

Contents

Acknowledgements — V

Notation — VII

Part I: Preliminary results

- 1 Elements of functional analysis — 3**
 - 1.1 Continuous functions, measures — 3
 - 1.2 Topological spaces — 5
 - 1.3 Differentiable functions, distributions — 6
 - 1.4 Integrable functions — 7
 - 1.5 Compactness and convergence of integrable functions — 9
 - 1.6 Sobolev spaces — 10
 - 1.7 Sobolev spaces of periodic functions — 12
 - 1.7.1 Hilbertian structure — 12
 - 1.7.2 L^p -structure — 13
 - 1.7.3 Regularization by convolution kernels — 14
 - 1.8 Bochner spaces — 15
 - 1.8.1 Time regularity — 15
 - 1.8.2 Compact embeddings — 16
 - 1.8.3 Regularization by convolution kernels — 18

- 2 Elements of stochastic analysis — 21**
 - 2.1 Random variables and stochastic processes — 21
 - 2.2 Random distributions — 30
 - 2.2.1 Measurability — 31
 - 2.2.2 Regularization — 32
 - 2.2.3 Equality in law — 34
 - 2.2.4 Progressive measurability — 36
 - 2.2.5 Special classes of random distributions — 39
 - 2.3 Stochastic Itô's integral — 40
 - 2.4 Itô's formula — 46
 - 2.5 Pathwise vs. martingale solutions — 47
 - 2.5.1 Pathwise uniqueness vs. uniqueness in law — 49
 - 2.6 Stochastic compactness method — 50
 - 2.7 Jakubowski–Skorokhod representation theorem — 55
 - 2.8 Random distributions in L^p and Young measures — 57
 - 2.9 Stochastic partial differential equations — 61

- 2.10 Gyöngy–Krylov lemma — 66
- 2.11 Stationarity — 70
- 2.12 Krylov–Bogoliubov method — 74

Part II: Existence theory

- 3 Modeling fluid motion subject to random effects — 81**
 - 3.1 Field equations — 82
 - 3.1.1 Constitutive relations – Navier–Stokes system — 83
 - 3.2 Random phenomena — 84
 - 3.2.1 Initial data — 84
 - 3.2.2 Driving force — 86
 - 3.3 Strong pathwise solutions — 88
 - 3.4 Dissipative martingale solutions — 90
 - 3.4.1 Weak formulation — 92
 - 3.4.2 Regularity properties of weak solutions — 94
 - 3.5 Stationary solutions — 97

- 4 Global existence — 101**
 - 4.1 Solvability of the basic approximate problem — 107
 - 4.1.1 Iteration scheme — 108
 - 4.1.2 The limit for vanishing time step — 110
 - 4.1.2.1 Regularity for the viscous approximation of the equation of continuity — 110
 - 4.1.2.2 Bounds on the approximate velocities — 111
 - 4.1.2.3 Hölder continuity of approximate velocities — 112
 - 4.1.2.4 Solvability of the first level approximate problem — 113
 - 4.1.3 Pathwise uniqueness — 117
 - 4.1.4 Strong solutions — 121
 - 4.1.5 General initial data — 123
 - 4.1.6 Energy balance — 124
 - 4.2 Solvability of the Galerkin approximation — 126
 - 4.2.1 Uniform energy bounds — 128
 - 4.2.2 Passage to the limit — 129
 - 4.3 The limit in the Galerkin approximation scheme — 131
 - 4.3.1 Uniform bounds — 133
 - 4.3.2 Asymptotic limit — 135
 - 4.4 Vanishing viscosity limit — 146
 - 4.4.1 Uniform energy bounds — 148
 - 4.4.2 Pressure estimates — 150
 - 4.4.3 Limit $\varepsilon \rightarrow 0$ — 153

- 4.4.3.1 Stochastic compactness method — 155
- 4.4.3.2 Deterministic compactness method — 164
- 4.5 Vanishing artificial pressure limit — 169
- 4.5.1 Uniform energy bounds — 171
- 4.5.2 Pressure estimates — 172
- 4.5.3 Limit $\delta \rightarrow 0$ – stochastic compactness method — 175
- 4.5.4 Limit $\delta \rightarrow 0$ – deterministic compactness method — 181
- 4.5.4.1 Compactness of the density — 181

- 5 Local well-posedness — 187**
- 5.1 Preliminary considerations — 191
- 5.1.1 Rewriting the equations as a symmetric hyperbolic-parabolic problem — 193
- 5.1.2 Outline of the proof of Theorem 5.0.3 — 194
- 5.2 The approximate system — 195
- 5.2.1 The Galerkin approximation — 198
- 5.2.2 Uniform estimates — 200
- 5.2.3 Compactness — 204
- 5.2.4 Identification of the limit — 206
- 5.2.5 Pathwise uniqueness — 207
- 5.2.6 Existence of a strong pathwise approximate solution — 209
- 5.3 Proof of Theorem 5.0.3 — 211
- 5.3.1 Uniqueness — 211
- 5.3.2 Existence of a local strong solution for bounded initial data — 212
- 5.3.3 Existence of a local strong solution for general initial data — 213
- 5.3.4 Existence of a maximal strong solution — 214

- 6 Relative energy inequality and weak–strong uniqueness — 217**
- 6.1 Relative energy inequality — 220
- 6.2 Weak–strong uniqueness — 223
- 6.2.1 Pathwise weak–strong uniqueness — 224
- 6.2.2 Weak–strong uniqueness in law — 228

Part III: Applications

- 7 Stationary solutions — 235**
- 7.1 Basic finite-dimensional approximation — 240
- 7.1.1 Approximate field equations — 240
- 7.1.2 Basic energy estimates — 241
- 7.1.3 Regularity of the density — 244
- 7.1.4 Approximate invariant measures — 246

7.2	First limit procedures: $R \rightarrow \infty, m \rightarrow \infty$ —	250
7.3	Vanishing viscosity limit —	255
7.4	Vanishing artificial pressure limit —	266
8	Singular limits —	271
8.1	Incompressible limit —	273
8.1.1	Incompressible Navier–Stokes equations —	275
8.1.2	Main result —	278
8.1.3	Convergence in law – the proof of Theorem 8.1.6 —	280
8.1.3.1	Uniform bounds —	280
8.1.3.2	Acoustic equation —	284
8.1.3.3	Compactness —	285
8.1.3.4	Identification of the limit —	289
8.1.4	Convergence in probability – the proof of Theorem 8.1.7 —	295
8.2	Inviscid–incompressible limit —	297
8.2.1	Solutions of the Euler system —	298
8.2.2	Main result —	300
8.2.3	Proof of Theorem 8.2.4 —	302
A	Appendix —	305
A.1	Elliptic equations and related problems —	305
A.2	Regularity for parabolic equations —	309
A.3	Renormalized solutions of the continuity equation —	312
A.4	A generalized Itô formula —	313
B	Bibliographical remarks —	317
	Bibliography —	319
	Index —	327