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2 Model order reduction in mechanical engineering

Abstract: This chapter describes several “success stories” of model order reduction (MOR) in mechanical engineering. First, the reader will be given an overview of specific model representations, MOR requirements, and reduction techniques relevant in the different fields of mechanical engineering. Then, four applications are presented: the reduction of a thermo-mechanical machining tool, the reduction of models of a car body and driver’s seat, the reduction of an elastic crankshaft, and the reduction of a leaf spring model.

Keywords: Model order reduction, mechanical engineering, second-order systems, projection

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2.1 Introduction

In the past 50 years, the number of applications of model reduction in mechanical engineering has increased massively, while the first applications in structural mechanics and acoustics ([94, 55, 56, 24], see also [18]) go back even further in time. *Modal methods* can be applied either to second-order models (common in multibody systems [48, 24, 1, 33, 84]) or to state-space representations. Corresponding methods were intensively developed from the 1960s onwards ([25, 73, 23, 20] and Chapter 4 of Volume 1 of *Model order reduction*) and are applied in other areas of engineering as well, for instance in the reduction of power systems [64] and for the purpose of control design, e. g., [71, 66, 58, 7]. With the advent of *balanced truncation* and of *Krylov subspace methods* ([78, 44], [42, 45], overviews in [4, 8, 10], and Chapters 2 and 3 of Volume 1 of *Model order reduction*) the approximation quality and the applicability to high- and very high-order linear systems improved significantly and opened numerous fields of applications. *Proper orthogonal decomposition* (POD) methods based on snapshots

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[63, 70, 60] and *hyper-reduction* techniques [22, 30, 31] were developed for the reduction of *nonlinear* models.

Today, the successful applications of model order reduction (MOR) in mechanical engineering deal with different system and problem classes from different physical domains, like:

- *structural and multibody dynamics*, modeled by linear or nonlinear differential equations [18, 17, 21, 67, 69, 35, 34, 53, 3, 31, 91, 90, 5, 104, 19, 49, 47, 95, 110];
- *fluid dynamics*, including fluid–structure interaction and aerodynamics [93, 26, 89, 88, 83, 2, 31];
- *thermo-mechanical, thermo-fluid, thermo-acoustic, and thermo-electrical systems* [68, 12, 14, 11, 97, 75, 54, 46].

An overview on *coupled* problems is [15]. It should be emphasized that this list of references is incomplete. In view of the huge number of publications on MOR applications, it seems rather impossible to give a complete overview. Besides the many scientific publications, there exist several collections of benchmark problems, like the “Oberwolfach Benchmark Collection,” the “Niconet Benchmark Collection,” and the “MOR-Wiki.” These collections include system models from different domains and can serve for the validation process.

In Sections 2.2 to 2.5 of this chapter, four successful applications are presented as examples of MOR of industrial problems:

- The reduction of a coupled thermo-mechanical machining tool model applies Krylov subspace-based reduction to a first- or second-order formulation of the linear model.
- The reduction of industrial models of a car body and driver’s seat applies a Craig–Bampton method and a Gramian-based method to second-order models of very high order.
- The error-controlled reduction of an elastic crankshaft applies a combination of methods to a second-order model.
- The reduction of a leaf spring model applies simulation-free projection and hyper-reduction to a nonlinear second-order model.

In the following, an overview of common representations for mechanical systems, modeling aspects, and basic properties will be given.

Typically, mechanical models originate either from direct discrete modeling, spatial discretization of partial differential equations, system identification, or a combination of those.

Direct discrete modeling uses discrete, linear, or nonlinear mass, damper, and spring elements to build up a model. This process normally leads to small- or medium-sized models.

Most models originate from spatial discretization of partial differential equations describing the laws of physics. For solids, a continuum mechanics approach – equi-

librium of forces, kinematic equations and the constitutive equations – describe the physics. Well-established methods like the finite element method (FEM), the finite difference method, the finite volume method, the boundary element method, and others are applied for spatial discretization. All procedures are based on the method of weighted residuals and differ only in the specific choices of test and weighting functions. The classical displacement-based FEM, for example, uses first- or higher-order polynomial test and weighting functions from identical function spaces, i. e., a Ritz–Galerkin projection from the continuous to the discrete space is performed. This typically leads to very large models comprising thousands up to millions of degrees of freedom.

Another way of modeling mechanical systems is a data-driven identification of their dynamics. This typically leads to rather small models.

All models have in common that the equilibrium of inertia forces, $\mathbf{M}\ddot{\mathbf{q}}(t)$, damping and internal restoring forces, $\hat{\mathbf{f}}(\dot{\mathbf{q}}(t), \mathbf{q}(t))$, and external forces, $\hat{\mathbf{F}}(t)$, determines the basic dynamics of mechanical systems:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \hat{\mathbf{f}}(\dot{\mathbf{q}}(t), \mathbf{q}(t)) = \hat{\mathbf{F}}(t), \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \quad (2.1)$$

with (generalized) displacements $\mathbf{q}(t) \in \mathbb{R}^N$, mass matrix $\mathbf{M} \in \mathbb{R}^{N \times N}$, nonlinear damping and internal forces $\hat{\mathbf{f}} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, and external forces or loadings $\hat{\mathbf{F}}(t) \in \mathbb{R}^N$.

External forces can also be considered explicitly as a space- and a time-dependent part, $\hat{\mathbf{F}}(t) = \mathbf{B}\mathbf{F}(t)$:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \hat{\mathbf{f}}(\dot{\mathbf{q}}(t), \mathbf{q}(t)) = \mathbf{B}\mathbf{F}(t), \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0. \quad (2.2)$$

This allows for a system-theoretic point of view where the input–output behavior is of importance: The input matrix $\mathbf{B} \in \mathbb{R}^{N \times p}$ contains weights and allocations to the degrees of freedom of the time-dependent forces, i. e., the input signals $\mathbf{F}(t) \in \mathbb{R}^p$ ($p \leq N$). The corresponding system outputs are given by equation (2.5).

Modeling of the dominating damping mechanisms is mostly not straightforward. Therefore, one frequently gets by with assuming simple linear viscous damping $\mathbf{D}\dot{\mathbf{q}}(t)$. Additionally excluding gyroscopic effects allows for writing $\hat{\mathbf{f}}(\dot{\mathbf{q}}(t), \mathbf{q}(t)) = \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{f}(\mathbf{q}(t))$, such that

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{f}(\mathbf{q}(t)) = \mathbf{B}\mathbf{F}(t), \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \quad (2.3)$$

with damping matrix $\mathbf{D} \in \mathbb{R}^{N \times N}$ and nonlinear internal forces $\mathbf{f} : \mathbb{R}^N \rightarrow \mathbb{R}^N$.

For sufficiently small displacements around an equilibrium position, used as initial configuration, only the linear part of the internal restoring forces, $\mathbf{f}(\mathbf{q}(t)) \approx \mathbf{K}\mathbf{q}(t)$, has to be considered. This leads to the well-known linear second-order representation

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{B}\mathbf{F}(t), \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \quad (2.4)$$

with stiffness matrix $\mathbf{K} \in \mathbb{R}^{N \times N}$.

Typically, displacements, velocities, or linear combinations, e. g., stresses, at specific nodes constitute the system outputs:

$$\mathbf{y}(t) = \mathbf{C}_q \mathbf{q}(t) + \mathbf{C}_{\dot{q}} \dot{\mathbf{q}}(t), \quad (2.5)$$

with output matrices $\mathbf{C}_q \in \mathbb{R}^{q \times N}$ and $\mathbf{C}_{\dot{q}} \in \mathbb{R}^{q \times N}$ considering displacements and velocities, respectively.

The transfer behavior – inputs to outputs – is commonly represented as

$$\mathbf{G}(s) = (\mathbf{C}_q + s\mathbf{C}_{\dot{q}})(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B} \quad (2.6)$$

in the frequency domain, such that

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{F}(s), \quad (2.7)$$

where $\mathbf{Y}(s)$ and $\mathbf{F}(s)$ are the Laplace-transformed outputs $\mathbf{y}(t)$ and inputs $\mathbf{F}(t)$, respectively.

While second-order representations are common in mechanics, state-space representations are often used in systems and control theory. Different implicit state-space representations exist, possessing desired properties for their system matrices. Here only, except for basis transformations, unique explicit representations will be given, while implicit ones are to be favored when, e. g., preservation of the matrix sparsity pattern is desired. The general nonlinear system representation (2.2) can be reformulated as

$$\begin{bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{q}}(t) \\ -\mathbf{M}^{-1}\hat{\mathbf{f}}(\dot{\mathbf{q}}(t), \mathbf{q}(t)) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \mathbf{F}(t), \quad \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{q}_0 \\ \dot{\mathbf{q}}_0 \end{bmatrix} \quad (2.8)$$

and the linear system representation (2.4) as

$$\begin{bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \mathbf{F}(t), \quad \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{q}_0 \\ \dot{\mathbf{q}}_0 \end{bmatrix}, \quad (2.9)$$

together with the output equation (2.5) rewritten as

$$\mathbf{y}(t) = [\mathbf{C}_q \quad \mathbf{C}_{\dot{q}}] \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}. \quad (2.10)$$

Systems involving coupling of either different components, i. e., multibody systems, or of different physical domains, i. e., multiphysics systems, can show mass matrices $\mathbf{M}$ in second-order representations (or left-hand side matrices $\mathbf{E}$ in implicit state-space representations $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$) with rank deficiencies due to the algebraic interface equations. Thus, the system dynamics are described by differential algebraic equations (DAEs). An example is given in Section 2.2.

Mass ($\mathbf{M}$) and stiffness ($\mathbf{K}$) matrices are symmetric and positive (semi-)definite for typical mechanical systems with appropriate boundary conditions suppressing rigid body modes. Implying those matrix properties directly results in the following system properties:

- *Passivity*: A mechanical system is passive for collocated inputs and velocity only outputs, i. e., $\mathbf{B} = \mathbf{C}_q^T$, $\mathbf{C}_q = \mathbf{0}$ [109].
- *Stability*: A mechanical system is always Lyapunov stable. It is also asymptotically stable for positive definite $\mathbf{D}$ [59].

Commonly, linear damping is realized via modal damping. A simple and popular choice is the special case of proportional or Rayleigh damping, where $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$ with $\alpha, \beta \geq 0$, i. e., the damping matrix ($\mathbf{D}$) is computed as a linear combination of mass ($\mathbf{M}$) and stiffness ($\mathbf{K}$) matrices [41]. With this, $\mathbf{D}$ is symmetric and positive definite, the same as $\mathbf{M}$ and $\mathbf{K}$. The conjugate complex eigenvalue pairs $s_{i,i+1} = \sigma \pm i\omega$ of undamped systems are all located on the imaginary axis. A mass proportional part, i. e., $\alpha \geq 0$, shifts all eigenvalues with equal amount to the left, $\Delta\sigma = \text{const.}$, while a stiffness proportional part, i. e., $\beta \geq 0$, shifts all eigenvalues to the left with an amount proportional to the frequency of the eigenvalue squared, $\Delta\sigma \propto \omega^2$.

In order to reduce the computational effort involved in numerically solving (2.4), a reduced-order model (ROM) that accurately approximates the behavior of the original full-order model (FOM) is aimed. This is usually achieved by *projective* MOR. To this end, the full displacements $\mathbf{q}(t) \in \mathbb{R}^N$ are first approximated by a linear combination of reduced displacements $\mathbf{q}_r(t) \in \mathbb{R}^n$ via the ansatz $\mathbf{q}(t) = \mathbf{V} \mathbf{q}_r(t) + \mathbf{e}(t)$, where $\mathbf{V} \in \mathbb{R}^{N \times n}$ and $n \ll N$. Inserting this ansatz in (2.4), (2.5) yields an overdetermined system with the residual $\boldsymbol{\varepsilon}(t) \in \mathbb{R}^N$,

$$\begin{aligned} \mathbf{M} \mathbf{V} \ddot{\mathbf{q}}_r(t) + \mathbf{D} \mathbf{V} \dot{\mathbf{q}}_r(t) + \mathbf{K} \mathbf{V} \mathbf{q}_r(t) &= \mathbf{B} \mathbf{F}(t) + \boldsymbol{\varepsilon}(t), \\ \mathbf{y}_r(t) &= \mathbf{C}_q \mathbf{V} \mathbf{q}_r(t) + \mathbf{C}_{\dot{q}} \mathbf{V} \dot{\mathbf{q}}_r(t). \end{aligned} \quad (2.11)$$

To obtain a square system, we premultiply (2.11) by $\mathbf{W}^T \in \mathbb{R}^{n \times N}$,

$$\begin{aligned} \mathbf{M}_r \ddot{\mathbf{q}}_r(t) + \mathbf{D}_r \dot{\mathbf{q}}_r(t) + \mathbf{K}_r \mathbf{q}_r(t) &= \mathbf{B}_r \mathbf{F}(t), \quad \mathbf{q}_r(0) = \mathbf{q}_{r,0}, \quad \dot{\mathbf{q}}_r(0) = \dot{\mathbf{q}}_{r,0}, \\ \mathbf{y}_r(t) &= \mathbf{C}_{q_r} \mathbf{q}_r(t) + \mathbf{C}_{\dot{q}_r} \dot{\mathbf{q}}_r(t), \end{aligned} \quad (2.12)$$

thus enforcing the *Petrov–Galerkin* condition $\mathbf{W}^T \boldsymbol{\varepsilon}(t) = \mathbf{0}$, where the residual $\boldsymbol{\varepsilon}(t)$ vanishes. The reduced matrices are given by $\{\mathbf{M}_r, \mathbf{D}_r, \mathbf{K}_r\} = \mathbf{W}^T \{\mathbf{M}, \mathbf{D}, \mathbf{K}\} \mathbf{V}$, $\mathbf{B}_r = \mathbf{W}^T \mathbf{B}$, $\mathbf{C}_{q_r} = \mathbf{C}_q \mathbf{V}$, and $\mathbf{C}_{\dot{q}_r} = \mathbf{C}_{\dot{q}} \mathbf{V}$, and the initial conditions are $\{\mathbf{q}_r(0), \dot{\mathbf{q}}_r(0)\} = (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \{\mathbf{q}_0, \dot{\mathbf{q}}_0\}$. Therefore, the main task of any projective MOR technique consists of finding suitable reduction bases $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{N \times n}$ that span appropriate subspaces $\mathcal{V} = \text{span}(\mathbf{V})$ and $\mathcal{W} = \text{span}(\mathbf{W})$.

MOR in mechanical engineering typically aims at achieving the following goals:

1. *Good approximation*. The reduction technique should yield a ROM which captures the most dominant dynamics and well approximates the state vector or input–output behavior of the FOM either in the time or in the frequency domain. The approximation quality can, for instance, be measured pointwise in time by $\|\mathbf{y}(t) - \mathbf{y}_r(t)\|_{(\cdot)}$ or $\|\mathbf{x}(t) - \mathbf{V} \mathbf{x}_r(t)\|_{(\cdot)}$, or pointwise in frequency by $\|\mathbf{G}(i\omega) - \mathbf{G}_r(i\omega)\|_{(\cdot)}$ using

suitable matrix and vector norms $(\cdot) = \{1, 2, \infty, \text{Fro}, \dots\}$. Another possibility is to use normwise error measures as $\|\mathbf{y} - \mathbf{y}_r\|_{(*)}$ or $\|\mathbf{x} - \mathbf{V} \mathbf{x}_r\|_{(*)}$ with $(*) = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_\infty, \dots\}$ in the time domain, or $\|\mathbf{G} - \mathbf{G}_r\|_{(*)}$ with $(*) = \{\mathcal{H}_2, \mathcal{H}_\infty, \text{Hankel} \dots\}$ in the frequency domain.

2. *Preservation of system properties.* Basic features of the original model (e. g., stability, passivity, second-order structure, port-Hamiltonian structure, etc.) should be preserved in the ROM. This requirement is achieved by applying special or adapted reduction methods tailored to address these demands.
3. *Numerical efficiency.* Model reduction pays off if the benefit of having multiple, cheap online evaluations (required, e. g., for design analysis, optimization, and control) outweighs the upfront offline cost needed for the computation of the reduced model. Thus, the reduction methods should be as numerically efficient and stable as possible. Expensive offline, and especially online, computations should be avoided. In addition, reduction approaches should preferably be applicable to large-scale models and industrial problems.

Depending on the application and the characteristic behavior of the FOM that should be approximated during the reduction, two categories can be distinguished to meet requirement 1:

- (i) *Initial condition-state-based reduction.* This category is especially interesting for mechanical engineering, where the eigendynamics, i. e., the state dynamics described for different initial conditions, are particularly relevant. Modal reduction techniques, such as modal truncation [94], mainly focus on the approximation of the homogeneous problem, i. e., the eigendynamics of the underlying model. POD also falls under this category, since it is based on snapshots of the simulated state trajectory.
- (ii) *Input-output-based reduction.* This category is especially interesting for control engineering or for problems where inputs and outputs are defined. In this regard, approaches such as balanced truncation [78, 44] and Krylov subspace methods [42, 10] exploit the information contained in the input and output matrices $\mathbf{B}$, $\mathbf{C}$ to obtain a reduced model which is tailored to approximate the input–output behavior.

To meet requirement 2, so-called *structure-preserving model reduction* is applied. For instance, if the principle of virtual work known from mechanical systems should be fulfilled, then the reduction should be performed by a *Galerkin projection* with $\mathbf{W} = \mathbf{V}$ rather than by a two-sided (oblique) Petrov–Galerkin projection. Note, however, that most second-order balancing approaches do not underlie a Galerkin projection. In any case, with $\mathbf{W} = \mathbf{V}$, the reduced matrices preserve the symmetry and definiteness properties of the original matrices. Furthermore, this choice leads to the foremost aim of preserving crucial properties such as the stability, passivity, and structure of the original FOM. Note that second-order models could also be reduced by first reformulating them into the state-space/first-order representation (with $\tilde{N} = 2N$), and

then by reducing the first-order models. However, unless special care is taken during the reduction of the reformulated state-space model, it can be difficult to gain back a structure-preserving, second-order ROM out of the first-order ROM [102, 103, 85, 21]. Consequently, MOR for mechanical systems is often performed directly on the second-order FOM by applying *second-order reduction techniques* suited to meet the mentioned requirements and needs. Hereby, either (i) classic modal-based procedures (such as second-order modal truncation) or (ii) *adapted* input–output-based approaches (e. g., second-order Arnoldi [SOAR] [6, 9, 102], SO-IRKA [112], or second-order balanced truncation [76, 108, 17, 16]) can be applied. Note, however, that some of these reduction techniques still use the first-order representation to compute, e. g., the Gramians or the optimal interpolation data, in order to build afterwards second-order reduction bases $\mathbf{V}$, $\mathbf{W}$ to project the second-order model.

It is also worth mentioning that in many engineering applications multiphysics problems arise, yielding *coupled domain models*. Similarly, in structural dynamics and industrial applications, the mechanical systems often consist of several separable components or substructures. In this context, the concept of *substructuring* and *component mode synthesis* (CMS) [48, 56, 24, 1] plays a key role, allowing to partition a large structural model into multiple substructures that are individually reduced, and then compatibly coupled along the component interfaces after reduction. Typical CMS techniques are Guyan condensation [48], the Craig–Bampton method [24], and their derivatives [56, 1]. Craig–Bampton, e. g., represents the combination of (i) a modal-based and (ii) an input–output-based procedure, since (i) “component eigenmodes” are combined with so-called (static/dynamic) (ii) “constraint modes” obtained for a unit displacement applied to the interface degrees of freedom. In addition to the classic CMS techniques, system-level *interface reduction* approaches that reduce the number of interface degrees of freedom are also very common [51]. Finally, note that input–output-based reduction approaches such as balanced truncation and Krylov subspace methods can naturally be applied instead of the modal-based “component eigenmodes.” This has been done, e. g., in [33, 84, 51], where (static/dynamic) (i) “constraint modes” have been combined with (ii) “input–output-based component modes.”

2.2 Model order reduction of a thermo-mechanical machine tool model aimed for position control

For modern machine tools, the accuracy, stability, and repeatability are the key performance factors. Thermally induced displacement errors at the tool center point (TCP) are among the main causes of work piece defects. The heat generated by motors, the friction-caused heat, the influence of the environment temperature, etc., are the fac-

tors which lead to unwanted thermal expansion of the machine tool [29, 77, 82] and require a real-time compensation (position control). In the past few years, the collaborative research center Transregio 96 [43] contributed a great deal to the optimal thermo-energetic design of machine tools. Numerical simulation via, e. g., the FEM is a common tool to compute the thermal expansion. However, the accurate numerical model is too large to be co-simulated with the control circuitry.

In the following, the successful application of Krylov subspace-based MOR from [39, 6, 102] is presented for creating a reduced-order thermo-mechanical machine tool model. Our case study is an academic model, displayed in Figure 2.1.

Figure 2.1: Structure of the machine tool model.

For demonstrating purpose, we observe a single stage movement in the positive x -direction and a single heat generation in the spindle of the tool. The heat generation is due to electrical drive only, whereas the friction-caused heat is neglected. Furthermore, ideal thermal contacts and temperature-independent material thermal parameters (volumetric heat capacity and heat conductivity) are assumed. The mechanical parts of the model are connected by linear springs, instead of using more realistic ball bearings, in which the frictional and thermal effects should be considered [65]. These assumptions lead to a linear thermo-mechanical model, which can be reduced by Krylov subspace-based MOR. Please note that as long as the above linearity conditions are fulfilled, this methodology can be applied to more realistic models as well. The primary goal of this section is to demonstrate that the position control can be implemented based on such an ROM. Further applications of MOR to machine tool models can be found in [32, 96, 81, 68].

2.2.1 Coupled domain thermo-mechanical models

In the thermo-mechanical model, both thermal and mechanical domains are taken into account during the simulation. The thermal and mechanical domains can be cou-

pled via the input vector or via the system matrices (strong coupling). As the thermal expansion is a transient process, which requires real-time compensation, the coupling via the system matrices is recommended. Such a strongly coupled linearized machine tool model has the following form:

$$\Sigma_{N+k} : \begin{cases} \underbrace{\begin{bmatrix} \mathbf{M}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_M \underbrace{\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{T}} \end{bmatrix}}_D + \underbrace{\begin{bmatrix} \mathbf{D}_q & \mathbf{0} \\ \mathbf{D}_{Tq} & \mathbf{D}_T \end{bmatrix}}_D \underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{T}} \end{bmatrix}}_K + \underbrace{\begin{bmatrix} \mathbf{K}_q & \mathbf{K}_{qT} \\ \mathbf{0} & \mathbf{K}_T \end{bmatrix}}_K \underbrace{\begin{bmatrix} \mathbf{q} \\ \mathbf{T} \end{bmatrix}}_K = \mathbf{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{q} \\ \mathbf{T} \end{bmatrix}, \end{cases} \quad (2.13)$$

where $\mathbf{M}_q, \mathbf{D}_q, \mathbf{K}_q \in \mathbb{R}^{N \times N}$ are the mass matrix, the structural damping matrix, and the stiffness matrix, respectively; $\mathbf{D}_T, \mathbf{K}_T \in \mathbb{R}^{k \times k}$ are the specific heat matrix and the conductivity matrix; $\mathbf{D}_{Tq} \in \mathbb{R}^{k \times N}$ is the thermo-elastic damping matrix and $\mathbf{K}_{qT} \in \mathbb{R}^{N \times k}$ is the thermo-elastic stiffness matrix. Matrices $\mathbf{D}_{Tq}$ and $\mathbf{K}_{qT}$ couple the thermal and mechanical domains. The state vector contains two parts, i. e., the structural displacement vector $\mathbf{q}(t) \in \mathbb{R}^N$ and the temperature vector $\mathbf{T}(t) \in \mathbb{R}^k$. In this model a single force input u_1 is applied onto the moving stage in the positive x -direction and a single heat generation input u_2 is assumed in the spindle of the tool. The output matrix $\mathbf{C} \in \mathbb{R}^{2 \times (N+k)}$ defines the observed outputs, displacement of the moving stage at the selected node in the positive x -direction and the temperature at the TCP. The machine tool is fixed at the bottom ground and the bottom temperature is set to 20 °C. The heat exchange between system and environment is modeled by the convection boundary condition $\phi_T = h(\mathbf{T} - \mathbf{T}_{\text{ambient}})$, where ϕ_T is a heat flux and h is the heat transfer coefficient, which is set to 5 $\frac{\text{W}}{\text{m}^2}$. The convection is considered over the whole surface of the machine tool model and the ambient temperature is set to 20 °C. The Rayleigh damping is considered, that is, the damping matrix $\mathbf{D}_q$ is proportional to the mass matrix $\mathbf{M}_q$ and the stiffness matrix $\mathbf{K}_q$ as follows: $\mathbf{D}_q = \alpha \mathbf{M}_q + \beta \mathbf{K}_q$. In this case, based on experience, parameters $\alpha = 0$ and $\beta = 1.5915 \cdot 10^{-4}$ are chosen to cause the damping ratio of 1 % at a frequency of 20 Hz.

The advantage of model (2.13) is that all physical effects are included. However, as will be shown in Section 2.2.2, MOR of this model leads to unnecessarily large ROMs, whose time integration within the controller loop might be prohibitive. On the other hand, as the time scale of structural dynamics (≈ 1 s) is much smaller than that of the heat transfer ($\approx 10^5$ s), one might think of separating physical domains and integrating them at the system level with different time steps, as suggested in [15].

Therefore, the so-called quasi-static approximation of structural mechanics is used, in which the mass matrix $\mathbf{M}_q$ and the structural damping matrix $\mathbf{D}_q$ from equa-

tion (2.13) are ignored. The resulting model has the following form:

$$\Sigma_{N+k\text{-Quasi}} : \begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_T \end{bmatrix}}_{\mathbf{D}} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{T}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{K}_q & \mathbf{K}_{qT} \\ \mathbf{0} & \mathbf{K}_T \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} \mathbf{q} \\ \mathbf{T} \end{bmatrix} = \mathbf{B} \begin{bmatrix} 0 \\ u_2 \end{bmatrix}, \\ \mathbf{y} = \mathbf{C} \begin{bmatrix} \mathbf{q} \\ \mathbf{T} \end{bmatrix}, \end{cases} \quad (2.14)$$

and accounts for thermal expansion of the mechanical structure and the transient heat transfer. Furthermore, the thermo-elastic damping $\mathbf{D}_{Tq}$, which describes the influence of structural dynamics upon the temperature field, can be neglected as well, because the dissipated heat is negligible compared to the motor-generated heat. Please note that the reduction of the DAE system (2.14) with Krylov subspace-based MOR is numerically well conditioned, since those methods solely impose that the matrix $\mathbf{K} + s_0 \mathbf{D}$ is regular for some value s_0 of the Laplace variable [6].

At the system level, (2.14) has to be combined with the purely mechanical model:

$$\Sigma_N : \begin{cases} \mathbf{M}_q \ddot{\mathbf{q}} + \mathbf{D}_q \dot{\mathbf{q}} + \mathbf{K}_q \mathbf{q} = \mathbf{B}_q u_1, \\ \mathbf{y} = \mathbf{C}_q \mathbf{q}, \end{cases} \quad (2.15)$$

that is, with its reduced counterpart, to reflect the dynamical behavior of the structure within a control loop.

2.2.2 Application of Krylov subspace-based MOR

The main issue in reducing coupled domain thermo-mechanical machine tool models lies in the fact that both physical domains have very different time constants, as the time scale of the structural part is much smaller than that of the heat transfer. In the following, the Krylov subspace-based MOR is applied, which matches the moments (coefficients of the Taylor series) of the transfer functions of the full and reduced models. For the reduction of model (2.13), the SOAR algorithm from [6] and [102] is used. For the first-order system (2.14) and for the proportionally damped second-order system (2.15), the first-order block Arnoldi algorithm from [39] is implemented (for the application to (2.15), see [28]).

Two inputs (mechanical force applied at a selected surface of the moving stage and heat source in spindle) and two outputs (displacement at selected node of the moving stage and the temperature at TCP) are defined, which leads to the following 2×2 transfer function matrix:

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}, \quad (2.16)$$

where $G_{11}(s) = \frac{Y_{\text{displ}}(s)}{U_{\text{force}}(s)}$, $G_{12}(s) = \frac{Y_{\text{temp}}(s)}{U_{\text{force}}(s)}$, $G_{21}(s) = \frac{Y_{\text{displ}}(s)}{U_{\text{heat}}(s)}$, $G_{22}(s) = \frac{Y_{\text{temp}}(s)}{U_{\text{heat}}(s)}$, and $Y_{\text{displ}}(s)$ and $Y_{\text{temp}}(s)$ are the Laplace transforms of the displacement and temperature outputs, respectively, defined by matrix $\mathbf{C}$ in (2.13); $U_{\text{force}}(s)$ and $U_{\text{heat}}(s)$ are the Laplace transforms of the force and heat inputs, respectively, defined as $u_1(t)$ and $u_2(t)$ in (2.13).

The goal is to investigate the convergence behavior of each reduced model for different model sizes and compare them with respect to their applicability for position control. The relative error between the full and the reduced model transfer functions at a specific frequency f is defined as

$$\epsilon(f) = \|\mathbf{G}(i2\pi f) - \mathbf{G}_r(i2\pi f)\|_2 / \|\mathbf{G}(i2\pi f)\|_2. \quad (2.17)$$

For the harmonic simulations of the full-scale model, we use the FEM simulator ANSYS Academic Research, Rel. 18.0, whereas the reduced models are created with model reduction inside ANSYS [98].

Firstly, the purely mechanical proportionally damped model (2.15) with the single force input (force applied at a selected surface of the moving stage in the positive x -direction in Figure 2.1) and a single output (displacement at selected node of the moving stage) is investigated.

Figure 2.2 (left) shows an excellent match between the transfer function $G_{11}(s)$ of the full model with 51,378 degrees of freedom and the reduced-order 100 model. The expansion point 0 Hz is chosen for the Arnoldi-based reduction and for matching the frequency range between 1 Hz and 1000 Hz, which is the frequency range of interest for the particular application. Note that other frequency ranges of interest can be matched by employing different expansion points [42]. From Figure 2.2 (right), it is observed that for matching the transfer function with maximal relative error of 1 % over the whole frequency range of interest, a moderate number of 30 Arnoldi vectors – which represent the orthonormal basis of the Krylov subspace $\mathcal{K}(-\mathbf{K}_q^{-1}\mathbf{M}_q, -\mathbf{K}_q^{-1}\mathbf{B}_q)$ – is required if no other expansion points are to be employed.

Figure 2.2: Transfer function $G_{11}(s)$ of purely mechanical full-scale model (2.15) and corresponding reduced-order 100 model, gained by Arnoldi-based reduction with expansion point 0 Hz (left). Relative error between the transfer functions of full-scale model (2.15) and corresponding reduced models of different sizes at different frequencies (right).

The coupled domain model (2.13) suffers from different time constants of structural dynamics and heat transfer, as the ROM has to simultaneously match the frequency ranges from 0 Hz to 10 Hz (thermal expansion effects occur here) and from 1 Hz to 1000 Hz (quick changes in dynamics occur here).

Figure 2.3 (top) shows an excellent match of the transfer functions $G_{11}(s)$ and $G_{12}(s)$ of the thermo-mechanical model (2.13) exposed solely to a single unit force input. Figure 2.3 (bottom) shows an excellent match of the transfer functions $G_{21}(s)$ and $G_{22}(s)$ of the thermo-mechanical model (2.13) exposed solely to a single unit heat input; $G_{21}(s)$ describes the thermal expansion. In this case we used the unit input for computing the transfer function, as the model is linear. At the system level, an arbitrary input can be used.

Figure 2.3: Transfer functions $G_{11}(s)$ (top-left), $G_{12}(s)$ (top-right) with single force input and $G_{21}(s)$ (bottom-left), $G_{22}(s)$ (bottom-right) with single heat input of strongly coupled full-scale model (2.13) and corresponding reduced models of order 400, gained by the SOAR algorithm with expansion points 0 Hz, 10 Hz, 100 Hz, and 1000 Hz.

In order to match the full required frequency range from 0 Hz to 1000 Hz, with maximal relative error of 1 %, 300, respectively, 250 Arnoldi vectors and four expansion points (0 Hz, 10 Hz, 100 Hz, and 1000 Hz) are required (Figure 2.4). This high number of 250 vectors is the main disadvantage of reducing the strongly coupled model (2.13). Note that more modern methods allow for an automatic choice of the expansion points and

Figure 2.4: Relative error (2.17) between the transfer functions of full and reduced model at different frequencies for different sizes of reduced models with only force input (left) and with only heat input (right) in strongly coupled thermo-mechanical model (2.13).

the corresponding number of moments. In this work, however, we adopted the practical engineering approach of “trial and error,” since our goal was to apply the classical MOR methodology to our coupled physics, realistic industrial application rather than to apply or improve adaptive reduction procedures. Future research should include more modern methods based on adaptive approaches such as, e. g., IRKA, TSIA, and CUREd SPARK from [45, 27, 113, 85, 111].

It is expected that the reduction of the coupled model with quasi-static approximation of the mechanical domain (2.14) leads to reduced models with lower order than the reduction of (2.13). Furthermore, (2.14) can be reduced with the first-order Krylov subspace algorithm from [39]. Figure 2.5 shows an excellent match of the transfer functions $G_{21}(s)$ and $G_{22}(s)$ of the full-scale and reduced-order 100 thermo-mechanical model with quasi-static approximation of the mechanical domain (2.14). It is exposed to a single heat input and $G_{21}(s)$ describes the thermal expansion, which should be accounted for within the position control. In this case, however, there is no need to cover the frequency range until 1000 Hz, as the structural dynamics are not included. Rather, the frequency range of interest from 0 Hz to 10 Hz should be approximated. For this, three expansion points (chosen by “trial and error”) are still needed: 0 Hz, 0.1 Hz, and 10 Hz.

Figure 2.6 shows that for matching the transfer function $\mathbf{G}(s)$ of the fully coupled model with quasi-static approximation of the mechanical domain and the single heat input, with maximal relative error of 1 %, within the frequency range of interest, solely 50 Arnoldi vectors and three expansion points are needed, which outperforms the reduced models of system (2.13).

In conclusion, by introducing the quasi-static approximation of the mechanical domain, the accuracy of the reduced thermo-mechanical model could be improved and the dimension of the reduced model could be decreased. For the complete approximation of the frequency range of interest, this model has to be combined with

Figure 2.5: Transfer functions $G_{21}(s)$ (left) and $G_{22}(s)$ (right) for single heat input of strongly coupled full-scale model with quasi-static approximation of mechanical domain (2.14) and corresponding reduced-order 100 models, gained by Arnoldi-based reduction with expansion points 0 Hz, 0.1 Hz, and 10 Hz.

Figure 2.6: Relative error (2.17) between the transfer functions of full and reduced model at different frequencies for different sizes of reduced model with only heat input in strongly coupled thermo-mechanical model with quasi-static approximation of the mechanical domain (2.14).

the (reduced) purely mechanical model at the system level, as will be shown in the next section.

2.2.3 Position control scenario based on reduced-order model

For position control of machine tools usually a cascade controller structure is applied [40]. Our goal is to parameterize such a controller based on an ROM. Figure 2.7 shows the ANSYS Twin Builder simulation setup.

In Figure 2.8 (left) the impact of the position control to the moving stage, when taking into account only structural dynamics of the machine, is displayed and Figure 2.8 (right) shows the gain-parameter optimization of the controller, based on the reduced-order purely mechanical model (2.15) with dimension 30. Note that timing

Figure 2.7: Co-simulation of the cascade controller and the reduced-order model (ROM) of the machine tool in ANSYS Twin Builder.

Figure 2.8: Output node displacement of the reduced purely mechanical model (2.15) with and without controller displayed in Figure 2.7 (left) and optimization of the controller gain based on the reduced-order numerical model (right).

for five simulations with different controller gain values amounts to solely 24 s on an Intel(R) Core(TM) i7 @ 2.50GHz RAM 8 GB.

In order to take into account thermal expansion, the reduced strongly coupled thermo-mechanical model can be applied within the control loop as schematically shown in Figure 2.9 (left).

The main disadvantage of this approach is that, due to the fact that structural dynamics and heat transfer have very different time constants and, hence, a large frequency range has to be matched, the size of the reduced model is relatively large for a system-level simulation (order 250 up to 300, as shown in the previous section). A remedy is to apply the quasi-static approximation of the mechanical domain, as described in equation (2.14). Based on the reduced-order quasi-statically approximated coupled model, it is suggested to use the structure for the positioning controller shown in Figure 2.9 (right). The advantage over the controller from Figure 2.9 (right) is that both

Figure 2.9: Position controller based on the reduced strongly coupled thermo-mechanical model (left) and on the reduced thermo-mechanical model with quasi-static approximation of the mechanical domain (right).

ROMs are of smaller size and can also be integrated in time separately. Also in this setup, it is possible to parameterize the controller based on the ROMs.

2.3 Coupling of reduced elastic bodies in vehicle dynamics

2.3.1 Requirements on the model reduction software and model reduction method

One important aspect in the development of vehicles is to ensure a comfortable ride feeling for the driver. In this context, it is necessary to investigate the effect of external disturbances, e. g., from potholes, on the driver in transient simulations over a long period of time [72]. Therefore, models of a driver's seat and a car body are used as an industrial automotive example in this contribution. The driver's seat and the car body, the coupling nodes, and the input and output nodes are shown in Figure 2.10.

In industrial applications, the elastic components are usually modeled with the FEM, where complex geometries require fine spatial discretizations to obtain meaningful models. For the driver's seat this leads to a model with $1.43 \cdot 10^5$ degrees of freedom and for the car body to a model with $1.99 \cdot 10^6$ degrees of freedom. The model properties are summarized in Table 2.1. The discretization and model generation are usually performed with commercial FEM software packages. Obviously, one challenge in MOR for industrial models is to access the model data, as the system matrices or geometry information from proprietary data structures, for further usage in the actual MOR. In this application, the MOR toolbox *Morembs* is used. It contains interfaces to many commercial FEM software packages and allows a user-friendly import of the model data. In a further step various MOR algorithms for second-order systems can be applied and the reduced models can be exported for further use in an industrial simulation chain. The

Figure 2.10: Finite element mesh of the driver’s seat and the car body with inputs, outputs, and interface nodes, kindly provided by Daimler AG.

Table 2.1: Model properties of the driver’s seat and the car body.

	Number of elements	Number of degrees of freedom	Number of inputs
Driver’s seat	23,968	142,941	24
Car body	511,156	1,993,167	36

typical MOR work flow with *Morembs* consisting of import, reduction, and export is illustrated in Figure 2.11. An exemplary MATLAB code for import, reduction, and export with *Morembs* is given in Figure 2.12. The toolbox is freely available for academic institutions. Further information can be found in [38] and in Chapter 13 of Volume 3 of *Model order reduction*.

Figure 2.11: Import, reduction, and export work flow with MOR toolbox *Morembs*.

```

% start the toolbox
setupMatMoremb;
% path to data from FEM software
dataDir = 'FE/VehicleStabilizer';
% set the interaction nodes
Interactions(1).node = 13;
Interactions(1).dof = 1:6;
% import the system matrices from the FEM data
sysdata = importFEdata('workdir',dataDir,'id','VehicleStabilizer',...
    'input',Interactions);
% reduce the system with a Craig-Bampton approach to a reduced order of 10
redSysdata = MOR('sysdata',sysdata,'redmethod','Craig-Bampton','nred',10);
% export the reduced system for further use in multibody software Neweul-M^2
ElasticBody = RedExport('sysdata',redSysdata,'savedir',pwd,...
    'filename','redbody','target','neweulm2');

```

Figure 2.12: Exemplary MATLAB code for import, reduction, and export with *Moremb*.

Another challenge is that the reduced component models should be reusable in different mounting situations or product variants. This means that a MOR method must be able to perform the reduction on component level first and to deliver reduced models which can be combined afterwards with other reduced models, e. g., from a database.

One well-established method meeting these requirements is the Craig–Bampton method as suggested in [24]. This method uses a splitting into an internal part and a boundary part of the elastic body. Static constraint modes are then combined with fixed interface eigenmodes of the internal part of the elastic body. One feasible modern method meeting these requirements as well is the so-called CMS-Gram method as presented in [52]. The method uses a splitting into an internal part and a boundary part, too. Then, a modified input matrix is formulated to allow the application of input–output-based MOR methods for the internal dynamics. In this case, the internal dynamics are reduced using a two-step approach. The first step uses *moment matching* to reduce the large-scale model to a medium-size model. The second reduction to a small system size is done using *frequency-weighted balanced truncation*. Both MOR steps are directly applied to the second-order system described by equation (2.4) to obtain a reduced second-order system. First, this obviously simplifies the physical interpretation of the ROM. Second, in industrial applications it is often even necessary to preserve the second-order structure, for example if the ROMs shall be used in a commercial simulation environment. In the following, both approaches will be compared.

2.3.2 Comparison of the approximation quality of the noncoupled and the coupled system

First, the driver’s seat as a single component is investigated. The Frobenius norm of the transfer function $\|\mathbf{G}(f)\|_{\text{Fro}}$ of the original model and of two reduced models of order

$n_{\text{seat}} = 50$ are shown in Figure 2.13. One reduced model is obtained using the Craig–Bampton approach while the second reduced model is obtained using the CMS-Gram method. It can be seen that all transfer functions are in good agreement. The corresponding Frobenius norm relative error $\epsilon(f) = \|\mathbf{G}(f) - \mathbf{G}_r(f)\|_{\text{Fro}} / \|\mathbf{G}(f)\|_{\text{Fro}}$ of the reduced models is shown in Figure 2.13 for the frequency range of interest from 0 Hz to 60 Hz. Both approaches show a good approximation quality in the lower frequency range. However, the error of the Craig–Bampton reduced model exceeds 10 % around 60 Hz, which does not fulfill the accuracy requirements of the ride simulation. In contrast, the approximation error of the CMS-Gram reduced model is about one to two magnitudes smaller and, therefore, shows a satisfying approximation quality over the entire frequency range.

Figure 2.13: Frobenius norm of the transfer function of the driver's seat for the full model, a Craig–Bampton reduced model, and a CMS-Gram reduced model with $n_{\text{seat}} = 50$ and corresponding Frobenius norm relative errors $\epsilon(f)$.

Another aspect is the improvement of the approximation quality for an increasing order of the reduced model. For the Craig–Bampton model this can be done by adding more eigenmodes to the basis. However, identifying additional eigenmodes which are important for the transfer behavior is not always a trivial task, especially for industrial models. As for the CMS-Gram approach, the approximation can be simply improved by adding more approximated eigenvectors of the Gramian controllability matrix to the basis.

Figure 2.14 shows the Frobenius norm maximum error

$$\epsilon_{\max} = \max_{f \in [0, 60] \text{ Hz}} (\|\mathbf{G}(f) - \mathbf{G}_r(f)\|_{\text{Fro}}) \quad (2.18)$$

between 0 Hz and 60 Hz for an increasing order n_{seat} of the reduced system. The error for the Craig–Bampton method decreases from $\epsilon_{\max} = 2.46 \cdot 10^{-1}$ at $n_{\text{seat}} = 50$ to

Figure 2.14: Maximum Frobenius norm error ϵ_{max} and mean Frobenius norm relative error ϵ_{mean} for a Craig–Bampton reduction and a CMS-Gram reduction from 0 Hz to 60 Hz.

$\epsilon_{\text{max}} = 2.19 \cdot 10^{-2}$ at $n_{\text{seat}} = 150$. The convergence of the input–output error is therefore rather slow, which is typical for modal-like MOR methods. However, the CMS-Gram method shows a very rapid convergence, leading to an error which is several magnitudes smaller, showing its superiority compared to the Craig–Bampton method. A similar behavior can be observed when investigating the mean error

$$\epsilon_{\text{mean}} = \frac{1}{60 \text{ Hz}} \int_0^{60 \text{ Hz}} \epsilon(f) df \quad (2.19)$$

from 0 Hz to 60 Hz, as shown in Figure 2.14. It can be seen that the CMS-Gram method delivers reduced models with much faster decreasing mean errors in the frequency range of interest compared to the Craig–Bampton method.

Next, the complete system where the driver’s seat is mounted to the car body is investigated. The transfer function of the connected, damped system is depicted in Figure 2.15. It can be seen that there are more eigenfrequencies from 0 Hz to 60 Hz compared to the single model of the driver’s seat, making a good approximation in this frequency range more difficult. The reduced coupled system is derived by a kinematic coupling of the reduced model of the driver’s seat to the reduced model of the car body. The relative error $\epsilon(f)$ for the coupled system is shown in Figure 2.16 for a reduced order of $n_{\text{seat}} = 50$ for the driver’s seat and a reduced order of $n_{\text{car}} = 200$ for the car body. Since both methods are interface-compatible, both methods deliver satisfying coupled ROMs, as well. However, the error of the reduced model obtained by the CMS-Gram method is about two to five magnitudes smaller compared to that of the Craig–Bampton reduced model. So it can be seen again that the CMS-Gram method delivers more accurate reduced models. A convergence analysis for increasing orders of the reduced models shows that the approximation error of the coupled CMS-Gram models also decreases faster.

Figure 2.15: Frobenius norm of the transfer function of the coupled system for the full model, a Craig–Bampton reduced model, and a CMS-Gram reduced model reduced model with $n_{\text{seat}} = 50$ and $n_{\text{car}} = 200$.

Figure 2.16: Frobenius norm relative error $\epsilon(f)$ of the coupled system for a Craig–Bampton reduced model and a CMS-Gram reduced model with $n_{\text{seat}} = 50$ and $n_{\text{car}} = 200$.

2.3.3 Concluding remarks

Two MOR methods, the modal-based Craig–Bampton method and the CMS-Gram method, are compared in this section in the context of an industrial application. Both methods deliver interface-compatible ROMs of single components. This allows the user to combine different ROMs to conduct, e. g., product variant studies. However, it is also shown that the CMS-Gram method yields ROMs with smaller relative errors, confirming the superiority of nonmodal-based MOR methods. Using nonmodal-based MOR methods as the CMS-Gram method allows the user therefore either to benefit from smaller approximation errors or to use reduced models of a smaller order with a similar approximation quality. Therefore, it became clear that it is worth considering modern input–output-based MOR methods such as frequency-weighted balanced truncation for industrial applications.

2.4 Error-controlled model order reduction of an elastic crank drive

A small-scaled, four-stroke internal combustion engine with two pistons in V configuration is depicted in Figure 2.17. The system consists of pistons, the crankshaft, and piston rods connecting each piston to the crankshaft. This mechanical model of the combustion engine is part of a multiphysics system since the gas force acting on the pistons results from a chemical reaction. In the model of this section, the gas force is approximated by an analytical function. As mentioned by [84], the elastic effects which superimpose the overall rigid body movements have a significant influence on the behavior of the crank drive.

Figure 2.17: Flexible multibody system of a crank drive as major moving parts of a combustion engine.

Figure 2.18: Elastic piston rod of the crank drive. Rigid connection elements (RBE2) are used to create an interface reduction.

Very often the mechanical parts of the crank drive are modeled as a flexible multibody system (FMBS) with the floating frame of reference formulation. FMBS consist of flexible and rigid bodies which are coupled by joints and coupling elements. Advantages of FMBSs are their inherently modular fashion and the description of the flexible motion with respect to the reference frame, which allows a linear description of the elasticity. One single flexible body is described with a nonlinear second-order ordinary differential equation (ODE),

$$\begin{bmatrix} \mathbf{M}_f(\mathbf{q}) & \mathbf{M}_{ef}^T(\mathbf{q}) \\ \mathbf{M}_{ef}(\mathbf{q}) & \mathbf{M}_e \end{bmatrix} \begin{bmatrix} \mathbf{a}(t) \\ \ddot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{K}_e \mathbf{q}(t) + \mathbf{D}_e \dot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{h}_f(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{h}_e(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}, \quad (2.20)$$

which can be split into two parts: The part belonging to the motion of the floating frame, quantities with the subscript f, and the often high-dimensional flexible part, quantities with the subscript e, describing the linear elastic motion with respect to the reference frame (see, e. g., [33]). The subparts of the equation of motion are ex-

plained, e. g., in [33]. The linear elastic part usually stems from a finite element description of continua. Since it is very high-dimensional, it needs to be reduced by MOR. The procedure is made in a modular fashion. First, only the linear elastic part of (2.20),

$$\mathbf{M}_e \ddot{\mathbf{q}}(t) + \mathbf{D}_e \dot{\mathbf{q}}(t) + \mathbf{K}_e \mathbf{q}(t) = \mathbf{B}_e \mathbf{u}_e(t), \quad (2.21)$$

$$\mathbf{y}_e(t) = \mathbf{C}_e \mathbf{q}(t) \quad (2.22)$$

is considered as a second-order, time-invariant multiple-input multiple-output (MIMO) system. All reaction, applied, and coupling forces – the latter especially need to be taken into account due to nonlinear rigid body motion – acting on the elastic body are considered as inputs $\mathbf{B}_e \mathbf{u}_e(t)$ and outputs $\mathbf{y}_e(t) = \mathbf{C}_e \mathbf{q}(t)$ to the elastic body. Using the Laplace transformation, the transfer matrix of the system $\mathbf{G}(s) = \mathbf{C}_e (s^2 \mathbf{M}_e + s \mathbf{D}_e + \mathbf{K}_e)^{-1} \mathbf{B}_e$ is obtained.

For this second-order system, an appropriate subspace is generated by second-order structure-preserving reduction techniques, e. g., by a Galerkin ansatz $\mathbf{q}(t) \approx \mathbf{V} \mathbf{q}_r(t)$, where $\mathbf{q}_r \in \mathbb{R}^n$, $\mathbf{V} \in \mathbb{R}^{N \times n}$ and $n \ll N$. The reduced second-order MIMO system consisting of the matrices $\{\mathbf{M}_{er}, \mathbf{D}_{er}, \mathbf{K}_{er}\} = \mathbf{V}^T \{\mathbf{M}_e, \mathbf{D}_e, \mathbf{K}_e\} \mathbf{V}$, $\mathbf{B}_{er} = \mathbf{V}^T \mathbf{B}_e$, and $\mathbf{C}_{er} = \mathbf{C}_e \mathbf{V}$ is never simulated. Instead, the calculated ansatz space $\mathcal{V} = \text{span}(\mathbf{V})$ is used to calculate the reduced nonlinear equations of motion for one body,

$$\begin{bmatrix} \mathbf{M}_{fr}(\mathbf{q}_r) & \mathbf{M}_{efr}^T(\mathbf{q}_r) \\ \mathbf{M}_{efr}(\mathbf{q}_r) & \mathbf{V}^T \mathbf{M}_e \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \ddot{\mathbf{q}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{V}^T \mathbf{K}_e \mathbf{V} \mathbf{q}_r + \mathbf{V}^T \mathbf{D}_e \mathbf{V} \dot{\mathbf{q}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{fr}(\mathbf{q}_r, \dot{\mathbf{q}}_r) \\ \mathbf{h}_{er}(\mathbf{q}_r, \dot{\mathbf{q}}_r) \end{bmatrix}. \quad (2.23)$$

It is worth mentioning that by this MOR procedure a perfect hyper-reduction is achieved, i. e., the nonlinear terms in (2.23) only depend on the reduced quantities. All quantities labeled $[\cdot](\mathbf{q}_r, \dot{\mathbf{q}}_r)$ depend linearly or quadratically on the reduced elastic quantities. Different from many other nonlinear MOR procedures, no additional hyper-reduction scheme (see Section 2.5 of this chapter of this volume and Chapter 5 of Volume 2 of *Model order reduction*) like the discrete empirical interpolation method from [22] is necessary.

The quantity parts of (2.23), e. g., $\mathbf{h}_{fr}(\mathbf{q}_r, \dot{\mathbf{q}}_r)$ or $\mathbf{h}_{er}(\mathbf{q}_r, \dot{\mathbf{q}}_r)$, are called mass invariants and can be calculated prior to the simulation (see, e. g., [105]). Therefore, no back projection into the original high-dimensional space is necessary during the simulation, which is one benefit of EMBS simulations.

The described procedure is applied per body, which allows for a modular setup. The single reduced or rigid bodies are later coupled using a minimal coordinate approach or Lagrange multipliers (see, e. g., [105]).

As shown in Figure 2.17 for the crank drive example, only the piston rod is considered flexible. In other settings (see, e. g., [84]), other parts of the crank drive, e. g., the crankshaft are analyzed. As mentioned in [84], elastic effects and the chosen reduction method are important for realistic simulations for a crank drive system. Due

to the fact that the input/output behavior is essential, special care needs to be taken to approximate the interfaces in a correct way in the modeling and reduction process. Nevertheless, we know that every approximation introduces an error. Therefore, we are especially concerned to find measures for the evaluation of the error.

In the next section, a short explanation about the utilized reduction methods is given. Afterward, the error in the frequency and time domains is analyzed. Furthermore, we will also mention some possible error estimators and error bounds. We will finish the section with an outlook.

2.4.1 Used MOR methods

Since the piston rod undergoes only small deformations, it is modeled as a linear system. Several linear reduction techniques can be used to approximate the elastic coordinates of the piston rod, e. g.:

- Krylov method, i. e., matching the moments of the transfer function at defined frequencies up to defined orders [42];
- Craig–Bampton, i. e., static interface constraint modes in combination with fixed interface normal modes;
- CMS-Gram, i. e., frequency-weighted balanced truncation for the internal dynamics together with static correction modes [52];
- POD-Gram, i. e., balanced truncation with POD approximated frequency-weighted Gramian matrices [36].

All these methods have in common that they can be tuned for specific loading scenarios, e. g., by specifying frequencies in the Krylov method. In the previous section some information about the Craig–Bampton and CMS-Gram methods is given. Furthermore, the same work flow is used. For MOR of EMBS a correct consideration of boundary conditions and static correctness are essential steps; therefore, CMS-based or interpolation-based methods are favored. For systems with many connection points, e. g., gearboxes, the Krylov reduction or the CMS-based reduction turns out to be challenging. Therefore, an interface reduction is necessary very often in a first step. Here a model-based interface reduction is performed by inserting rigid connection elements (RBE2) at the two bearings of the piston rod (Figure 2.18). The slave nodes (nodes at the bearing seats) and the master node (the central node) behave like a rigid body.

Due to the boundary conditions, the connecting points at the piston rod can only move in the yz -plane. Therefore, only the inertia forces in these directions are considered as excitations in the CMS-Gram method and collocated outputs $\mathbf{C}_e = \mathbf{B}_e^T$ are considered. The six static modes of the interface node between crankshaft and piston rod correspond to the rigid body modes. For the CMS-Gram approach, the frequency range of interest is $\mathcal{I}_f = [0 \text{ kHz}, 8 \text{ kHz}]$.

2.4.1.1 Results in the frequency domain

We are interested in the elastic movements; therefore, we only analyze the nonrigid body movement, which is the elastic dynamics of the system. This neglects the error due to coupling of the rigid body movement and internal dynamics. In a first step the separation between interface and internal nodes is conducted by a CMS-based approach, and later the Gramian matrix is used to approximate only the internal dynamics, which belongs to the internal nodes. In Figure 2.19, the transfer function of the unreduced elastic dynamic of the piston rod is plotted. The error system $\mathbf{G}_e = \mathbf{G} - \tilde{\mathbf{G}}_r$ with the reduced second-order system $\tilde{\mathbf{G}}_r(s) = \mathbf{C}_{eR}(s^2\mathbf{M}_{eR} + s\mathbf{D}_{eR} + \mathbf{K}_{eR})^{-1}\mathbf{B}_{eR}$ is used to evaluate different reduction methods. As in Section 2.3, the relative error

$$\epsilon_r(s = i\omega) = \|\mathbf{G}(\omega) - \tilde{\mathbf{G}}_r(\omega)\|_{\text{Fro}} / \|\mathbf{G}(\omega)\|_{\text{Fro}} \quad (2.24)$$

over the angular frequency ω measured in the Frobenius norm of models reduced with different reduction methods is plotted in Figure 2.20. The interesting frequency range between 0 kHz and 20 kHz is equidistantly sampled with 500 points.

Figure 2.19: Norm of the transfer function of the unreduced ($N = 7332$) internal dynamic (black) and with CMS-Gram ($n = 6$) reduced internal dynamic (blue).

Figure 2.20: Different relative errors for different reduction methods and different reduction levels. Blue: CMS-Gram; green: classical Craig–Bampton. Lighter colors label models with smaller reduction size.

It can be seen that the relative error for the CMS-Gram-based reductions is smaller than the error of the Craig–Bampton reductions.

If we compare the CMS-based approach with reduction not based on a separation between interface and internal nodes, e. g., modal or POD-Gram reduction, a far worse approximation of the elastic/internal dynamics is achieved even with higher dimensions. The eigenvectors of the free system have a very low correlation to the internal dynamics. The generalized inertia forces do not excite these eigenmodes. Therefore, they are unimportant for the problem at hand. The POD-based reduction is not tuned

to approximate the internal dynamics. Therefore, no fair comparison is possible and we decided not to show these results.

Typically, the error is measured in specific norms, e. g., the $\mathcal{H}_\infty$ - or the $\mathcal{H}_2$ -norm [85]. For Gramian-based reductions, an error bound based on the sum of neglected Hankel singular values (HSVs) is available.

In Figure 2.21, the relative error in the $\mathcal{H}_2$ -norm of the internal dynamics is plotted. For a reduction size of 6, only static modes are used; therefore, the CMS-Gram and the Craig–Bampton approach behave exactly the same. Already with one more reduction mode, the CMS-Gram approach shows better results. After 12 and 16 modes, we again see a big improvement (Figure 2.21). The Craig–Bampton reduction has a step-like decay, some eigenmodes of the bounded system have only a minor influence on the approximation quality. Therefore, after we achieved a reduction size of 14 no further improvements were made with the Craig–Bampton approach. In comparison to the Craig–Bampton reduction, the CMS-Gram-based reduction explicitly considers the generalized inertia forces, and therefore has a more steady and rapid decay of the error and the HSVs. A more elaborate description of this behavior is given by [51] for a slightly different crank drive example.

Figure 2.21: $\mathcal{H}_2$ -error of the internal dynamics.

We saw that Gramian-based approximation in combination with static condensations is favorable. However, for a large-scale system, a direct calculation of the Gramian matrices is not possible. Two of the many approaches to calculate Gramian matrix-based reduction spaces are (i) the POD-based approximation of the frequency-weighted Gramian matrices and (ii) a two-step approach in which a Krylov-based approach calculates an intermediate size model in a first step. The reduction quality of the POD approach depends on the location of the frequency snapshots and the Krylov-based reduction depends on the location of the expansion points. The intermediate models

need to guarantee an acceptable approximation quality; however, due to their large size we cannot calculate the $\mathcal{H}_\infty$ -norm, the $\mathcal{H}_2$ -norm, or ϵ_r .

2.4.1.2 Error estimators in the frequency domain

As mentioned in [33] and [36], error estimators can be used to gradually select good positions of the frequency snapshots – respectively expansion points – at the most suitable position and estimate the error. Different error estimators for a given frequency range $[\omega_{\min}, \omega_{\max}]$ are available to replace the time needed to evaluate the original system in calculating the relative error ϵ_r defined in (2.24).

One possibility introduced by [42] is the replacement with a second reduced system $\tilde{\mathbf{G}}$. The approximation error is then estimated by the relative error between the two reduced-order systems (see, e. g., [42]):

$$\epsilon_{\text{Grimme}}(\omega) = \|\tilde{\mathbf{G}}(\omega) - \tilde{\mathbf{G}}(\omega)\|_{\text{Fro}} / \|\tilde{\mathbf{G}}(\omega)\|_{\text{Fro}}. \quad (2.25)$$

A second error estimator was introduced by [13]: Instead of using a second ROM, the ROM from the previous iteration $\tilde{\mathbf{G}}_{k-1}$ is used:

$$\epsilon_{\text{BRK}}(f) = \|\tilde{\mathbf{G}}_k(i2\pi f) - \tilde{\mathbf{G}}_{k-1}(i2\pi f)\|_{\text{Fro}} / \|\tilde{\mathbf{G}}_k(i2\pi f)\|_{\text{Fro}}. \quad (2.26)$$

In the provable error estimator of [62], the error is separated into two complementary subspaces $\mathcal{S}_\Phi$ and $\mathcal{S}_\perp$ of the eigenmodes in the interesting frequency range. This error estimator has a provable error bound for lossless systems in addition to an extension to second-order systems (see, e. g., [33]):

$$\epsilon_{\text{KFD}}(\omega) = V_2(R, \eta, \xi_{p_{\max}}) \|\mathbf{K}_c \mathbf{B}_e - \mathbf{K}_\omega(\omega) \mathbf{V} \mathbb{L}_r^{-1}(i\omega) \mathbf{B}_{er}\|. \quad (2.27)$$

The error can be calculated efficiently by an offline-online decomposition. The term $\mathbf{K}_c$ is independent of ω and $\mathbf{V}$, the term $\mathbf{K}_\omega(\omega)$ is independent of the reduction matrix $\mathbf{V}$, the term $\mathbb{L}_r^{-1}$ is based on reduced quantities and is further explained in [33], and the term $V_2(R, \eta, \xi_{p_{\max}})$ is a maximum bound of the magnification function of a second-order elementary vibrating system well known in linear vibrations theory (see, e. g., [79]).

In Figure 2.22, the different error estimators are plotted. These error estimators could, e. g., be used in a greedy-based selection process to add new expansion points for a Krylov subspace used for reduction or in a POD-like approximation of the second-order frequency-weighted Gramian matrices [36]. For this example, the error estimator ϵ_{KFD} has a slightly different behavior from the other error estimators but both share the same form.

Mechanical systems are not very sensitive to the location of the expansion points as other systems may be, especially in this example, where there is not much dynamic

Figure 2.22: Comparison of the three error estimators and the relative error.

in the interesting frequency range. Therefore, all of the error estimators are helpful in automating the reduction process. Even if the absolute values of the error estimators are not consistent with the real error, the frequency location of maximum values and the convergence behavior of the error estimators are consistent with the real error.

2.4.1.3 Results in the time domain

In the time domain, we compare the well-established Craig–Bampton method with a mediocre basis size of 25 to the other reduction methods with basis size of only 7 (Figure 2.23). The Craig–Bampton model of size 25 is considered as a converged ROM. For the results an explicit first-order Runge–Kutta solver with a fifth-order automatic time step size control for stiff ODEs (MATLAB ode15s) is used. Modal reduction shows a phase shift and a larger error at the time of highest deformation. Despite having no phase shift, the Craig–Bampton method with only seven basis vectors shows bad conformance with the reference solution at the zoom-in view due to the small basis size. Only CMS-Gram accomplishes almost no error compared to the reference solution with a basis size of only 7.

2.4.1.4 Error estimators in the time domain

Error bounds based on the residual

$$\mathbf{R}_m(t) = \mathbf{M}_e \mathbf{V} \ddot{\mathbf{q}}_r + \mathbf{D}_e \mathbf{V} \dot{\mathbf{q}}_r(t) + \mathbf{K}_e \mathbf{V} \mathbf{q}_r(t) - \mathbf{B}_e \mathbf{u}_e(t) \quad (2.28)$$

between the reduced and the original model can be used to deliver a posteriori error bounds in the time domain, which account for the current excitation. The general, a posteriori error estimator $\tilde{\Delta}$ of [99] for second-order mechanical systems is used in

Figure 2.23: Deformation of one node in the z -direction for various reduction methods. The zoom-in view shows the time of the largest deformation.

the following. It has been analyzed in [37]. We apply this error estimator to the single, clamped piston rod excited with an approximated gas force. The true error is compared to the error estimator in Figure 2.24 for different model reduction methods.

It is of no surprise that modal reduction with 100 modes produces the smallest error. Looking at reductions with smaller basis size, Krylov 6 seems to have the best results. While all other methods have a practically indistinguishable error bound, the error estimator of Krylov 6 is two orders lower. This phenomenon still needs to be investigated.

2.4.2 Outlook

For EMBS simulations, MOR is one essential step to create fast-to-calculate but still convincing simulation models. If there are many inputs to the system, the interface reduction process often plays a far more critical role than the used MOR schemes. Error estimators in the frequency domain are helpful in automating the reduction process. Error estimation in the time domain with a priori error bounds for coupled multibody systems is a nontrivial task since coupling terms influence the behavior of a single part. New strategies such as to rewrite the multibody system as DAEs to consider all reaction forces as inputs need to be developed and rigorously tested.

With the CMS-Gram methods, a method is at hand which combines the benefit of static correctness, with an error-based, Gramian matrix-based approximation of the internal dynamics. Furthermore, the generalized inertia forces are considered in the reduction process. Good results are achieved in the frequency as well as in the time domain.

Figure 2.24: True error $\Delta(t) = \|\mathbf{V}\mathbf{q}_r(t) - \mathbf{q}(t)\|_2$ in solid lines and the error estimator $\tilde{\Delta}_q$ of the state in dashed lines of a clamped piston rod excited with an approximated gas force; $\tilde{\Delta}_{\text{all}}$ represents the error estimator of all methods except Krylov 6.

2.5 Nonlinear model order reduction for a leaf spring model

2.5.1 Nonlinear model reduction in structural dynamics

In engineering, one finds many applications where exact knowledge of the dynamic behavior of the structures during operation is essential. This is mainly due to the fact that the dynamics influence design goals such as fatigue, vibration comfort, or noise emission. In this section, we present the MOR of the leaf spring of a truck. This part undergoes large deformations in certain maneuvers like strong braking and exhibits hence geometrically nonlinear behavior. Since the transfer path of the excitation force goes through the spring, an accurate model of the nonlinear spring is crucial for the dynamics assessment of the otherwise linear chassis model.

The semi-discretized equations of motion from finite element models that are able to describe the dynamics of structures with large deflections are described by equation (2.2). To reduce the computational effort, one needs to perform two steps. First, a Galerkin projection is carried out in order to reduce the number of unknowns. Assuming a linear viscous damping matrix, this yields

$$\mathbf{V}^T \mathbf{M} \mathbf{V} \ddot{\mathbf{q}}_r(t) + \mathbf{V}^T \mathbf{D} \mathbf{V} \dot{\mathbf{q}}_r(t) + \mathbf{V}^T \mathbf{f}(\mathbf{V} \mathbf{q}_r(t)) = \mathbf{V}^T \mathbf{B} \mathbf{F}(t). \quad (2.29)$$

This approximation is similar to the projection methods used in linear model reduction for structural dynamics as described in Section 2.1. However, this reduction step alone does not reduce computation time significantly. The solution time for the leaf spring model, which will be shown below, can only be sped up 1.6 times by this

step, despite the fact that the solution vector has been reduced from 200,000 degrees of freedom to 100. The reason for this is the nonlinear restoring force term $\mathbf{V}^T \mathbf{f}(\mathbf{V} \mathbf{q}_r)$, which is evaluated by an assembly of all element forces. Unlike in the reduction of linear systems, one cannot build a reduced matrix since the restoring forces are nonlinear. Hence, the evaluation of the nonlinear term is the new bottleneck of the solution process. The methods reducing the evaluation costs of this nonlinear term are called “hyper-reduction” methods. The application of a hyper-reduction is the second step that needs to be performed. Different hyper-reduction methods can be found in the literature. One prominent example is the discrete empirical interpolation method [22]. Other approaches are more suitable for nonlinear structural dynamics, such as the energy-conserving sampling and weighting (ECSW) method [31, 30] and polynomial expansion [50], which will be used here.

2.5.2 Basis generation

One approach to get a reduction basis is the so-called POD. It is based on the singular value decomposition $\mathbf{q}_\tau = \mathbf{U} \mathbf{\Sigma} \mathbf{W}^T$ of some training displacements $\mathbf{q}_\tau$, which usually come from results of a time integration of the full model. The reduction basis $\mathbf{V}$ is then built by stacking the first n left singular vectors into $\mathbf{V}$ so that $\mathbf{V}$ consists of the first n columns of $\mathbf{U}$.

Another approach generates the reduction matrix from two parts,

$$\mathbf{V} = [\mathbf{V}_{\text{lin}} \quad \mathbf{V}_{\text{nl}}], \quad (2.30)$$

where $\mathbf{V}_{\text{lin}}$ contains some vibration modes $\boldsymbol{\phi}$ of the linearized system and $\mathbf{V}_{\text{nl}}$ contains some modal derivatives $\mathbf{v}_{ij}$. The modal derivatives describe how the mode $\boldsymbol{\phi}_i$ changes if the system is perturbed in the direction of another mode $\boldsymbol{\phi}_j$. A slightly modified version of the modal derivatives are the static modal derivatives (SMD) [107, 57] that are calculated by solving

$$\mathbf{K} \mathbf{v}_{ij} = -\nabla_{\boldsymbol{\phi}_j} \mathbf{K}(\mathbf{q}) \boldsymbol{\phi}_i, \quad (2.31)$$

which often perform better than the modal derivatives.

2.5.3 Hyper-reduction

2.5.3.1 Polynomial expansion

When linear materials are used, the internal force taking into account geometric nonlinear effects due to large deformations and rotations can be written using Einstein summation convention as

$$\mathbf{f}_i(\mathbf{q}) = \mathbf{K}_{ij}^{(1)} \mathbf{q}_j + \mathbf{K}_{ijk}^{(2)} \mathbf{q}_j \mathbf{q}_k + \mathbf{K}_{ijkl}^{(3)} \mathbf{q}_j \mathbf{q}_k \mathbf{q}_l. \quad (2.32)$$

Thus, the nonlinear restoring force vector can be described by three tensors $\mathbf{K}^{(1)}$, $\mathbf{K}^{(2)}$, and $\mathbf{K}^{(3)}$.

Applying a Galerkin projection with the reduction basis $\mathbf{V}$, one gets

$$(\mathbf{V}^T \mathbf{f}(\mathbf{V} \mathbf{q}_r))_i = \bar{\mathbf{K}}_{ij}^{(1)} \mathbf{q}_{r,j} + \bar{\mathbf{K}}_{ijk}^{(2)} \mathbf{q}_{r,j} \mathbf{q}_{r,k} + \bar{\mathbf{K}}_{ijkl}^{(3)} \mathbf{q}_{r,j} \mathbf{q}_{r,k} \mathbf{q}_{r,l}, \quad (2.33)$$

with

$$\begin{aligned} \bar{\mathbf{K}}_{ij}^{(1)} &= (\mathbf{V}^T)_{ik} \mathbf{K}_{kl}^{(1)} \mathbf{V}_{lj}, \\ \bar{\mathbf{K}}_{ijk}^{(2)} &= (\mathbf{V}^T)_{il} \mathbf{K}_{lmn}^{(2)} \mathbf{V}_{mj} \mathbf{V}_{nk}, \\ \bar{\mathbf{K}}_{ijkl}^{(3)} &= (\mathbf{V}^T)_{im} \mathbf{K}_{mnop}^{(3)} \mathbf{V}_{nj} \mathbf{V}_{ok} \mathbf{V}_{pl}. \end{aligned}$$

Therefore, one only needs to identify $\bar{\mathbf{K}}^{(1)}$, $\bar{\mathbf{K}}^{(2)}$, and $\bar{\mathbf{K}}^{(3)}$ to get a reduced model whose nonlinear term can be evaluated very fast. For the computation of these three tensors, several techniques exist which can be classified as intrusive and nonintrusive methods. Intrusive methods identify the tensors by computing the coefficients on element level [92, 106] requiring access to the element formulation inside the finite element code. Nonintrusive methods, however, do not require the element formulation and can hence be used with commercial finite element software, where the access to internal computations is limited. Some techniques identify the tensors by prescribing displacements and evaluating the resulting nonlinear forces [80, 61], others by prescribing forces and evaluating displacements [74]. The implicit condensation and expansion method [50] computes both polynomial tensors and the reduced basis in one step. In our case study, we use the nonintrusive identification as proposed in [101, 86, 87], which uses multiple evaluations of the reduced tangential stiffness matrix $\mathbf{K}(\mathbf{q})$ at different displacements.

2.5.3.2 Energy-conserving sampling and weighting

Another approach is the ECSW method [31, 30], which is based on a reduced assembly of a subset of the elements and extrapolates their contribution to the full force vector:

$$\mathbf{V}^T \mathbf{f}(\mathbf{V} \mathbf{q}_r) = \sum_{e \in E} \mathbf{V}^T \mathbf{L}_e^T \mathbf{f}_e(\mathbf{L}_e \mathbf{V}_e \mathbf{q}_r) \approx \sum_{e \in \tilde{E} \subset E} \xi_e \mathbf{V}^T \mathbf{L}_e^T \mathbf{f}_e(\mathbf{L}_e \mathbf{V}_e \mathbf{q}_r). \quad (2.34)$$

The matrix $\mathbf{L}_e$ is a Boolean mapping matrix from the local degrees of freedom of element e to the global degrees of freedom and ξ_e are positive weights for extrapolation. The weights ξ_e and the element subset $\tilde{E}$ are chosen by using training displacements for which the virtual work of the restoring force in the direction of the reduction basis shall be retained in the hyper-reduced model. As in the POD, these training displacements are often computed from simulations of the high-dimensional model. Another approach to gain training displacements is presented in the next section, since performing full simulations is numerically expensive.

2.5.3.3 Nonlinear stochastic Krylov training sets for ECSW

An alternative approach to build training vectors for ECSW, which avoids the full simulations, is called nonlinear stochastic Krylov training sets (NSKTS), proposed in [100]. The idea of this method is to build a subspace whose vectors are able to approximate the nonlinear force vector $\mathbf{f}$. If the viscous damping term $\mathbf{D}$ is neglected, one finds

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{f}(\mathbf{q}(t)) = \mathbf{B}\mathbf{F}(t) \rightsquigarrow \mathbf{f} \in \text{span}(\mathbf{B}, \mathbf{M}\ddot{\mathbf{q}}(t_1), \mathbf{M}\ddot{\mathbf{q}}(t_2), \dots, \mathbf{M}\ddot{\mathbf{q}}(t_n)).$$

As the accelerations $\ddot{\mathbf{q}}(t_i)$ are unknown, an approximation for the subspace described above must be made. In the NSKTS, the subspace is approximated with the Krylov subspace

$$\mathbf{F}_{\text{kry}} = \text{span}\{\mathbf{B}, \mathbf{M}\mathbf{K}^{-1}\mathbf{B}, (\mathbf{M}\mathbf{K}^{-1})^2\mathbf{B}, \dots\} = \mathcal{K}(\mathbf{M}\mathbf{K}^{-1}, \mathbf{B}), \quad (2.35)$$

which is orthogonalized and normalized so that $\mathbf{F}_{\text{kry}}^T \mathbf{K}^{-1} \mathbf{F}_{\text{kry}} = \mathbf{I}$.

Then, some random vectors $\mathbf{f}_{\text{NSKTS}}^\tau$ living in this Krylov subspace are generated and applied as external force to the nonlinear static problem

$$\mathbf{f}(\mathbf{q}_\tau^{(k)}) = k\mathbf{f}_{\text{NSKTS}}^\tau, \quad k \in (0, 1].$$

This equation must be solved by a nonlinear solver such as Newton–Raphson. The training set is then built by saving the solution $\mathbf{q}_\tau^{(k)}$ for some load steps k for each external force $\mathbf{f}_{\text{NSKTS}}^\tau$ as training vector. This procedure reduces the computation cost for obtaining a set of training vectors by avoiding direct time integration of the full model. All steps can also be carried out in the reduced subspace spanned by $\mathbf{V}$. This has the advantage that the nonlinear static problems can be solved faster and the resulting training vectors live in the subspace of the displacements for the ROM [100]. There are no a priori error estimates for the NSKTS method and, therefore, the necessary number of training vectors is evaluated by varying its number and checking the convergence of the solution.

2.5.4 Case study: leaf spring

In the following, the results of the reduction of a truck chassis leaf spring are summarized. The case study is carried out with the Finite Element Package AMfe, developed by the Chair of Applied Mechanics at Technical University of Munich. The code is available at <https://github.com/AppliedMechanics/AMfe>. The full study is published in [101].

Figure 2.25 shows the leaf spring, which consists of two leafs (top and bottom), a central fixture that joins the leafs, and two rubber pads that keep the distance between the leafs at the ends. The front eye is fixed with the frame via a joint allowing

Figure 2.25: Leaf spring obtained from [101] consisting of two leaves, rubber pads, and central fixture.

rotations about the y -axis while the rear eye allows both rotations about the y -axis and a translation in the x -direction. The rubber pads are fixed with the top leaf and have a sliding contact with the lower leaf. The model is meshed with 85,762 linear elements (tetrahedrons and hexahedrons). A load case is applied that stems from a multibody simulation of a brake maneuver. Time-varying forces and loads are applied on the top of the central fixture.

We compare the simulation time and accuracies of a simulation with 1,500 time steps carried out with a generalized α scheme ($\rho_\infty = 0.8$). As accuracy measure we define the relative error

$$\text{RE} = \frac{\sqrt{\sum_i \Delta \mathbf{q}(t_i)^T \Delta \mathbf{q}(t_i)}}{\sqrt{\sum_i \mathbf{q}_{\text{ref}}(t_i)^T \mathbf{q}_{\text{ref}}(t_i)}} \cdot 100 \% \quad \text{with } \Delta \mathbf{q}(t_i) = \mathbf{q}(t_i) - \mathbf{q}_{\text{ref}}(t_i). \quad (2.36)$$

We compare two kinds of errors: First, the error of the Galerkin projection, where the non-hyper-reduced but projected system is compared with the full solution (RE_f), and, second, the error of the hyper-reduction, where the hyper-reduced solution is compared to the non-hyper-reduced but projected solution (RE_{hr}).

Table 2.2 lists the relative errors and simulation times for different simulations. The wall time for a full simulation run with 216,499 degrees of freedom is 40,022 s (≈ 11 h). As reduction basis, 100 basis vectors consisting of 40 vibration modes and 60 static modal derivatives are chosen. One can see that only a small speedup of 1.66 is gained while the error is quite small. To further speed up the simulation, hyper-reduction is needed.

Therefore, two hyper-reduction methods are considered. First, the ECSW is carried out with 160 nonlinear stochastic Krylov training sets. The hyper-reduced mesh is shown in Figure 2.26. Only 816 elements are evaluated while the relative error of the hyper-reduction is 0.13 %. The hyper-reduced model gains a speedup of 38.23 compared to the full simulation. Second, the polynomial expansion is carried out. The identification of the tensor coefficients needs 28,856 s (≈ 8 h), but the hyper-reduced model has a wall time of just 113 s, which is a speedup of 354.18 compared to the full simulation. Since the model has a linear elastic material and the nonlinear strain measure is quadratic in $\mathbf{q}$, the cubic polynomial expansion of the internal restoring force

Table 2.2: Errors and speedup rates of the different reduction techniques for the leaf spring example obtained from [101].

Reduction method	DOFs	Elements	RE_f [%]	RE_{hr} [%]	t_w [s]	Speedup [-]
full	216,499	85,762	–	–	40,022	–
Modes & SMDs	100	85,762	0.68	–	24,127	1.66
Modes & SMDs + ECSW with NSKTS	100	816	0.78	0.13	1 047	38.23
Modes & SMDs + polynomial Exp.	100	–	0.68	0.00	113	354.18

Figure 2.26: Hyper-reduced mesh obtained from [101]. The reduced mesh consists of 816 elements that have different weights ξ .

(2.32) is exact. Hence, the relative error of the polynomial expansion hyper-reduction method is zero in this case.

2.5.5 Conclusion

The geometric nonlinear leaf spring has been reduced by a reduction basis of dimension 100. The reduction basis has been computed by using the properties of the system without the need for results from full dynamic simulations. The combination of 40 vibration modes and 60 static derivatives leads to accurate results although the full model has 216,499 degrees of freedom. Then, two different hyper-reduction techniques, ECSW and polynomial expansion, are carried out. Nonlinear stochastic Krylov training sets are used for ECSW which avoid full simulation runs and lead to relative small reduction costs of about 73 minutes. The ECSW-reduced model leads to a speedup factor of 38.23. The polynomial expansion gives the best speedup, which is about 10 times higher than with ECSW, while the reduction time is much higher (about 8 h). Thus, one can conclude that polynomial expansion is best suited if offline costs

do not matter or if one needs to run many simulations or many time steps with the reduced model. However, the polynomial (cubic) expansion is only valid for models with linear elastic materials. Another issue is the computational cost and memory demand for the identification of the polynomial coefficients. Both depend highly on the dimension n of the reduction basis, since they increase with $\mathcal{O}(n^4)$. In our test case, the reduction basis of dimension 100 was suitable. For models requiring a larger reduced basis, hyper-reduction using the ECSW is a good option.

Author contributions

M. Cruz Varona, C. Lerch, and B. Lohmann (Technical University of Munich) wrote Section 2.1. C. D. Yuan, E. B. Rudnyi, and T. Bechtold (University of Rostock) wrote Section 2.2. B. Fröhlich, P. Holzwarth, and P. Eberhard (University of Stuttgart) wrote Section 2.3. D. Grunert and J. Fehr (University of Stuttgart) wrote Section 2.4. And C. H. Meyer, J. B. Rutzmoser, and D. J. Rixen (Technical University of Munich) wrote Section 2.5.

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