{y}_1, \dots, \boldsymbol{y}_q)$ is a multi-committee of system (5.6), then the sequence $(\Phi^*(\boldsymbol{y}_1), \dots, \Phi^*(\boldsymbol{y}_m))$ of the images of its elements under the map Φ^*, the adjoint of Φ, is a multi-committee of system (5.4).

Let $\{J^{\mathbf{0}}_1, \dots, J^{\mathbf{0}}_q\}$ and $\{J_1, \dots, J_r\}$ be the families of the multi-indices of MFSs of systems (5.5) and (5.6), respectively; see Section 2.4. The hypergraph of MFSs of system (5.5) is denoted by $\mathtt{MFSH}(\mathfrak{S}_{\mathbf{0}}, \Phi) := ((J^{\mathbf{0}}_1, \dots, J^{\mathbf{0}}_q), \mathcal{E}^{\mathbf{0}})$.

Let us consider the family

$$W := 2^{\{J^{\mathbf{0}}_1, \dots, J^{\mathbf{0}}_q\}} - (\emptyset, \{J^{\mathbf{0}}_1\}, \dots, \{J^{\mathbf{0}}_q\}) \,.$$

For each element $w := \{J^{\mathbf{0}}_{i_1}, \dots, J^{\mathbf{0}}_{i_s}\}$ there exists $k \in [s]$ such that $(i_{(k \pmod s))+1} - i_k) \pmod q > t + 1$, where $q := 2t + 1$. Let us define a map $\Lambda\colon\ W \to \mathbb{Z}$ by $\Lambda(w) := (i_k - i_{(k \pmod s))+1}) \pmod q$, the family

$$W' := \Big\{ w := \{J^{\mathbf{0}}_{i_1}, \dots, J^{\mathbf{0}}_{i_s}\} \colon\ \forall k \in [s]\ \exists J_{j_k}\colon\ J_{j_k} \supseteq J^{\mathbf{0}}_{i_k},\ \bigcup_{k \in [s]} J_{j_k} = [m] \Big\}$$

and the quantity

$$\delta(\mathfrak{S}_{\boldsymbol{b}}) := \begin{cases} \min\{\Lambda(w)\colon\ w \in W'\}, & \text{if } |W'| > 0\,, \\ t, & \text{if } |W'| = 0\,. \end{cases}$$

Proposition 5.5. *The number of elements in a minimal multi-committee of system* (5.4) *does not exceed* $2\delta(\mathfrak{S}_{\boldsymbol{b}}) + 1$.

In this proposition, a bound on the number of elements in a minimal multi-committee of systems $\mathfrak{S}_{\boldsymbol{b}}$ is justified, which depends on the set $\boldsymbol{A}(\mathfrak{S}_{\boldsymbol{b}})$ of determining vectors and on the vector $\boldsymbol{b}$. In the proof of Proposition 5.5, which we omit, one finds for system (5.6) a multi-committee, minimal among all the multi-committees formed from the solutions to MFSs whose multi-indices contain the multi-indices of MFSs of system (5.5), that compose a chain in the hypergraph $\mathtt{MFSH}(\mathfrak{S}_{\mathbf{0}}, \Phi)$ of maximal feasible subsystems of system (5.5).

5.3 Alternative covers

Let X be a nonempty set of any kind, and $\mathcal{M} \subseteq \mathbf{2}^X$ some family of subsets of the set X. Let $\mathcal{A}, \mathcal{B} \subset X$ be nonempty disjoint subsets of the set X.

An ordered pair $(\mathfrak{A}, \mathfrak{B})$ *of families* $\mathfrak{A}, \mathfrak{B} \subset \mathcal{M}$ *of subsets of the set X, picked from the admissible family* $\mathcal{M}$*, will be called an* alternative cover *of the pair* $(\mathcal{A}, \mathcal{B})$ *if* $\mathcal{A} = \bigcup_{A \in \mathfrak{A}} A$, $\mathcal{B} = \bigcup_{B \in \mathfrak{B}} B$*, and* $A \cap B = \emptyset$ *for any sets* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{B}$.

An alternative cover will be called finite *if each of the families* $\mathfrak{A}$ *and* $\mathfrak{B}$ *is finite.*

By the cardinality *of a finite alternative cover* $(\mathfrak{A}, \mathfrak{B})$ *we mean the quantity* $|\mathfrak{A}| + |\mathfrak{B}|$.

This construction has a close relation with the pattern recognition subject. Suppose that in a space X two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ are fixed, and some admissible family $\mathcal{M}$ of subsets of the space X is predetermined. If there exists an alternative cover

$(\mathfrak{A}, \mathfrak{B})$ of the pair $(\mathcal{A}, \mathcal{B})$ then it *separates* the sets $\mathcal{A}$ and $\mathcal{B}$ in the space X. As a consequence, the problem of *efficient* separation of the subsets $\mathcal{A}$ and $\mathcal{B}$ of the space X, in the class of subsets from $\mathcal{M}$, can be stated as the problem of the search for a finite alternative cover, of *minimal cardinality*, of the pair $(\mathcal{A}, \mathcal{B})$.

Thus, we can symbolically write down the pattern recognition problem, in its geometric setting, as follows:

$$\mathrm{R}_1\colon\ \big(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X\big) \to (\mathfrak{A}, \mathfrak{B})\ .$$

Substantially, alternative covers can differ; the next clarification of the pattern recognition problem consists in that a quality functional $f(\mathcal{A}, \mathcal{B})$ for an alternative cover is introduced which should be optimized, say minimized:

$$\mathrm{R}_2\colon\ \big(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X,\ f\colon\ \mathbf{2}^{\mathcal{M}} \times \mathbf{2}^{\mathcal{M}} \to \mathbb{R}\big) \xrightarrow{\min f} (\mathfrak{A}, \mathfrak{B})\ .$$

One natural criterion of the quality of an alternative cover is its cardinality. Let us denote by f_{card} the map of the form $f\colon\ \mathbf{2}^{\mathcal{M}} \times \mathbf{2}^{\mathcal{M}} \to \mathbb{N}$, for which we have $f_{\mathrm{card}}(\mathcal{A}, \mathcal{B}) := |\mathcal{A}| + |\mathcal{B}|$. We state the problem

$$\mathrm{R}_3\colon\ \big(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X,\ f_{\mathrm{card}}\big) \xrightarrow{\min f_{\mathrm{card}}} (\mathfrak{A}, \mathfrak{B})\ .$$

Interpretation of committees in terms of alternative covers

Suppose that in a space X finite disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ are fixed, as well as some class F of real-valued functions over X. Let us consider the inequality system

$$\begin{cases} f(\boldsymbol{x}) > 0, & \text{if } \boldsymbol{x} \in \mathcal{A}\ , \\ f(\boldsymbol{x}) < 0, & \text{if } \boldsymbol{x} \in \mathcal{B}\ . \end{cases} \tag{5.7}$$

Recall that by a *committee* of inequality system (5.7) we mean a finite collection of maps $\mathcal{K} := \{f_1, f_2, \ldots, f_q\} \subset F$, such that each inequality of system (5.7) is satisfied by more than half maps from $\mathcal{K}$.

For some subset of real-valued functions $F_0 \subseteq F$, the sets

$$\mathbf{C}_>(F_0) = \bigcap_{f \in F_0} \{\boldsymbol{x} \in X\colon\ f(\boldsymbol{x}) > 0\}$$

and

$$\mathbf{C}_<(F_0) = \bigcap_{f \in F_0} \{\boldsymbol{x} \in X\colon\ f(\boldsymbol{x}) < 0\}$$

will be called *F-polyhedra* of the space X. Let us denote by $\mathcal{M}(F, X)$ the class of all F-polyhedra of the space X. The pattern recognition problems R_1–R_3 can then be regarded in the situation when the class $\mathcal{M}$ is the class $\mathcal{M}(F, X)$, that is,

$$\mathrm{R}_1'\colon\ \big(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)\big) \to (\mathfrak{A}, \mathfrak{B})\ ,$$

and analogously for R_2 and R_3.

With each committee $\mathcal{K}$ of inequality system (5.7) can be put in correspondence an alternative cover $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$ of the pair $(\mathcal{A}, \mathcal{B})$ as follows:

$$\mathfrak{A} := \left\{\mathbf{C}_>(\mathcal{K}')\colon\ |\mathcal{K}'| > \tfrac{1}{2}|\mathcal{K}|,\ \mathcal{K}' \subseteq \mathcal{K}\right\}\,,$$
$$\mathfrak{B} := \left\{\mathbf{C}_<(\mathcal{K}')\colon\ |\mathcal{K}'| > \tfrac{1}{2}|\mathcal{K}|,\ \mathcal{K}' \subseteq \mathcal{K}\right\}\,.$$

It follows from the definitions of a committee and family $(\mathfrak{A}, \mathfrak{B})$, that $(\mathfrak{A}, \mathfrak{B})$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$. It is evident that $\mathbf{C}_>(\mathcal{K}') \cap \mathbf{C}_<(\mathcal{K}'') = \emptyset$ when $|\mathcal{K}'| > \frac{1}{2}|\mathcal{K}|$ and $|\mathcal{K}''| > \frac{1}{2}|\mathcal{K}|$. Let us denote by $\mathbf{max}_{\subseteq}\mathfrak{A}$ the set of maximal elements of the poset $(\mathfrak{A}, \subseteq)$, and suppose

$$\mathfrak{A} = \mathbf{max}_{\subseteq}\mathcal{A}\,, \quad \mathfrak{B} = \mathbf{max}_{\subseteq}\mathcal{B}$$

and

$$\mathfrak{A}(\mathcal{K}) := \{A' \in \mathbf{max}_{\subseteq}\mathfrak{A}\colon\ A' \cap \mathcal{A} \neq \emptyset\}\,,$$
$$\mathfrak{B}(\mathcal{K}) := \{B' \in \mathbf{max}_{\subseteq}\mathfrak{B}\colon\ B' \cap \mathcal{B} \neq \emptyset\}\,.$$

Thus, $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra.

With a committee $\mathcal{K}$ of system (5.7) we have put in one-to-one correspondence the alternative cover $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$. Let us depict symbolically the scheme of constructing alternative covers of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra on the basis of the committee method:

$$\mathrm{R}'_{\mathrm{com}}\colon\ (X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)) \to \mathcal{K} \to (\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))\,.$$

There exist systems of the form (5.7) such that for two distinct committees $\mathcal{K}_1$ and $\mathcal{K}_2$ of the same cardinality, the following relations hold:

$$|\mathcal{K}_1| = |\mathcal{K}_2|\,,$$
$$|\mathfrak{A}(\mathcal{K}_1)| + |\mathfrak{B}(\mathcal{K}_1)| \neq |\mathfrak{A}(\mathcal{K}_2)| + |\mathfrak{B}(\mathcal{K}_2)|\,.$$

As an example, let X be the space $\mathbb{R}^2$, and F the class of linear functionals; let us suppose

$$\mathcal{A} := \{\boldsymbol{a}_1 = (-1.5, 1.5),\ \boldsymbol{a}_2 = (1.5, 1.5),\ \boldsymbol{a}_3 = (0, -1)\}\,,$$
$$\mathcal{B} := \{\boldsymbol{b}_1 = (-8, 3),\ \boldsymbol{a}_2 = (0, -5),\ \boldsymbol{a}_3 = (8, -3)\}\,,$$
$$\mathcal{K}_1 := \{f_1 = \boldsymbol{x}_1 - \boldsymbol{x}_2 + 2,\ f_2 = -\boldsymbol{x}_2,\ f_3 = -\boldsymbol{x}_1 - \boldsymbol{x}_2 + 2\}\,,$$
$$\mathcal{K}_2 := \{g_1 = \boldsymbol{x}_1 - \boldsymbol{x}_2 + 4,\ g_2 = -\boldsymbol{x}_2 - 2,\ g_3 = -\boldsymbol{x}_1 - \boldsymbol{x}_2 + 4\}\,,$$
$$\mathfrak{A}(\mathcal{K}_1) := \{\mathbf{C}_>(\{f_1, f_2\}),\ \mathbf{C}_>(\{f_1, f_3\}),\ \mathbf{C}_>(\{f_2, f_3\})\}\,,$$
$$\mathfrak{B}(\mathcal{K}_1) := \{\mathbf{C}_<(\{f_1, f_2\}),\ \mathbf{C}_<(\{f_1, f_3\}),\ \mathbf{C}_<(\{f_2, f_3\})\}\,,$$
$$\mathfrak{A}(\mathcal{K}_2) := \{\mathbf{C}_>(\{f_1, f_2, f_3\})\}\,,$$
$$\mathfrak{B}(\mathcal{K}_2) := \{\mathbf{C}_<(\{f_1, f_2\}),\ \mathbf{C}_<(\{f_1, f_3\}),\ \mathbf{C}_<(\{f_2, f_3\})\}\,.$$

The committees $\mathcal{K}_1$ and $\mathcal{K}_2$ contain an equal and minimal possible number of members (namely, three), but the alternative covers corresponding to these committees have different cardinalities: $|\mathfrak{A}(\mathcal{K}_1)| + |\mathfrak{B}(\mathcal{K}_1)| = 6$, while $|\mathfrak{A}(\mathcal{K}_2)| + |\mathfrak{B}(\mathcal{K}_2)| = 4$.

Thus, we can conclude that a committee of system (5.7) represents a concise form of determining an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra, but not necessarily of minimal cardinality. This conclusion is rather important for the committee method, because it motivates us to take into account a very natural and relevant additional criterion of decision rule optimization. A simplest application of such an approach is as follows: if several committees of system (5.7) are found, then one should choose a committee such that the corresponding alternative cover has minimal cardinality.

In the case when $X := \mathbb{R}^n$, and F is the class of linear functionals, let us consider subsystems of system (5.7), defined as follows. Let $\mathcal{L} \subset \mathcal{A} \cup \mathcal{B}$; then an inequality $f(\boldsymbol{x}) > 0$ belongs to the subsystem under consideration when $\boldsymbol{x} \in \mathcal{A}$, and an inequality $f(\boldsymbol{x}) < 0$ belongs to this subsystem when $\boldsymbol{x} \in \mathcal{B}$. Let us denote this subsystem by $\mathrm{S}(\mathcal{A}', \mathcal{B}')$, where $\mathcal{A}' := \mathcal{L} \cap \mathcal{A}$ and $\mathcal{B}' := \mathcal{L} \cap \mathcal{B}$. Suppose that inequality system (5.7) is infeasible, and $\mathcal{K} := \{f_1, f_2, \ldots, f_q\}$ is its committee. Let $(\mathfrak{A}', \mathfrak{B}')$ be the alternative cover that corresponds to the committee $\mathcal{K}$. The pair $(\mathfrak{A}', \mathfrak{B}')$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of convex cones.

Alternative covers do not necessarily possess committees that generate them.

The following algorithmic problem can be stated. For a given inequality system of the form (5.7), to find a committee that generates an alternative cover of cardinality, minimal among all committees of system (5.7).

Let $\{\mathbf{J}(\mathcal{A}_i, \mathcal{B}_i) : \ i \in [q]\}$ be the family of all inclusion-maximal feasible subsystems of system (5.7). Let us form the $|\mathcal{A} \dot\cup \mathcal{B}| \times q$ incidence matrix $\mathbf{E} = (e_{ij})$: its first $|\mathcal{A}|$ rows are marked by the elements from $\mathcal{A}$, the last $|\mathcal{B}|$ rows are marked by the elements from $\mathcal{B}$, and the columns are marked by the maximal feasible subsystems of system (5.7).

In terms of the matrix $\mathbf{E}$, we can restate the problem as follows:

1. The problem of constructing a committee with the minimal number of members is reduced to choosing an inclusion-minimal collection of columns of the matrix $\mathbf{E}$ such that in each subrow, obtained as the intersection of the corresponding entire row with the above-mentioned columns, the number of units exceeds the number of zeros.
2. The problem of constructing a committee that generates an alternative cover of minimal cardinality is reduced to choosing a collection of columns of the matrix $\mathbf{E}$, such that condition 1 is fulfilled and, moreover, the total number of inclusion-minimal subrows of the upper and lower semimatrices is minimal among all such collections.

Interpretation of logical decision trees in terms of alternative covers

Suppose that in a space X finite disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ are fixed, as well as some class F of real-valued functions over X. Let us consider a binary tree $\mathcal{T} := (V, E)$ with root v_0, with each node of which a function from F is associated, that is, a map $\psi\colon\ V \to F$ is determined. Let v be a leaf of the tree $\mathcal{T}$ and $(v_0, v_1, \dots, v_{k-1}, v)$ the path from the root of the tree to the current node v. Then the leaf v is assigned the F-polyhedron

$$\mathbf{C}_v := \mathbf{C}_>\left(\left\{(-1)^{\sigma(v_0)}\psi(v_0), (-1)^{\sigma(v_1)}\psi(v_1), \dots, (-1)^{\sigma(v_{k-1})}\psi(v_{k-1})\right\}\right) ,$$

where $\sigma(v_i) := \begin{cases} 1, & \text{if } v_i \text{ is the left child of } v_{i-1}, \\ 0, & \text{if } v_i \text{ is the right child of } v_{i-1}. \end{cases}$

Let V' be the set of leaves of the tree $\mathcal{T}$. Let us suppose

$$\begin{aligned} \mathbf{C}_{V'}(\mathcal{T})\colon\ &= \{\mathbf{C}_v\colon\ v \in V'\} , \\ \mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T})\colon\ &= \{\mathbf{C}_v\colon\ v \in \mathbf{C}_{V'}(\mathcal{T}) \cap \mathcal{A}\} , \\ \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\colon\ &= \{\mathbf{C}_v\colon\ v \in \mathbf{C}_{V'}(\mathcal{T}) \cap \mathcal{B}\} . \end{aligned}$$

If $\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}) \cap \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T}) = \emptyset$ and $\mathcal{A} \cup \mathcal{B} \subset \bigcup_{v \in V'} \mathbf{C}_v$, then $\left(\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}), \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\right)$ is an alternative cover of $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra.

Let us depict symbolically the scheme of constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra on the basis of logical decision trees:

$$\mathrm{R}'_{\text{tree}}\colon\ (X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)) \to (\mathcal{T} := (V, E)) \to \left(\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}), \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\right) .$$

Constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in accordance with the scheme $\mathrm{R}'_{\text{tree}}$ has, compared to the scheme R'_{com}, the following advantages:

(1) Suppose that for the pair $(\mathcal{A}, \mathcal{B})$ two trees $\mathcal{T}_1 := (V_1, E_1)$ and $\mathcal{T}_2 := (V_2, E_2)$ are given, such that $(\mathbf{C}^{\mathcal{A}}_{V'_1}(\mathcal{T}_1), \mathbf{C}^{\mathcal{B}}_{V'_1}(\mathcal{T}_1))$ and $(\mathbf{C}^{\mathcal{A}}_{V'_2}(\mathcal{T}_2), \mathbf{C}^{\mathcal{B}}_{V'_2}(\mathcal{T}_2))$ are alternative covers of the pair $(\mathcal{A}, \mathcal{B})$, and they are inclusion-minimal with respect to this property. If $|V_1| = |V_2|$, then the corresponding alternative covers also have the same cardinality.

(2) The obtained decision and the process itself have good substantial interpretation.

(3) In view of (2), organization of the constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$, in accordance with the scheme $\mathrm{R}'_{\text{tree}}$, in interactive mode appears to be natural and efficient.

(4) When constructing an alternative cover in accordance with the scheme $\mathrm{R}'_{\text{tree}}$, the problem of missing fragments of initial data, difficult in the case of the scheme R'_{com}, is solved naturally and quite easily.

Notes

The notion of committee of a linear inequality system was first formulated, in the explicit form, in notes [1, 2]. A systematical study of various committee constructions appeared in works [94, 95] and transformed over the past decades into an important independent branch of pure and applied mathematical investigations that exercise a significant influence on the arsenal of efficient methods in optimization and pattern recognition. We refer the reader to monograph [96] for a comprehensive review of fundamental results in the committee theory, as well as their applications. Key advances in the ever widening bounds of the theory of committee constructions are the research subject, for example, in surveys [76, 97–99].

On page 115, we briefly describe the two-class pattern recognition problem following [96].

The material in Section 5.1 is particularly based on the results of works [47, 49–51]. The combinatorial algorithm of extracting MFSs of an infeasible system of linear inequalities, as noted on page 117, repeatedly builds the blockers of set families; recall that constructing the blockers can also be interpreted as the finding of inclusion-minimal covers of the columns of (0, 1)-matrices that determine the families. See, for example, works [27, 61, 68, 109, 114, 157, 164] on the constructing representative systems.

The connectedness of the squares of subgraphs of the graphs of MFSs, mentioned on page 122, was proved by the second author of work [58].

In Section 5.2, we adopt fragments of surveys [76, 98]. We use the term *multi-committee* for committee constructions that represent, in the general case, multisets; the properties of the hypergraphs of MFSs are analyzed, in particular, in [74, 75].

The outline of alternative covers presented in Section 5.3 follows article [48].