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5 Complexity reduction of electromagnetic systems

Abstract: This chapter has two main objectives: first, to propose a computer-aided consistent and accurate description of the behavior of electromagnetic devices at various speeds or frequencies and, second, to describe procedures to generate compact electrical circuits for them, with an approximatively equivalent behavior. The extracted models should have a finite complexity as low as possible, while yielding an acceptable accuracy, as well as preserve essential characteristics, such as passivity. A successful complexity reduction can be obtained if *a priori* and *on-the-fly* reduction strategies are applied before and during the model discretization, followed by *a posteriori* complexity reduction.

Keywords: electromagnetic devices, electromagnetic systems, conceptual models, mathematical models, numerical models, a priori reduction, on-the-fly reduction, a posteriori reduction

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5.1 Introduction

An electromagnetic (EM) device is a system in which the EM field plays an essential role. In order to extract a model of finite complexity, the first objective is to reduce the complexity of the physical and mathematical models, which are initially of infinite dimension. The reduction from infinity to a finite order can be achieved by discretization with a numerical method. However, due to the structural and geometrical complexity of devices encountered now in real-life, the discrete models thus obtained have an extremely high order of complexity, reaching even orders of magnitude above millions, requiring additional complexity reduction. That is why it is important to reduce the model complexity not only after numerical discretization, but also prior to it, during the modeling stage.

The modeling procedure consists of seven successive steps, each dedicated to building a particular form of the model: (1) *Conceptual modeling* (conceptual model = geometrical model + physical model) establishes consistent geometrical models and physical models of the device including simplifying hypotheses and justified approximations of geometrical and physical nature. (2) *Mathematical modeling* formulates the mathematical equations that describe the operation of the device, presenting the con-

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ceptual model in a mathematical language as a properly formulated problem. (3) *Approximate analytical modeling* (AAM) determines the approximate analytical relationship between the input and output physical quantities by solving an approximate, simplified version of the model equations. (4) *Numerical modeling* aims to build an algorithm to obtain the solutions of the mathematical model equations. (5) *Computational modeling* (simulation software) aims to create and test a computer program to implement the numerical algorithm conceived in the previous step. (6) Model order reduction (MOR) finds the simplest input/output system which approximates a posteriori, i. e., after the discretization, the behavior of the modeled device, while maintaining the original behavior with an acceptable accuracy. (7) Verification and validation is the final step of the procedure, consisting of the verification of the solution obtained by simulation and the validation of the extracted model, by comparison with experiments. Without this final check the entire procedure is completely useless.

The discretization methods used in the fourth step and their main characteristics are given in Table 5.1. Since the most used one is the finite element method (FEM), the final section of this chapter is devoted solely to it.

Table 5.1: Characteristics of the fundamental numerical methods for EM field analysis.

Method	Discretization mesh	Form of discretized equations
FEM Finite element	Unstructured, formed with triangles, quadrilaterals, tetrahedrons, hexahedrons, etc.	Weak form, variational equations
FDM, FIT Finite differences, Finite integrals	Grid – mesh with regular topology, Pair of staggered dual grids in the case of hyperbolic equations	Differential (FDM) or global equations (FIT)
BEM Boundary elements	Unstructured two-dimensional mesh, on the domain boundary	Integral equations

The complexity of the EM model in steps 1 and 2, which has distributed parameters and is described by partial differential equations (PDEs), can be reduced not only a posteriori, after discretization, but also on-the-fly, during the discretization, or even a priori. All these complexity reduction steps contribute equally to obtain, eventually, a model appropriate for the designer's needs. From this perspective, it becomes apparent that the efficient reduction of the model complexity cannot be limited to a reduction applied to the discrete model with classical MOR methods, or by matrix condensation, and that the best final results require reductions before and during discretization.

This chapter is structured according to the modeling procedure above.

5.2 Fundamental quantities and equations

In the macroscopic electromagnetism, the spatial distribution of the EM field is described at each moment in time by a first set of four *local primitive* physical quantities [43], which are the alphabet of this theory: (1) *electric field strength* $\vec{E} = \vec{f}_1(\vec{r}, t)$ (V/m); (2) *electric flux density* $\vec{D} = \vec{f}_2(\vec{r}, t)$ (C/m²); (3) *magnetic field strength* $\vec{H} = \vec{f}_3(\vec{r}, t)$ (A/m); and (4) *magnetic flux density* $\vec{B} = \vec{f}_4(\vec{r}, t)$ (T). Mathematically, they are three-dimensional time-dependent vector fields, defined over unbounded spatial and temporal domains. However, in order to be represented on computers, their restrictions are used instead, defined on a so-called bounded “computational” domain $\vec{r} \in \mathcal{D} \subset \mathbb{R}^3$, and on a specified bounded time interval $0 \leq t < T$. A second set of two primitive physical quantities describes locally the EM state of the material objects: (5) *volume charge density* $\rho = f_5(\vec{r}, t)$ (C/m³); and (6) *volume current density* $\vec{J} = \vec{f}_6(\vec{r}, t)$ (A/m²).

The *derived global* quantities, defined as spatial integrals on specified lines (open $\mathcal{C}$, closed Γ), surfaces (open $\mathcal{S}_\Gamma$, closed Σ), or domains ($\mathcal{D}_\Sigma$) of the local quantities provide the global description of the EM field and objects: (1) *electric voltage* $u = \int_{\mathcal{C}} \vec{E} \cdot d\vec{r}$ (V); (2) *electric flux* $\psi = \int_{\mathcal{S}_\Gamma} \vec{D} \cdot d\vec{A}$ (C); (3) *magnetic voltage* $u_m = \int_{\mathcal{C}} \vec{H} \cdot d\vec{r}$ (A); (4) *magnetic flux* $\varphi = \int_{\mathcal{S}_\Gamma} \vec{B} \cdot d\vec{A}$ (Wb); (5) *electric charge* $q = \int_{\mathcal{D}_\Sigma} \rho \, dv$ (C); and (6) *electric current intensity* $i = \int_{\mathcal{S}_\Gamma} \vec{J} \cdot d\vec{A}$ (A).

The global quantities are time-dependent scalar functions, associated with corresponding one-, two-, or three-dimensional manifolds. The curves and surfaces considered for modeling have to be sufficiently smooth (i. e., Lipschitz manifolds) and oriented. A closed curve Γ is oriented according to the right-hand rule as related to the supported surface, and a closed surface Σ is oriented outwards. Open lines or surfaces are oriented arbitrarily. These definitions highlight that the local quantities are actually associated with differential forms in external calculus. Using a simplified language, we might say that the field strengths ($\vec{E}$, $\vec{H}$) are 1-forms, the fluxes ($\vec{D}$, $\vec{B}$, $\vec{J}$) are 2-forms, and the volume charge density ρ is a 3-form. A p -form is a quantity to be integrated on a corresponding p -manifold (curve, surface, domain for $p = 1, 2, 3$, respectively), by using a corresponding differential dx (elementary length $d\vec{r}$, area $d\vec{A}$, and volume dv , respectively). If we denote the space of p -forms with W^p , then the primitive EM quantities are the p -forms: $\vec{E}, \vec{H} \in W^1$, $\vec{D}, \vec{B}, \vec{J} \in W^2$, $\rho \in W^3$.

The *general laws* of the EM field describe quantitatively the EM fundamental phenomena [33]. Their global forms, valid for any three-dimensional domain $\mathcal{D}_\Sigma$ and any surface $\mathcal{S}_\Gamma$, with $\Sigma = \partial\mathcal{D}_\Sigma$ and $\Gamma = \partial\mathcal{S}_\Gamma$, are relationships between global primitive quantities, stated as follows:

1. *electric flux law* (Gauss)

$$\psi_\Sigma = q_{\mathcal{D}_\Sigma}, \quad (5.1)$$

2. *magnetic flux law* (Gauss)

$$\varphi_\Sigma = 0, \quad (5.2)$$

3. *EM induction law* (Faraday)

$$u_\Gamma = -\frac{d\varphi_{S_\Gamma}}{dt}, \quad (5.3)$$

4. *magnetic circuit law* (Ampere–Maxwell)

$$u_{m\Gamma} = i_{S_\Gamma} + \frac{d\psi_{S_\Gamma}}{dt}. \quad (5.4)$$

The local forms of (5.1)–(5.4) are called *Maxwell's equations*:

$$\operatorname{div} \vec{D} = \rho, \quad \nabla \cdot \vec{D} = \rho, \quad (5.5)$$

$$\operatorname{div} \vec{B} = 0, \quad \nabla \cdot \vec{B} = 0, \quad (5.6)$$

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (5.7)$$

$$\operatorname{curl} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (5.8)$$

These are general, fundamental, first-order PDEs of the macroscopic electromagnetism, valid only for media at rest, whereas (5.1)–(5.4) are valid also for moving media. Their proofs use the Gauss–Ostrogradsky and Stokes theorems and they can be written using the *del* operator ∇ .

On motionless surfaces of discontinuity, the local forms are

$$\operatorname{div}_s \vec{D} = \rho_s, \quad \vec{n}_{12} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s, \quad (5.9)$$

$$\operatorname{div}_s \vec{B} = 0, \quad \vec{n}_{12} \cdot (\vec{B}_2 - \vec{B}_1) = 0, \quad (5.10)$$

$$\operatorname{curl}_s \vec{E} = 0, \quad \vec{n}_{12} \times (\vec{E}_2 - \vec{E}_1) = \vec{0}, \quad (5.11)$$

$$\operatorname{curl}_s \vec{H} = \vec{J}_s, \quad \vec{n}_{12} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s, \quad (5.12)$$

where $\vec{n}_{12}$ is the unit vector normal to the discontinuity surface, the subscripted vectors indicate their values at opposite sides of the interface, at the same point on it, $\rho_s = f_7(\vec{r}, t) = dq/dA$ (C/m²) is the *surface charge density*, and $\vec{J}_s = \vec{f}_8(\vec{r}, t) = \vec{t} di/d\vec{l}$ is the *surface current density*.

To ensure the completeness of the system of equations (5.1)–(5.4), which describe the EM field, three material-dependent *constitutive relationships* of the macroscopic electromagnetism are added [33]:

1. *Electric conduction law* (Ohm):

$$\vec{J} = f_a(\vec{E}). \quad (5.13)$$

In conductors with affine characteristics $\vec{J} = \sigma(\vec{E} + \vec{E}_i)$ or $\vec{J} = \sigma \vec{E} + \vec{J}_i$.

2. Polarization law:

$$\vec{D} = f_b(\vec{E}). \quad (5.14)$$

In dielectrics with affine characteristics $\vec{D} = \epsilon \vec{E} + \vec{P}_p$.

3. Magnetization law:

$$\vec{B} = f_c(\vec{H}). \quad (5.15)$$

In magnetic materials with affine characteristics $\vec{B} = \mu \vec{H} + \mu_0 \vec{M}_p$.

For the particular case of linear media ($\vec{E}_i = \vec{0}$, $\vec{P}_p = \vec{0}$, $\vec{M}_p = \vec{0}$), these relationships are linear functions described by the material coefficients: conductivity σ (S/m), permittivity ϵ (F/m), and permeability μ (H/m), which are scalar quantities in isotropic media (function of position in nonhomogeneous ones and constant otherwise). In anisotropic media, they are second-order tensors, described by symmetrical and positive definite 3×3 matrices. In affine media, the characteristics include, along with the linear term, a vector constant: the intrinsic electric field $\vec{E}_i$ or intrinsic current density $\vec{J}_i = \sigma \vec{E}_i$, the permanent polarization $\vec{P}_p$, and the permanent magnetization $\vec{M}_p$, respectively. The affine model is obtained through linearization of the nonlinear model, by using the truncated Taylor series expansion of the characteristic function around a given operating point (usually the origin), by retaining the first two terms only. Polarization $\vec{P}$ (C/m²) and magnetization $\vec{M}$ (A/m) are physical quantities describing the deviation of the electric displacement and of the magnetic flux density in substance from their values in vacuum condition: $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$, $\vec{M} = \vec{B}/\mu_0 - \vec{H}$. These quantities have a temporal component, dependent (linear or nonlinear) on the field strength, and a permanent component (constant, independent of the field strength): $\vec{P}(\vec{E}) = \vec{P}_t(\vec{E}) + \vec{P}_p$, $\vec{P}_t(\vec{0}) = \vec{0}$, $\vec{P}(\vec{0}) = \vec{P}_p$, $\vec{M}(\vec{H}) = \vec{M}_t(\vec{H}) + \vec{M}_p$, $\vec{M}_t(\vec{0}) = \vec{0}$, $\vec{M}(\vec{0}) = \vec{M}_p$.

In addition to the four general laws and the three constitutive laws, the macroscopic electromagnetism includes two *transfer laws* which describe how the energy or the substance is transferred between an EM and other physical systems:

1. The *power transfer law* (Joule) describes the energy transfer occurring between the EM field and conducting substances during the conduction process:

$$p = \vec{J} \cdot \vec{E} \quad (\text{W/m}^3), \quad (5.16)$$

where p is the power volume density transferred from field to substance.

2. The *mass transfer law* (Faraday) describes the mass transfer in electrolytes

$$\vec{\delta} = k \vec{J} \quad (\text{kg}/(\text{m}^2\text{s})), \quad (5.17)$$

where $\vec{\delta}$ is the mass density of the flux rate.

Figure 5.1: Diagram of electromagnetism relationships. Notations x' and x'' refer to distinct terms of the corresponding equation.

The diagram shown in Figure 5.1 illustrates the causal relationships, as described by the nine laws of electromagnetism. This image highlights the fundamental EM phenomena and their intertwining, thus providing a physical meaning to the laws. The dotted arrows represent phenomena occurring only in time-varying regimes, whereas the thick, solid arrows represent phenomena occurring in both dynamic and stationary regimes.

A different graphic representation of fundamental relations of electromagnetism, this time from a mathematical perspective as opposed to the physical/causal viewpoint, is the Tonti diagram, also known as “Maxwell’s House” [9], as shown in Figure 5.2 (left). The four pillars in this diagram are perfect De Rham sequences (Figure 5.2

Figure 5.2: Left: Maxwell’s house. Right: De Rham sequence = chain containing spaces of differential forms, with increasing order. The sequence is perfect: the kernel of an operator is in the domain of the previous operator.

(right)), each one being associated with an EM field primitive quantity ($\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$). The arrows representing Maxwell's relations are emphasized.

On the basis of these nine laws, two important theorems can be proved [33].

1. The *electric charge conservation theorem* highlights the strong link existent between charge and current:

$$i_{\Sigma} = -\frac{dq_{\mathcal{D}_{\Sigma}}}{dt} \quad \forall \mathcal{D}_{\Sigma}. \quad (5.18)$$

2. The *EM energy conservation theorem* has the following form in motionless media:

$$P_{\Sigma} = P_{c\mathcal{D}_{\Sigma}} + \frac{dW_{\mathcal{D}_{\Sigma}}}{dt} \quad \forall \mathcal{D}_{\Sigma}, \quad (5.19)$$

where $P_{\Sigma} = -\oint_{\Sigma} (\vec{E} \times \vec{H}) \cdot \vec{n} \, dA$ is the power transferred inward through the boundary $\Sigma = \partial\mathcal{D}_{\Sigma}$ of the considered domain, $P_{c\mathcal{D}_{\Sigma}} = \int_{\mathcal{D}_{\Sigma}} p \, dv$ is the power transferred to the conducting substance inside the domain, and, in the case of linear media, $W_{\mathcal{D}_{\Sigma}} = \int_{\mathcal{D}_{\Sigma}} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B})/2 \, dv$ is the EM energy inside the domain.

The *fundamental problem of EM field analysis* consists of the computation of the EM field, as a solution to Maxwell's equations, and of the constitutive relations for materials. In the general transient case, this problem can be formulated as follows. **Given**

- the shape and dimensions of the *computing domain* $\mathcal{D}_{\Sigma}$;
- the values of σ , ε , μ in each point of $\mathcal{D}_{\Sigma}$ or these *material constants* in each assumed linear or affine and homogenous subdomain;
- the *internal field sources* at each point of the computing domain, provided by $\vec{E}_i$, $\vec{P}_p$, and $\vec{M}_p$ vectors;
- *boundary conditions*: at each moment in time $0 \leq t < T$, at each point on the domain's boundary $\Sigma = \partial\mathcal{D}_{\Sigma}$ the tangential component, either of the electric field $\vec{E}_t$ or of the magnetic field $\vec{H}_t$;
- *initial conditions*: at each point in $\mathcal{D}_{\Sigma}$ the electric flux density $\vec{D}$ and the magnetic flux density $\vec{B}$ at the initial moment $t = 0$, with $\text{div } \vec{B}(\vec{r}, 0) = 0$.

Find the solution represented by the fields $\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$, ρ , $\vec{J}$, defined on the computing domain $\mathcal{D}$ and on the time interval $0 \leq t < T$.

Such a PDE problem is mathematically correctly formulated (“well-posed” in the Hadamard sense) if a solution exists, is unique, and is continuously dependent on the problem data (more precise, if the problem is well-conditioned). The mathematical formulation of the field problem requires a selection of the most adequate functional framework and the reformulation of the EM field problem within that framework.

The *theorem of the solution uniqueness* states that if $\sigma > 0$, $\varepsilon > 0$, $\mu > 0$, the solution of the Maxwell equations (5.1)–(5.4) together with the constitutive relations in affine media (5.13)–(5.15) is unique if the following conditions are given:

- (CD): internal sources of the field in domain $\mathcal{D}_\Sigma$:

$$\vec{E}_i(\vec{r}, t), \vec{P}_p(\vec{r}, t), \vec{M}_p(\vec{r}, t), \quad \vec{r} \in \mathcal{D}_\Sigma, \quad 0 \leq t < T, \quad (5.20)$$

- (CS): boundary conditions on Σ :

$$\vec{n} \times \vec{E}(\vec{r}, t) \times \vec{n} = \vec{E}_t, \quad \vec{r} \in S_E \subset \partial\mathcal{D}_\Sigma, \quad (5.21)$$

$$\vec{n} \times \vec{H}(\vec{r}, t) \times \vec{n} = \vec{H}_t, \quad \vec{r} \in S_H = \partial\mathcal{D}_\Sigma - S_E, \quad 0 \leq t < T, \quad (5.22)$$

- (C0): initial conditions:

$$\vec{D}(\vec{r}, 0) = \vec{D}_0(\vec{r}), \vec{B}(\vec{r}, 0) = \vec{B}_0(\vec{r}), \quad \vec{r} \in \mathcal{D}_\Sigma. \quad (5.23)$$

The proof is based on the EM energy theorem and on the null-solution lemma, which states that a linear equation has a unique solution if the associated homogenous equation (i. e., a field problem with null uniqueness conditions) has only the null solution.

Most often, the boundary conditions are null. The condition $\vec{E}_t = \vec{0}$, called perfect electric conductor, relates to a perfect conductor (superconductor with $1/\sigma = 0$) located on the boundary.

If, to facilitate the understanding, we assume that $\vec{E}_t$ is given over the entire boundary, then the result of the field analysis is a unique distribution of the EM field in the computing domain. This result is uniquely determined including the component $\vec{H}_t$ on the boundary. Thus, a correct formulation of the problem of EM field analysis guarantees a proper definition of an operator which links the two input and output vector fields $\vec{E}_t$ and $\vec{H}_t$, respectively, defined on the boundary $\Sigma = \partial\mathcal{D}_\Sigma$ and on the time interval $0 \leq t < T$. This transfer operator defines a *dynamic input-output (I/O) system*, which describes the response of the entire domain $\mathcal{D}_\Sigma$ under several excitations.

Although from the perspective of system theory the transfer function refers only to the boundary quantities, treating the studied object as a black box, in fact the transfer function depends on the internal structure and material behavior. Thus, to extract this transfer operator the fundamental EM field problem has to be solved in the entire $\mathcal{D}_\Sigma$. If the medium of this domain is linear, then this I/O system is also linear and the I/O transfer operator is a linear one, since the solved equations are linear. From the perspective of system theory, this I/O dynamical system has the Maxwell equations as state equations. Its state variables (those describing the initial conditions) are the electric and magnetic flux densities $\vec{D}$, $\vec{B}$ in the entire domain $\mathcal{D}_\Sigma$. The I/O system thus defined, which is an EM system, has input and output signals, as well as state variables, all of infinite dimensions, as they are functions defined on Σ and $\mathcal{D}_\Sigma$, respectively.

These mathematical objects have only theoretical value, since in practice, in the computer representations their finite approximations are used. For example, in the case of uniform cubic grid with n nodes along every direction, the system has 2 components $\times 6$ faces $\times n^2$ nodes $= 12n^2$ I/O signals and 3 components $\times 2$ vectors $\times n^3$ nodes $= 6n^3$ state variables. These numbers of order $O(n^2)$, $O(n^3)$ tend towards infinity when

$n \rightarrow \infty$. It can be said that this EM field system is of “infinite-inputs, infinite-outputs” (IIIO) type. It would be multiple-inputs multiple-outputs (MIMO) if the number of input and output signals would be greater than one, but finite. Such a MIMO EM system is discussed in the next section.

In the classical system theory, when they have a finite number of state variables, these kinds of systems are finite, with lumped parameters. In the EM field systems, since they have an infinite number of state variables, distributed in space, we say that these systems are with distributed parameters.

5.3 Coupled field-circuit problems and EMCE boundary conditions

In this section we consider the reduction of an EM system from an infinite number of I/O signals to one with a finite number of I/O signals. This approach is necessary when EM devices with distributed parameters (which include field effects such as eddy currents) are coupled with external circuits.

By definition, electric circuits have a finite number of components interconnected at nodes, characterized by their electric potentials. A subcircuit with m terminals is characterized at each terminal node by a pair of scalar quantities: current and potential. Depending on the excitation mode, one of these quantities is the input signal, the other one being the output. Consequently, an m -polar circuit is an I/O dynamic system of MIMO type, with $m - 1$ inputs and $m - 1$ outputs, the m -th node being the reference potential.

In order to couple circuits and EM devices, the latter need boundary conditions compatible with external circuits. Specifically, on the boundary of the EM device model, a finite number m of equipotential patches represent the *terminals* connected to the nodes of the exterior circuit. Usually, these terminals are very good conductors (copper, silver, or even gold-plated), which may be modeled as perfect conductive parts without a significant variation of electric potential over their surfaces. Moreover, the boundary conditions need to ensure that for each terminal the associated potential and current can be correctly defined.

The *multipolar electric circuit element (ECE)* is a domain $\mathcal{D}$ (Figure 5.3) with boundary conditions that ensure the compatibility with the ECEs [42, 31].

The boundary $\Sigma = \partial\mathcal{D}$ of the domain comprises m disjoint parts $S_1, S_2, \dots, S_m$, with $S = \bigcup_{k=1}^m S_k$, called *electric terminals* on which:

$$\vec{n} \cdot \text{curl } \vec{E}(P, t) = 0 \quad (\forall) P \in \Sigma, \quad (5.24)$$

$$\vec{n} \cdot \text{curl } \vec{H}(P, t) = 0 \quad (\forall) P \in \Sigma - S, \quad (5.25)$$

$$\vec{n} \times \vec{E}(P, t) = \mathbf{0}, \quad (\forall) P \in S, \quad (5.26)$$

where $\vec{n}$ is the normal unitary vector in P .

Figure 5.3: The multipolar electric circuit element (ECE).

These boundary conditions ensure: the absence of magnetic coupling with the exterior (5.24); electric coupling only through terminals (5.25); and equipotentiality of each terminal (5.26). The *electric current of a terminal k* is defined as the total current (conduction and displacement) flowing through it: $i_k(t) = \oint_{\Gamma_k} \vec{H} \cdot d\vec{r}$, where $\Gamma_k = \partial S_k$ is the boundary of the terminal surface S_k . We assume that Γ_k are oriented so that the associated normal of S_k is inwards oriented. Due to (5.25), the sum of all terminal currents is zero and the Kirchhoff current law is a consequence. The *electric voltage of the terminal k* is defined as the integral $v_k(t) = \int_{C_k} \vec{E} \cdot d\vec{r}$ along an arbitrary curve C_k included in Σ , which is a path between a point on S_k and a point on a reference terminal, say, S_n . Condition (5.24) ensures the consistent definition of the terminal voltage, its independence of the shape of C_k , and the Kirchhoff voltage law as a consequence.

A uniqueness theorem can be stated and the power transferred by any ECE through its boundary, from outside to inside, is given by

$$\begin{aligned}
 P &= - \oint_{\Sigma} (\vec{E} \times \vec{H}) \cdot d\vec{A} = \oint_{\Sigma} (-\text{grad } v \times \vec{H}) \cdot (-d\vec{A}) = \oint_{\Sigma} (v \text{ curl } \vec{H}) \cdot (-d\vec{A}) \\
 &= v_k \sum_{k=1}^m \int_{S_k} \text{curl } \vec{H} \cdot (-d\vec{A}) = v_k \sum_{k=1}^m \int_{\Gamma_k} \vec{H} \cdot d\vec{r} = \sum_{k=1}^{m-1} v_k i_k.
 \end{aligned} \tag{5.27}$$

If the terminals are excited by known potentials, then the problem of EM field analysis in a linear domain with ECE boundary conditions has a unique solution as it can be probed by using the lemma of the null solution,

Consequently, the terminals' currents are output signals univocally defined by solving the field problem. As the domain is linear, the equations are linear, and the device with ECE boundary conditions is a linear, MIMO dynamic system with $m-1$ input signals and $m-1$ outputs. The field problem is also well formulated, if only $p < m$ terminals have known voltages, one is grounded, and the remaining $m-p-1$ have known currents. This is the case of the hybrid excitation. The ECE is *voltage-excited* if $p = m-1$ and *current-excited* if $p = 0$. Under null initial conditions, due to the linearity of the field equations, and by applying the Laplace transform, the output vector $\mathbf{v}(s) = [v_1(s), v_2(s), \dots, v_{m-1}(s)]^T$ of terminal potentials of a current-excited ECE will be

linearly dependent on the input current vector $\mathbf{i}(s) = [i_1(s), i_2(s), \dots, i_{m-1}(s)]^T$:

$$\mathbf{v}(s) = \mathbf{Z}(s)\mathbf{i}(s), \quad (5.28)$$

where $\mathbf{Z}(s)$ is the operational impedances matrix. Since the field problem has a state space of infinite dimension, it is expected that the operational impedances have an infinite number of poles, although it is a finite $(m-1) \times (m-1)$ matrix. In addition, it is expected that the complex impedances matrix, resulting from replacement of the complex frequency s with $j\omega$, will be positive real, since the complex power $\underline{S}$ has the real part P positive for any excitation since

$$\underline{S} = P + jQ = \sum_{k=1}^m v_k i_k^* = \underline{\mathbf{i}}^H \underline{\mathbf{v}} = \underline{\mathbf{i}}^H \underline{\mathbf{Z}} \underline{\mathbf{i}}, \quad (5.29)$$

where $\underline{S} = -\oint_{\Sigma} (\vec{E} \times \vec{H}^*) \cdot d\vec{A}$, $P = \int_{\mathcal{D}_{\Sigma}} \sigma \vec{E} \cdot \vec{E}^* dv$, $Q = \omega \int_{\mathcal{D}_{\Sigma}} (\vec{B} \cdot \vec{H}^* - \vec{E} \cdot \vec{D}^*) dv$. Since the material constants are positive scalars, the ECE device is reciprocal (with a symmetrical impedance matrix) and passive (with absorbed active power $P > 0$, regardless of the field distribution). When terminals are voltage-excited, the complex admittance $\underline{\mathbf{Y}} = \underline{\mathbf{Z}}^{-1}$ matrix is defined similarly. In general, some terminals can be current-controlled, and the others voltage-controlled, in which case the device is characterized by a hybrid operational matrix $\mathbf{H}$.

The generalization of the ECE concept is the multipolar EM circuit element (EMCE), which has not only electric terminals, but also magnetic terminals. The power transferred by an EMCE with n electric terminals and m magnetic terminals is

$$P = -\oint_{\Sigma} (\vec{E} \times \vec{H}) \cdot d\vec{A} = \sum_{k=1}^{n-1} v_k i_k + \sum_{k=1}^{m-1} u_{m_k} \frac{d\varphi_k}{dt}, \quad (5.30)$$

where u_{m_k} is the magnetic voltage of the magnetic terminal k and φ_k is the magnetic flux of flowing through the magnetic terminal k . This expression is completely compatible with the power transferred by a multipolar electric circuit connected to the electric terminals and a multipolar magnetic circuit connected to the magnetic terminals of the EMCE.

We used EMCE formulation in applications such as modeling RF passive components or blocks as in [12], RF models of microelectromechanical switches [14, 39], or even modeling of myelinated axonal compartments [27]. Other researchers also used this formulation in magnetoquasi-static (MQS) problems for inductance extraction [38]. Similar conditions, although with a different definition for the terminal voltages, are proposed in [23].

In what follows we will assume a simple connected domain. The case of multiply connected domains have been addressed in the early paper of Timotin in [52]. The very recent comprehensive study of Hiptmair and Ostrowski [24] proves the usefulness of these boundary conditions, currently not available in popular FEM software.

5.4 Simplified models for spatial distributions and transmission lines

The next step is to reduce the dimension of the state space of the EM system. In the general case, the complete description of the EM field state requires six scalar quantities (components of $\vec{D}$ and $\vec{B}$) at every point of the physical three-dimensional space, at any moment. A dramatic reduction, to four or even two scalar quantities per point, can be achieved if a space simplification to two dimensions or even one dimension is possible.

Transmission lines (TLs) are plan-parallel structures which guide waves, encountered for instance in microelectronics, as shown in Chapter 4 of this volume [5]. At low frequencies the transmission is mainly due to conduction, whereas at high frequencies it is much more due to EM induction and displacement current. The aim of this section is to define the I/O dynamic system with distributed parameters associated to multiconductor TLs.

In the simple case of a two-conductor line, the electric field has a two-dimensional transverse distribution similar to that of an electrostatic (ES) field, being determined by the conductors' potentials, with field lines normal to the surfaces of the conductors. At high frequencies, when the penetration depth of the EM field can be neglected, the current is distributed on the conductor's surfaces and the magnetic field is also transverse and perpendicular to the electric field (Figure 5.4). Under these conditions the current $i(z, t)$ carried by the two conductors in opposite directions and the voltage $v(z, t)$ between the two conductors completely describe the EM field distribution for any cross-section of the TLs. That is why these two scalar time-dependent functions are selected as state variables.

The TL equations, derived by Heaviside in 1880, can be obtained either starting from Maxwell's equations, splitting the phenomena according to transverse and longitudinal operators as below, or by using an approach typical to circuits with distributed

Figure 5.4: Electric (left) and magnetic (right) fields for a two-conductor transmission line.

Figure 5.5: TL as an EM device (left) and its circuit model with distributed parameters (right).

parameters (Figure 5.5). The line is characterized by the per unit length (p. u. l.) parameters: R (Ω/m), L (H/m), C (F/m), and G (S/m), which may be extracted by solving four field problems in the electroconductive stationary (EC), magnetostationary (MG), ES, and EC regimes, respectively. In the simplest case of a homogeneous dielectric, only one two-dimensional ES field problem has to be solved, since $G = C\sigma/\varepsilon$ and $L = C\mu/\varepsilon$.

TLs are multipolar elements of electric circuit, their modeling being essential in many RF applications.

In the case of a multiconductor line with n conductors, $2n$ scalar time-dependent signals are required at each space point along the line. For each conductor the state variables are the local current and potential. By solving the PDE equations of the EM field, the equations which describe the propagation along an n -conductor TL are obtained [16]:

$$\begin{aligned} -\frac{\partial v_k}{\partial z} &= r_k^0 i_k + \sum_{m=1}^n \left(l_{km}^0 \frac{\partial i_m}{\partial t} + \frac{\partial}{\partial t} \int_0^t \left(\frac{dl_{km}}{d\tau} \right)_{t-\tau} i_m(\tau, t) d\tau \right), \\ -\frac{\partial i_k}{\partial z} &= \sum_{m=1}^n \left(g_{km} v_m + c_{km} \frac{\partial v_m}{\partial t} \right), \end{aligned} \quad (5.31)$$

where r_k^0 is the p. u. l. DC resistance of the conductor k , l_{km}^0 are p. u. l. external inductances (self inductances for $k = m$ and mutual inductances for $k \neq m$) of the conductors k and m when the return current is distributed on the surface of the substrate, and l_{km} are “transient p. u. l. inductances,” defined as averaged values on the cross-section of the conductor k of the vector potential obtained in zero initial conditions by a unity step current injected in conductor m .

For current and voltage null initial conditions, the Laplace transform of the general propagation equations (5.31) are

$$\begin{aligned} -\frac{d\mathbf{v}(z, s)}{dz} &= \mathbf{Z}(s)\mathbf{i}(z, s), \\ -\frac{d\mathbf{i}(z, s)}{dz} &= \mathbf{Y}(s)\mathbf{u}(z, s), \end{aligned} \quad (5.32)$$

with $0 < z < l$, where l is the line length and $\mathbf{v}$ and $\mathbf{i}$ are n -dimensional vectors of voltages and currents, respectively. They are similar to the TL equations (also named

telegrapher's equations) in the frequency domain, obtained from Kirchhoff equations for RLCG lines with distributed parameters. The only major difference is that here resistances and inductances are dependent on the complex frequency s in order to model the skin effect in the line's conductors and the eddy current losses in substrate. To extract the frequency dependence of inductances, an EM field problem in MQS regime has to be solved. The field distribution inside the conductors inserts new state variables (an infinity, if their values are considered for each z). In [28] a method named *of the two fields* is proposed, applied to extract the frequency dependent p. u. l. parameters (admittance $\mathbf{Y}$ and impedance $\mathbf{Z}$) for lossy multiconductor transmission lines, from the field solutions. Based on them a method to extract a parametric reduced-order model for this system is generated. Another approach to include in the TL model the eddy currents effect is based on the substitution of the longitudinal p. u. l. resistance with the approximate ladder model developed in Section 5.6.

If we denote

$$\mathbf{u} = \begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & -\mathbf{Z}(s) \\ -\mathbf{Y}(s) & \mathbf{0} \end{bmatrix}, \quad (5.33)$$

then (5.32) can be written as

$$\frac{d\mathbf{u}}{dz} = \mathbf{A}\mathbf{u}, \quad (5.34)$$

and its solution is

$$\mathbf{u}(z, s) = \exp(\mathbf{A}z)\mathbf{u}(0, s). \quad (5.35)$$

If the quantities of the line ends are denoted by

$$\mathbf{u}_- = \begin{bmatrix} \mathbf{v}_- \\ \mathbf{i}_- \end{bmatrix} = \begin{bmatrix} \mathbf{v}(0, s) \\ \mathbf{i}(0, s) \end{bmatrix}, \quad \mathbf{u}_+ = \begin{bmatrix} \mathbf{v}_+ \\ \mathbf{i}_+ \end{bmatrix} = \begin{bmatrix} \mathbf{v}(l, s) \\ \mathbf{i}(l, s) \end{bmatrix}, \quad (5.36)$$

then

$$\mathbf{u}_+ = \mathbf{T}\mathbf{u}_-, \quad \text{where } \mathbf{T} = \exp(\mathbf{A}l) = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix}. \quad (5.37)$$

The terminal operational admittance matrix $\mathbf{Y}$ can be computed from $\mathbf{T}$ as

$$\begin{bmatrix} \mathbf{i}_- \\ \mathbf{i}_+ \end{bmatrix} = \mathbf{Y} \begin{bmatrix} \mathbf{v}_- \\ \mathbf{v}_+ \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -\mathbf{T}_{12}^{-1}\mathbf{T}_{11} & \mathbf{T}_{12}^{-1} \\ \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11} - \mathbf{T}_{21} & -\mathbf{T}_{22}\mathbf{T}_{12}^{-1} \end{bmatrix}. \quad (5.38)$$

Here, the terminals are considered voltage-controlled and, therefore, the multiconductor line with n conductors is a MIMO-type system with $2n$ input signals (potentials of the n close and n distant terminals) and $2n$ output signals (currents through these

terminals). However, being a distributed system (with an infinite number of state variables, one-dimensionally distributed along the line $0 < z < l$), the dimension of state space is infinite. Actually, the multiconductor line meets the ECE boundary conditions (5.24)–(5.26) for an element with $2n$ terminals that are the close and distant extremities of the n conductors. The EM field within the TL is distributed in space and time, but in harmonic state, the information is concentrated in a point in the frequency domain and its one-dimensional space distribution is given by one complex number (voltage or current) related to each line terminal.

For the particular case of a line with a single conductor over the ground, which is a two-port quadrupole described by a system with distributed parameters with two input and two output signals, the global admittance has the following expression:

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{\cosh(\gamma l)}{Z_c \sinh(\gamma l)} & -\frac{1}{Z_c \sinh(\gamma l)} \\ -\frac{1}{Z_c \sinh(\gamma l)} & \frac{\cosh(\gamma l)}{Z_c \sinh(\gamma l)} \end{bmatrix} = \mathbf{Y}^T, \quad (5.39)$$

where $Z_c = \sqrt{(R + sL)/(G + sC)}$, $\gamma = \sqrt{(R + sL)(G + sC)}$ are the characteristic impedance and the complex propagation constant, respectively, depending on p. u. l. parameters R , L , C , and G and frequency. The result of the presented approach related to this particular case has an analytic expression, and thus illustrates the AAM approach.

The TL approach is essential for the MOR of on-chip interconnect lines (see also Chapter 4 of this volume [5]). Although geometric reduction is very effective, this approach is not the final step in the complexity reduction; it needs to be sustained by an additional order reduction process, in order to obtain the desired reduction to a minimal finite order instead of an infinite order [30]. This section uncovered only one step in the process of complexity reduction of EM systems, that based on simplifications of geometrical structure, in which for adequate geometrical modeling it is assumed that each component of the EM field is dependent only on certain coordinates from a proper selected coordinates system. This choice, called geometrical modeling, reduces the complexity of particular EM systems in a dramatic manner.

5.5 Regimes of the electromagnetic field and models with lumped parameters

The complexity reduction of general EM systems can be done by simplifying the physics, i. e., by disregarding some phenomena. This can be acceptable under certain conditions, specific to each particular studied case. In practice such simplifications, called a *field regime*, are based on a series of hypotheses so that the theory remains coherently and rigorously mathematically formulated. This is the main advantage of using a field regime, even if the obtained model does not perfectly reflect the reality.

The EM field regime described by Maxwell's equations (5.5)–(5.8) and (5.13)–(5.15) is known as the *general electrodynamic (ED) regime* or the *full-wave regime*.

Since the magnetic flux density $\vec{B}$ is divergence-free, this field is solenoidal and thus a *magnetic vector potential* $\vec{A}$ can be defined so that

$$\vec{B} = \nabla \times \vec{A}. \quad (5.40)$$

In order to define a unique $\vec{A}$ a gauge condition is needed, which imposes a constraint on the divergence of $\vec{A}$. The simplest condition is the Coulomb gauge: $\nabla \cdot \vec{A} = 0$. By substituting (5.40) in the EM induction law (5.7) it follows that

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = \vec{0}. \quad (5.41)$$

Consequently, a *scalar potential* V can be defined so that

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}. \quad (5.42)$$

The scalar potential V and the magnetic vector potential $\vec{A}$ are called *electrodynamic potentials* in the ED regime of the EM field. In terms of differentials, with a simplified language, the scalar potential is a 0-form $V \in W^0$ and the vector potential is a 1-form $\vec{A} \in W^1$. The perfect character of the De Rham sequence (Figure 5.2) guarantees the existence of these two potentials.

In linear media with imposed current $\vec{J}_i$, Maxwell equations conduce to second-order PDEs, expressed in potentials which are state variables [33]:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times (\nu \nabla \times \vec{A}) + \sigma \frac{\partial \vec{A}}{\partial t} + \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} + \sigma \nabla V + \varepsilon \nabla \frac{\partial V}{\partial t} = \vec{J}_i, \quad (5.43)$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \nabla \cdot (\sigma \nabla V) = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J}_i, \quad (5.44)$$

where $\nu = 1/\mu$ and the gauge $\nabla \cdot (\sigma \vec{A}) = 0$ was used. Their solution is an EM field that propagates with a finite speed.

A complexity reduction of general EM systems can be achieved by eliminating some quantities that are irrelevant in certain conditions. For example, if the EM field varies sufficiently slow so that wave propagation can be ignored, one of the two field regimes known as *quasi-static* can be used [8].

The *electroquasi-static* (EQS) regime is intended especially for the study of capacitive effects, where the phenomenon of EM induction is neglected (Figure 5.6). Formally, $\mu = 0$ and thus $\vec{B} = \vec{0}$. This choice reduces by half the dimension of the state space, since the only state quantity is $\vec{D}$. An example of usage is an RCG transmission line (with null line inductance), used to model one-dimensionally distributed resistive-capacitive combined effects.

Figure 5.6: Causal diagram of the EQS regime.

1. $\nabla \cdot \vec{D} = \rho$
3. $\nabla \times \vec{E} = 0$
4. $\Rightarrow \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$
5. $\vec{D} = \varepsilon \vec{E} + \vec{P}_p$
7. $\vec{J} = \sigma(\vec{E} + \vec{E}_i)$

1. $\nabla \cdot \vec{D} = \rho$
2. $\nabla \cdot \vec{B} = 0$
3. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\nabla \times \vec{H} = \vec{J} \Rightarrow \nabla \cdot \vec{J} = 0$
5. $\vec{D} = \varepsilon \vec{E} + \vec{P}_p$
6. $\vec{B} = \mu \vec{H} + \mu_0 \vec{M}_p$
7. $\vec{J} = \sigma(\vec{E} + \vec{E}_i)$

Figure 5.7: Causal diagram of the MQS regime.

The *MQS* regime is intended especially for the study of inductive effects, where the displacement current is neglected (Figure 5.7). Formally, $\varepsilon = 0$ and thus $\vec{D} = \vec{0}$. This choice reduces by half the dimension of the state space, since the only state quantity is $\vec{B}$. An example of usage is an RLG transmission line (with null line capacitance), used to model one-dimensionally distributed resistive-inductive combined effects.

The electrostatic regime, capacitance extraction

A very efficient method to reduce the complexity of EM systems is by modeling them with circuits having lumped elements (resistors, inductors, capacitors). In these models, the current and the electric and magnetic fields are segregated, and thus their energies are concentrated: electric energy in capacitors, magnetic energy in inductors, whereas the resistors do not concentrate EM energy. The fields interact only through the circuit.

The extraction of the corresponding lumped parameters (R, L, C) is carried out by assuming stationary field distributions, even if the EM field is variable in time and, consequently, the circuit will be simulated in a dynamic regime. Thus, the ES regime is used to extract capacities (C), the MG regime for extracting inductances (L), and the EC regime for extracting resistances (R).

1. $\nabla \cdot \vec{D} = \rho$
3. $\nabla \times \vec{E} = 0$
5. $\vec{D} = \varepsilon \vec{E} + \vec{P}_p$
7. $\vec{J} = \sigma(\vec{E} + \vec{E}_i) = \vec{0}$

Figure 5.8: Causal diagram of the ES regime.

In a static regime there is no time variation, no motion, and no energy transfer, so that power and currents are zero. The causal diagram and the first-order fundamental differential ES equations are shown in Figure 5.8. The nonrotational character of the electric field allows definition of the *electrostatic scalar potential*

$$\vec{E} = -\nabla V, \quad (5.45)$$

which is the solution of the *elliptical second-order PDE of Poisson type*, with a *div-grad operator* [33]:

$$-\nabla \cdot (\varepsilon \nabla V) = \rho_t, \quad (5.46)$$

where $\rho_t = \rho - \nabla \cdot \vec{P}_p$. In homogeneous media, without charges and permanent polarization, the equation becomes of Laplace type, $\Delta V = 0$.

On discontinuity surfaces between two bodies with different dielectric characteristics, the interface conditions are

$$\vec{n}_{12} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s \Rightarrow \varepsilon_1 \frac{dV_1}{dn} - \varepsilon_2 \frac{dV_2}{dn} = \rho_s - \vec{n}_{12} \cdot (\vec{P}_{p2} - \vec{P}_{p1}), \quad (5.47)$$

$$\vec{n}_{12} \times (\vec{E}_2 - \vec{E}_1) = \vec{0} \Rightarrow V_1 = V_2. \quad (5.48)$$

These interface conditions highlight the continuity of potential due to the continuity of the tangential component of the electric field strength and the continuity of the normal component of the electric induction when there is no permanent polarization and the interface is not charged.

The boundary conditions that ensure the potential uniqueness are [8]

$$\text{Dirichlet: } V(P) = f_D(P), \quad P \in S_D \neq \emptyset, \quad S_D \subset \Sigma = \partial \mathcal{D}, \quad (5.49)$$

$$\text{Neumann: } \frac{\partial V}{\partial n} = f_N(P), \quad P \in S_N = \Sigma - S_D. \quad (5.50)$$

The capacitive interaction between $n + 1$ conductors located in a linear dielectric (Figure 5.9 (left)) is described by Maxwell's equations for capacitances:

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \Leftrightarrow \mathbf{q} = \mathbf{C}\mathbf{V}, \quad (5.51)$$

Figure 5.9: Capacitance extraction. Left: Excitation of conductors needed in the ES problem from which column j of the *nodal capacitance matrix* is computed. Right: Equivalent circuit for $n = 3$. C_{kj} are *partial capacitances* that are computed from the nodal capacitances.

where $\mathbf{q}, \mathbf{V} \in \mathbb{R}^n$ are the vectors of charges and potentials, respectively, and $\mathbf{C} = (c_{ij})_{i,j=1,n} \in \mathbb{R}^{n \times n}$ is the matrix of conductors' capacitance, also named *matrix of nodal capacitances*. The column j of the matrix $\mathbf{C}$ is extracted from the solution of a fundamental ES problem in the dielectric, with Dirichlet boundary conditions, null on the surface of all the conductors except for conductor j for which $V = V_j$ (Figure 5.9 (left)). The capacitance $c_{kj} = q_k/V_j|_{V_l=0, l \neq j}$, where $k = 1, n$ is computed from the charge

$$q_k = \oint_{\Sigma_k} \vec{D} \cdot d\vec{A} = - \oint_{\Sigma_k} \epsilon \frac{dV}{dn} dA, \quad (5.52)$$

which is linearly dependent on V_j . The computation of the matrix $\mathbf{C}$ needs the solving of n distinct ES field problems, obtained by exciting successively only one conductor, the other conductors being grounded.

Due to passivity and reciprocity, $\mathbf{C}$ is symmetrical, positively defined, and diagonal dominant. Its inverse $\mathbf{S} = \mathbf{C}^{-1}$ is called the *matrix of potential coefficients*. The electric energy stored in this system is

$$W_e = \frac{1}{2} \mathbf{V}^T \mathbf{q} = \frac{1}{2} \mathbf{V}^T \mathbf{C} \mathbf{V} = \frac{1}{2} \mathbf{q}^T \mathbf{V} = \frac{1}{2} \mathbf{q}^T \mathbf{S} \mathbf{q} > 0. \quad (5.53)$$

Relation (5.51) can be modeled with a capacitive circuit, having the topology of a complete polygon (Figure 5.9 (right)). By applying the nodal method, it follows that the capacitances of this circuit (which have positive values), called *partial capacitances*, can be computed from the nodal capacitances as

$$C_{kj} = -c_{kj} \quad \forall k, j = 1, \dots, n, \quad k \neq j, \quad (5.54)$$

$$C_{kk} = \sum_{j=1}^n c_{kj} \quad \forall k = 1, \dots, n. \quad (5.55)$$

This procedure is widely applied in practice to extract parasitic RC parameters in integrated circuits. The ES regime results provide the $\mathbf{C}$ parameters, whereas the

EC regime is similar to ES. If ε is replaced with σ , then the conductance values are obtained.

The advantage of having a model described as an equivalent capacitive circuit comes from the fact that its *complexity reduction* can be efficiently achieved by removing negligible equivalent capacitances connected between floating nodes (e. g., capacitances smaller than a threshold of 0.1 % relative to capacitances to ground). This leads to a *sparse nodal capacitance matrix*, which does not lose its symmetry and positive definite, thus passive, characteristic.

Since the capacitance between two conductors is inversely proportional with the distance between them, it follows that around each conductor there is a spherical “window” containing coupled conductors. Outside this window, the capacitive coupling may be ignored. This observation simplifies the ES field problem that has to be solved to extract a (sparse) column of the nodal capacitances matrix. Even more efficient methods can be conceived if the pattern of the sparse nodal capacitance matrix is known, e. g., it is banded [35]. Such strategies are important and compulsory for integrated circuits, where the number of conductors is huge, over one million.

The magnetic stationary regime, inductance extraction

In a stationary regime there is no time variation and no motion, but energy transfer is allowed, so that power and currents are nonzero. The causal diagram and the first-order fundamental differential equations of the magnetic stationary (MG) regime are shown in Figure 5.10. In the particular case where there are no currents, this becomes the magnetostatic (MS) regime. The rotational character of $\vec{H}$ does not allow a natural definition of a scalar potential, this being the main difference between MG and MS. The solenoidal character of $\vec{B}$ allows however the definition of the *magnetic vector potential*

$$\vec{B} = \nabla \times \vec{A}, \quad (5.56)$$

which is the solution of the *second-order PDE* with a *curl-curl operator* [33]:

$$\nabla \times (\nu \nabla \times \vec{A}) = \vec{J}_t, \quad (5.57)$$

where $\vec{J}_t = \vec{J} + \nabla \times (\nu \vec{B}_r)$. As discussed in the general ED case, a gauge condition is required to ensure the uniqueness of the magnetic vector potential. In homogeneous

Figure 5.10: Causal diagram of the MG regime.

media, with Coulomb gauge $\nabla \cdot \vec{A} = 0$, the magnetic vector potential satisfies a Poisson vector equation:¹

$$-\Delta \vec{A} = \mu \vec{J}_t. \quad (5.58)$$

If there are no sources, i. e., $\vec{J}_t = \vec{0}$, then the equation becomes Laplace, $\Delta \vec{A} = \vec{0}$.

On discontinuity surfaces between two bodies with different magnetic linear characteristics, the interface conditions are

$$\vec{n}_{12} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \Rightarrow \vec{n}_{12} \times (\nu_2 \nabla \times \vec{A}_2 - \nu_1 \nabla \times \vec{A}_1) = \vec{J}_s, \quad (5.59)$$

$$\vec{n}_{12} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \Rightarrow \vec{n}_{12} \cdot (\nabla \times \vec{A}_2 - \nabla \times \vec{A}_1) = 0. \quad (5.60)$$

If there is no current sheet at the interface, from (5.59) it follows that $\vec{H}_{t1} = \vec{H}_{t2}$, and if $\nu_1 = \nu_2$, then $(\nabla \times \vec{A}_1)_t = (\nabla \times \vec{A}_2)_t$. From the Coulomb gauge it can be derived that $A_{n1} = A_{n2}$ and from (5.60) $\vec{A}_{t1} = \vec{A}_{t2}$. Consequently, the magnetic vector potential is continuous, $\vec{A}_1 = \vec{A}_2$, across interfaces.

An important practical problem is the quantitative characterization of the inductive interaction between n conductive wires. This interaction is described by Maxwell's equations for inductivities, which give the magnetic fluxes produced by these n conductors placed in a linear magnetic medium:

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} \Leftrightarrow \boldsymbol{\varphi} = \mathbf{L} \mathbf{i}, \quad (5.61)$$

where $\boldsymbol{\varphi}, \mathbf{i} \in \mathbb{R}^n$ are the vectors of magnetic fluxes and currents, respectively, and $\mathbf{L} = (L_{ij})_{i,j=1,n} \in \mathbb{R}^{n \times n}$ is the matrix of inductances, holding *self and mutual inductances*. The column j of the matrix $\mathbf{L}$ is extracted from the solution of a fundamental MG problem in the magnetic medium, where all the currents are zero, except for conductor j for which $i_j \neq 0$. The mutual inductance between wires k and j is $L_{kj} = \varphi_k / i_j|_{i_i=0, i \neq j}$, where $k = 1, n$ is computed from the magnetic flux

$$\varphi_k = \int_{S_{r_k}} \vec{B} \cdot d\vec{A} = \oint_{\Gamma_k} \vec{A} \cdot d\vec{r}, \quad (5.62)$$

which is linearly dependent on i_j .

The computation of the matrix $\mathbf{L}$ needs the solving of n distinct MG field problems, obtained by current exciting successively only one conductor, the other conductors being open.

¹ $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \Delta \vec{A} = -\Delta \vec{A}$, due to the Coulomb gauge.

Due to passivity and reciprocity, the matrix $\mathbf{L}$ is symmetrical, positively defined, and diagonal dominant. Its inverse is denoted by $\mathbf{K} = \mathbf{L}^{-1}$. The magnetic energy stored in this system is

$$W_m = \frac{1}{2} \mathbf{i}^T \boldsymbol{\varphi} = \frac{1}{2} \mathbf{i}^T \mathbf{L} \mathbf{i} = \frac{1}{2} \boldsymbol{\varphi}^T \mathbf{i} = \frac{1}{2} \boldsymbol{\varphi}^T \mathbf{K} \boldsymbol{\varphi} > 0. \quad (5.63)$$

In the case of wire conductors, (5.62) can be applied only for the computation of mutual inductances, since the integral is not convergent for self-inductances. The self-inductances can be computed from (5.63), where $W_m = \int_{\mathbb{R}^3} \vec{A} \cdot \vec{J} \, dv$.

Relation (5.61) can be modeled with an inductive circuit with n coupled coils.

The modeling of inductive effects in integrated circuits is becoming increasingly important as their clock speeds increase. However, on-chip wiring is a large structure, which requires without doubt complexity reduction of its models. The inductance matrix is a full one, and its size is huge, in particular if wire segmentation is applied as in partial electric equivalent circuits [47]. The complexity reduction of the equivalent circuit and the matrix sparsification is more difficult for inductances than for capacitances. Sparsification of a dense or even full matrix means their approximation with a sparse matrix, which has very few nonzero elements. One straight approach to make the inductance matrix sparse is simply to discard those mutual coupling terms of $\mathbf{L}$, which are below a certain threshold. This approach, however, does not guarantee the positive definiteness of the resulting inductance matrix and therefore the model passivity is not preserved. A series of alternative methods for robust sparsification of the inductance matrix which preserve the passivity were proposed, such as the K-method [34], the vector potential equivalent circuit method [54], and the magneto-electric equivalent circuit method [29].

Reduced circuit models with lumped parameters

The simplest model of a system with n conductors in an ED regime, in which there are EM induction phenomena, and capacitive effects are neglected, can be modeled by an n -port circuit (Figure 5.11 (left)) characterized by the equations

$$\mathbf{u} = \mathbf{L} \frac{d\mathbf{i}}{dt} + \mathbf{R}\mathbf{i}, \quad (5.64)$$

where $\mathbf{R} = \text{diag}(R_1, \dots, R_n) \in \mathbb{R}^{n \times n}$, with R_k being the resistance of the conductor k ; $\mathbf{i}, \mathbf{u} \in \mathbb{R}^n$ are the vectors of currents and voltages along the wires, respectively. These matrices are extracted from the MG and EC field solutions. The number of state variables is equal to the number of inductors, i. e., n .

Similarly, the simplest model of a system with $n + 1$ conductors in an ED regime placed in an imperfect insulator, but where the EM induction phenomena are neglected, can be modeled by a multiport circuit (Figure 5.11 (right)), characterized by the equations

$$\mathbf{i} = \mathbf{C} \frac{d\mathbf{v}}{dt} + \mathbf{G}\mathbf{v}, \quad (5.65)$$

Figure 5.11: The simplest circuit model for n conductors in MQS (left – RL circuit) and EQS (right – GC circuit).

where $\mathbf{C}$ is the capacitance matrix and $\mathbf{G}$ is the matrix of conductances for resistive losses in the dielectric ($\mathbf{G} = \mathbf{C}\sigma/\varepsilon$ for homogeneous media). These matrices are extracted from the ES and EC field solutions. The number of state variables is also equal to n , being less than the number of capacitors since they are in excess, as can be easily seen (there are loops consisting solely of capacitors).

In the two models discussed above, which can be used in MQS or EQS regimes, the lumped parameters are extracted from static or stationary regimes. The results they give may be acceptable at some frequency ranges, but they might not be accurate enough at high frequencies, e. g., when eddy currents and skin effects become relevant. As will be explained in the next section, in this case circuit models with lumped parameters extracted from MQS and EQS fields can be used. Even so, neither these models are accurate enough beyond a certain frequency, where both inductive and capacitive effects have to be considered simultaneously.

It is evident that both inductive and capacitive effects are considered if the field regime used for extraction is the general ED. However, combined inductive and capacitive effects can be more easily modeled if the RL and CG circuits above are combined in T or Π models, without solving the ED field problem (Figure 5.12). These models have the order of complexity equal to $3n$.

Figure 5.12: The simplest RLC circuit model for conductor k , where $k = 1, \dots, n$: T (left) and Π (right). To simplify the figure, partial capacitances between conductors are not shown.

Even if they have inductive, capacitive, and conductive effects, the T or Π models are not able to describe the field propagation. If this effect is important, in the case of wire-shaped conductors, the line of length l may be decomposed in p segments. If each segment has a length $l/p \ll \lambda$, where λ is the wavelength (the rule of thumb is $p = 4l/\lambda$), then the propagation along a segment can be neglected. Consequently, each segment can be modeled with a T or Π schematic. Then, by connecting all segments' models, a reduced system with $(2p + 1)n$ state variables is obtained.

The great advantage of modeling with lumped parameters is that the original field problem (EQS, MQS, or ED) with an infinite state space is reduced to an I/O dynamical system with a finite number (from n up to $(2p + 1)n$) of state variables. Moreover, the number of I/O signals is also finite ($2n$ inputs and $2n$ outputs for n conductors). In three-dimensional devices, all three directions have to be "segmented," but this is carried out with numerical approaches, e. g., as discussed in Section 5.7.

Identification of the appropriate field regime

The discussion above revealed that the complexity reduction of EM system can be achieved by choosing an appropriate field regime. For this, the *characteristic times*, which depend on the material constants ϵ , μ , σ and the characteristic length l of the device under study, are computed and compared [44].

There are three characteristic times of an EM device: (1) *charge relaxation time* $\tau_e = \epsilon/\sigma$ is the relaxation time of an electric field in a conductor; (2) *magnetic field diffusion time* $\tau_m = \mu\sigma l^2$ is the diffusion time of the magnetic field in a conductor; and (3) *propagation time* $\tau_{em} = l/c$, with $c = 1/\sqrt{\epsilon\mu}$, is the time in which a wave propagating with speed c travels a distance l .

It follows that $\tau_{em}^2 = \tau_e \tau_m$. In very good conductors $\tau_e \ll \tau_{em} \ll \tau_m$, i. e., charge relaxation time is negligible and magnetic inductive effects are preponderant. In poor conductors, $\tau_m \ll \tau_{em} \ll \tau_e$, the time for magnetic field diffusion is insignificant, and the charge relaxation is important.

If we denote by τ the *characteristic time*, defined as the duration period or time constant of the phenomenon under study, then the appropriate regime can be decided by comparing τ with τ_{em} , τ_e , τ_m . The graphical representation shown in Figure 5.13 illustrates this comparison. A point in this diagram has as abscissa the decimal logarithm of τ/τ_{em} and the ordinate the decimal logarithm of τ_m/τ_{em} .

If $\tau_e < \tau_m$ (i. e., $\tau_e < \tau_{em} < \tau_m$), then the corresponding point on the map will be in the first "quadrant" of the map ($x > 1$, $y > 1$), and if $\tau < \tau_m$, an MQS regime has to be considered (e. g., the case of very good conductors), otherwise a static/stationary magnetic regime is appropriate.

If $\tau_m < \tau_e$ (i. e., $\tau_m < \tau_{em} < \tau_e$), then the corresponding point on the map will be in the fourth "quadrant" of the map ($x > 1$, $y < 1$), and if $\tau < \tau_e$, an EQS regime has to be considered (e. g., the case of poor conductors), otherwise a static/stationary electric regime is appropriate. The lines drawn on the map are not very strict. In fact,

Figure 5.13: Map of EM field regimes, as a function of characteristic times.

if the phenomena are very slow ($\tau \gg \tau_e > \tau_m$ or $\tau \gg \tau_m > \tau_e$), then the stationary/static regimes will be definitely used for modeling; otherwise (for intermediary speeds) quasi-stationary regimes may be chosen. In the case of fast phenomena, there is a small band region on the left side of the $x = 1$ axis, where both charge relaxation and EM induction phenomena are considered, but the propagation is ignored since the displacement current is neglected. This is called the EM quasi-stationary (EMQS) regime.

When the phenomena are extremely fast ($\tau \ll \tau_{em}$, usually $\tau < 10\tau_{em}$ is assumed, i. e., the left semi-space with respect to the $x = 1$ line), then the wavelength is much smaller than the dimensions of the device, and therefore the ED regime must be used to describe the field propagation.

If $\tau > 10\tau_{em}$, then the discussion depends on the ordering between τ_e and τ_m , τ_{em} having a value in-between these two characteristic times being their geometric mean.

5.6 Equivalent infinite circuits of devices with distributed parameters and their finite approximations

The models with lumped parameters shown in Figure 5.11 and Figure 5.12 are valid not only for wire-shaped conductors, but also for massive conductors if the operating frequency is low enough. However, with the frequency increase, the penetration depth of the EM field diminishes, making the field regime to move progressively to MQS, EMQS, and ED, eventually. This phenomenon, called “skin effect,” would make the lumped parameters of Figures 5.11 and 5.12 dependent on the frequency. However, equivalent circuits with constant lumped parameters are preferred, so that their be-

havior seen from the terminals is able to catch the high-frequency phenomena. This section shows how such an equivalent circuit can be obtained.

The infinite electric circuit equivalent to the two-pole ECE

Let us consider the ECE shown in Figure 5.3 with two terminals ($m = 2$). We will assume that the domain $\mathcal{D}$ is linear and the charge relaxation time is much smaller than the magnetic field diffusion time ($\tau_e \ll \tau_m$). Consequently, the appropriate field regime is MQS and satisfies

$$\frac{1}{\sigma} \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E}(\vec{r}, t) \right) = -\frac{\partial \vec{E}(\vec{r}, t)}{\partial t}, \quad (5.66)$$

with the ECE boundary conditions (5.24), (5.25), (5.26) expressed only in $\vec{E}$:

$$\vec{n} \cdot (\nabla \times \vec{E}(\vec{r}, t)) = 0 \quad (\forall) \vec{r} \in \Sigma, \quad (5.67)$$

$$\vec{n} \cdot \vec{E}(\vec{r}, t) = 0 \quad (\forall) P \in \Sigma - S_1 - S_2, \quad (5.68)$$

$$\vec{n} \times \vec{E}(\vec{r}, t) = \vec{0} \quad (\forall) \vec{r} \in S_1 \cup S_2. \quad (5.69)$$

The input signal is a known voltage imposed between the two terminals:

$$\int_{C_{AB} \subset \Sigma} \vec{E} \cdot d\vec{r} = u(t), \quad (5.70)$$

where $A \in S_1$ and $B \in S_2$. The output signal is the current through the terminal

$$i(t) = \int_{S_2} \sigma \vec{E} \cdot d\vec{A}. \quad (5.71)$$

We will use an approach specific to functional analysis, based on the *modal analysis* of this device. Let us define a linear operator $\mathcal{A}$

$$\mathcal{A}(\vec{E}) = \frac{1}{\sigma} \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E}(\vec{r}, t) \right). \quad (5.72)$$

Thus the equation to be solved is

$$\mathcal{A}(\vec{E}) = -\frac{\partial \vec{E}(\vec{r}, t)}{\partial t}. \quad (5.73)$$

The operator $\mathcal{A}$ is a kind of avatar of the differential curl-curl operator, densely defined on a Hilbert space $\mathcal{H}$, with compact resolvent [25]. If the domain under study is bounded and sufficiently smooth, the operator $\mathcal{A}$ with $\sigma > 0$ and $\mu > 0$ is symmetrical and positively defined in the space of solenoidal functions with null-type ECE boundary conditions. Consequently, its eigenvalues equation

$$\mathcal{A}(\vec{e}_k) = \lambda_k \vec{e}_k \quad (5.74)$$

has an infinite number of solutions $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty$. These solutions are the spectrum of differential operator $\mathcal{A}$, and the corresponding eigenfunctions make an orthonormal basis of $\mathcal{H}$

$$(\vec{e}_i, \vec{e}_j) = \delta_{ij}, \quad \text{where } (\vec{e}_i, \vec{e}_j) = \int_{\mathcal{D}} \vec{e}_i \cdot \vec{e}_j \, dv. \quad (5.75)$$

Thus, the solution of (5.73) for a step excitation $u(t) = U_0 h(t)$, where $h(t)$ is the unit step, can be expressed as a generalized Fourier series of eigenfunctions:

$$\vec{E}(\vec{r}, t) = \left[\vec{E}_0(\vec{r}) + \sum_{k=1}^{\infty} c_k(t) \vec{e}_k(\vec{r}) \right] U_0. \quad (5.76)$$

This is a convergent series, with separate spatial and temporal variables. The term $\vec{E}_0(\vec{r})$ is the stationary field in $\mathcal{D}$, generated by the voltage $u = U_0 = 1$, with $\mathcal{A}(\vec{E}_0) = 0$. From (5.73), (5.74), and (5.76) it follows that

$$\lambda_k c_k(t) + \frac{dc_k}{dt} = 0 \quad \Rightarrow \quad c_k(t) = -b_k e^{-t/\tau_k} \quad \text{with } \tau_k = 1/\lambda_k. \quad (5.77)$$

Consequently, (5.76) becomes

$$\vec{E}(\vec{r}, t) = \left[\vec{E}_0(\vec{r}) - \sum_{k=1}^{\infty} b_k e^{-t/\tau_k} \vec{e}_k(\vec{r}) \right] U_0. \quad (5.78)$$

The initial null condition $\vec{E}(\vec{r}, 0) = \vec{0}$ imposes that

$$\vec{E}_0(\vec{r}) = \sum_{k=1}^{\infty} b_k \vec{e}_k(\vec{r}). \quad (5.79)$$

This means that b_k are the Fourier coefficients of the solution for the stationary regime. Finally, the ECE current is derived from (5.71), (5.78), and (5.79):

$$i(t) = \left[\sum_{k=1}^{\infty} b_k (1 - e^{-t/\tau_k}) \int_{S_2} \sigma \vec{e}_k(\vec{r}) \cdot d\vec{A} \right] U_0. \quad (5.80)$$

If we denote

$$R_k = 1 / \left(b_k \int_{S_2} \sigma \vec{e}_k(\vec{r}) \cdot d\vec{A} \right) \quad \text{and} \quad L_k = R_k / \lambda_k, \quad (5.81)$$

then (5.80) can be written as

$$i(t) = \sum_{k=1}^{\infty} \frac{U_0}{R_k} (1 - e^{-tR_k/L_k}), \quad (5.82)$$

which represents the current of a ladder circuit with an infinite number of steps consisting of RL series circuits with the parameters given by (5.81).

Due to the linearity of the analyzed system, the obtained result is valid with null initial conditions for any excitation, not only for the step voltage. The infinite-ladder circuit (Figure 5.14 (left)) is therefore equivalent to the distributed parameter element $\mathcal{D}$ with MQS field and ECE boundary conditions, regardless of its excitation.

The finite approximation of infinite equivalent circuits

The modal analysis explained above is based on the computation of the problem's eigenvalues and eigenfunctions. The spectrum of the operator $\mathcal{A}$, which in theory can be continuous, is in reality a discrete one, being a numerable infinite set. Since the series (5.78) and (5.79) are convergent, the partial sums converge to the infinite series, the higher-order terms are less and less important.

The complexity of this system can be reduced by truncating the ladder to a limited number of steps, thus ignoring the irrelevant effect of the upper steps. This generates an error in the stationary regime, which can be eliminated by adding a resistance which, paralleled with those of the inferior steps, produces the stationary regime resistance R_0 (Figure 5.14 (right)): $R_{n+1} = 1/(1/R_0 - \sum_{k=1}^n 1/R_k)$.

Figure 5.14: Infinite ECE equivalent circuit (left) and its corrected truncation (right).

Thus a sequence of finite circuits is obtained, which converges towards the exact infinite model, all having a correct stationary behavior. Although simple, this reduction method based on truncation is not optimal. The applied compensation corrects the value of the admittance in the stationary regime, but it affects the limit to infinity, which becomes $1/R_{n+1}$ instead of the correct null value which occurs in the absence of this modification. Since R_k increases to ∞ , the error goes to zero when $k \rightarrow \infty$.

For the particular case when the element of the ECE circuit is made of a homogeneous material and has a cylindrical shape of arbitrary cross-section, the eigenvalues and eigenfunctions are precisely those of the two-dimensional Laplace operator in the cross-section with null Dirichlet boundary values. The truncation errors have been estimated for two special cases of practical significance, for which the analytic solutions are known – the plate (Figure 5.15) and the circular cylinder.

Figure 5.15: Dipolar ECE.

The expressions obtained for the lumped parameters of the infinite equivalent circuit for the plate are

$$R_0 = L/(2\sigma al), \quad (5.83)$$

$$R_k = R_0 \beta_k^2 / 2, \quad \text{where } \beta_k = (2k - 1)\pi/2, \quad k = 1, 2, \dots, \quad (5.84)$$

$$L_k = \mu a L / (4l), \quad k = 1, 2, \dots \quad (5.85)$$

The inductances L_k have a constant value, while the resistances R_k increase approximately proportional with the square of k . That means a truncation error of about 1%, if only 10 steps are retained in the finite ladder approximation.

The admittance of the plate $Y(s) = \tanh(\sqrt{s\tau})/(R_0 \sqrt{s\tau})$, with $\tau = \mu\sigma a^2$, is a transcendental function versus s , with an infinite number of poles $s_k = -R_k/L_k$. In the MQS regime, the poles and zeros of the admittance (impedance) are real and negative, in a series which tends to infinity. Thus, the system is passive and stable. Figure 5.16 represents with dots the Nyquist diagram of a plate admittance $\underline{Y}(j\omega)R_0$, for $\omega\tau = 10^{-3}, 10^{-2}, \dots, 10^3$.

The result of the presented approach is an analytical solution; therefore it is an illustration of the AAM. The ladder truncation is not the best method to reduce the infinite circuit to a finite one. In [32] another procedure is proposed, able to identify

Figure 5.16: Nyquist plot of the plate admittance.

reduced-order models for various orders, which ensures an error of minimum Hankel norm. It is determined that the errors of this optimal procedure are about 10 times smaller than those of models of the same order obtained through simple truncation and compensation. Hence, the circuits from Figures 5.11 and 5.12 can model also the skin effects in conductors, if ladder circuits with four steps, three RL series, and one a simple R substitute initial RL series groups.

5.7 FEMs and their reduction

Analytical approaches are not appropriate for EM devices with complex geometries encountered in practice, in which cases numerical methods have to be used. The FEM is the most popular one, due to its ability to adapt for the solving of intricate geometries and various types of PDE that correspond to any EM field regime, or to other multi-physics fields.

The theoretical foundation of FEM is the reformulation of the problem in weak form. In this section we will show how the EM equations for various field regimes can be reformulated in weak form. Then, we will review how a corresponding discrete model can be obtained and how its order can be reduced.

The weak form of the fundamental electrostatic problem

The ES potential V defined by (5.45) satisfies the generalized Poisson equation (5.46) in a linear domain $\mathcal{D}$, with $\varepsilon > 0$, to which boundary conditions (5.49)–(5.50) are added, to ensure the potential's uniqueness. Relations (5.46), (5.49), and (5.50) represent the *strong form* of the fundamental ES problem and V is its strong solution. The weak form is obtained by projecting (5.46) on a set of independent directions of a Hilbert space $\mathcal{H}$ of “test” functions u :

$$-\int_{\mathcal{D}} \nabla \cdot (\varepsilon \nabla v) u \, dx = \int_{\mathcal{D}} \rho u \, dx, \quad (5.86)$$

where the weak solution is denoted by v and ρ_t was renamed ρ . By applying the Gauss–Ostrogradsky formula² it follows from (5.86) that

$$\int_{\mathcal{D}} \varepsilon (\nabla v) \cdot (\nabla u) \, dx = \int_{\mathcal{D}} \rho u \, dx + \int_{\partial \mathcal{D}} \varepsilon u \frac{dv}{dn} \, dA. \quad (5.87)$$

This form of equation is also called *Galerkin projection*, and the procedure for obtaining it *method of moments*, since the moments are defined by the products between the residual and test functions, that have to be null.

² $\nabla \cdot (\varepsilon \nabla v) u = \nabla \cdot (\varepsilon \nabla v) u + \varepsilon (\nabla v) \cdot (\nabla u)$, then apply $\int_{\mathcal{D}}$ and use Gauss–Ostrogradsky: $\int_{\mathcal{D}} \nabla \cdot (\varepsilon \nabla v) u \, dx = \int_{\partial \mathcal{D}} (\varepsilon \nabla v) u \cdot \vec{n} \, dA$.

By defining a bilinear functional (form) $a(\cdot, \cdot)$ and a linear functional $f(\cdot)$, relation (5.87) imposes the equality between these two functionals for any test function u having zero values on the Dirichlet boundary:

$$a(u, v) = f(u) \quad \forall u \in \mathcal{H}, \quad (5.88)$$

where

$$a(u, v) = \int_{\mathcal{D}} \varepsilon(\nabla v) \cdot (\nabla u) \, dx, \quad (5.89)$$

$$f(u) = \int_{\mathcal{D}} \rho u \, dx + \int_{S_N} \varepsilon u f_N \, dA, \quad u(S_D) = 0. \quad (5.90)$$

The test functions u satisfy a null Dirichlet condition on S_D , while the solution v satisfies a nonnull condition on S_D . Therefore, in $\mathcal{H}$ there are two subspaces: one of *test* functions $\mathcal{H}_0$ with null Dirichlet conditions, which is a linear space, and the other one of *trial* functions $\mathcal{H}_D$, in which the solution is searched for; $\mathcal{H}_D$ is an affine (and not a linear) subspace of $\mathcal{H}$ containing the functions that satisfy the nonnull Dirichlet condition f_D in (5.49). For a correct definition of the bilinear functional $a(\cdot, \cdot)$, the space $\mathcal{H}$ will be considered a Sobolev space of square Lebesgue integrable functions on $\mathcal{D}$: $\mathcal{L}^2(\mathcal{D})$, which has generalized, square integrable derivatives (gradient), isomorphic with $\mathcal{W}^0$, the domain of the grad operator. The *inner product* in $\mathcal{H}$, denoted by $(u, v)_{\mathcal{H}}$, defines the norm in that space:

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}^1 = \{u \in \mathcal{L}^2(\mathcal{D}) \mid \nabla u \in (\mathcal{L}^2(\mathcal{D}))^3\}; \\ (u, v)_{\mathcal{H}} &= \int_{\mathcal{D}} uv \, dx + \int_{\mathcal{D}} (\nabla u) \cdot (\nabla v) \, dx \quad \Rightarrow \quad \|v\|_{\mathcal{H}} = \sqrt{(v, v)_{\mathcal{H}}}. \end{aligned} \quad (5.91)$$

The weak form of the ES problem is (5.88). It addresses the two boundary conditions Dirichlet and Neumann in a very different manner. The Dirichlet condition is “strongly” imposed from the start, such that the solution will be searched for in a set of functions, which satisfy this condition. On the other hand, the Neumann boundary condition, which is explicitly included in the expression of the linear functional (5.90), is “weakly” satisfied as well as possible. For this reason we say that the Dirichlet condition (strongly accomplished) is *essential*, while the Neumann condition (weakly accomplished) is *natural*. In the particular case of null Dirichlet conditions ($f_D = 0$), the two spaces, that of possible solutions (trial function space, named *Ansatz* in German publications) and that of test functions

$$\mathcal{H}_0^1 = \{u \in \mathcal{L}^2(\mathcal{D}) \mid \nabla u \in (\mathcal{L}^2(\mathcal{D}))^3, u(S_D) = 0\} \subset \mathcal{H}^1, \quad (5.92)$$

$$\mathcal{H}_D^1 = \{u \in \mathcal{L}^2(\mathcal{D}) \mid \nabla u \in (\mathcal{L}^2(\mathcal{D}))^3, u(S_D) = f_D\} \subset \mathcal{H}^1, \quad (5.93)$$

are identical, $\mathcal{H}_D = \mathcal{H}_0$. By comparing the two forms, strong and weak, for the ES equations, we observe the following *advantages of the weak form*:

- In the strong form there are second-order differentials, while in the weak form the differentials are only of the first order, which impose fewer restrictions to functions.
- In the strong form, the boundary conditions are separately written, while in the weak form they are included in the equation, in the definition of the functionals, explicitly for the natural condition and implicitly for the essential one.
- The strong equation is verified for each point in the computing domain, while in the weak form the equation is globally verified (for each basis test function).
- In the weak form, the interface conditions on discontinuity surfaces are automatically fulfilled and they do not need to be explicitly imposed, which is the case for the strong form of the equations.
- The material parameter may have an arbitrary spatial change in the weak format, since it does not need to be derivable.

These advantages make the use of functional analysis instead of classical calculus worthwhile.

Well-posed fundamental electrostatic problems

For the functional framework given by (5.88)–(5.93), the conditions required by the Lax–Milgram theorem (discussed in Chapter 1 of [4]) are satisfied [8]. This theorem states that if the bilinear functional $a(u, v)$ defined on the Hilbert space $\mathcal{H}$ is *bounded*, i. e., a positive constant C exists so that

$$|a(u, v)| \leq C \|u\| \|v\|, \quad (5.94)$$

and *coercive*, i. e., it a real positive constant c exists so that

$$|a(u, u)| \geq c \|u\|^2 \quad (5.95)$$

for any elements u and v in $\mathcal{H}$, then equation $a(u, v) = f(u)$ has a solution in $\mathcal{H}$ for any f from $\mathcal{H}'$ (dual space of $\mathcal{H}$, defined as the space of linear functionals over $\mathcal{H}$), which is unique and bounded, i. e., $\|v\| \leq \|f\|/c$, where c is the coercivity constant.

This theorem has a crucial role, since it guarantees proper expression of the ES problem in its weak form, providing the *existence*, *uniqueness*, and *well-conditioning* with the Lipschitz constant $L = c$ of the solution. Consequently, this result concludes the mathematical modeling of ES problems.

Since the bilinear functional $a(\cdot, \cdot)$ is symmetrical and positive, the solution of the weak form minimizes the *energy* convex functional $F(v)$ given by (5.96), a statement known as the *Dirichlet principle*:

$$F(v) = \frac{1}{2} a(v, v) - f(v) \quad \Rightarrow \quad dF = \frac{1}{2} (a(dv, v) + a(v, dv)) - f(dv), \quad (5.96)$$

$$\min F(v) \Leftrightarrow dF = 0 \quad \Leftrightarrow \quad a(u, v) = f(u), \quad \forall u = dv \in \mathcal{H}. \quad (5.97)$$

This is the proof that, in the case of elliptical div-grad PDE equations (e. g., Laplace and Poisson equations), the Galerkin projective formulation (5.88) and Ritz minimizing formulation (5.96) are equivalent. For this reason, the solution of the weak form is also called the *variational approach*.

Finite element method principles

The central concept of the FEM is the use of the weak form given by (5.88) for a finite-dimensional subspace of $\mathcal{H}$, denoted as $\mathcal{H}_h$, suitable for a computing system. The goal is to find a numeric, discrete solution v_h , characterized by a finite number of degrees of freedom. The numerical solution satisfies

$$a(u_h, v_h) = f(u_h), \quad \forall u_h \in \mathcal{H}_h \subset \mathcal{H}, \quad (5.98)$$

where $a(\cdot, \cdot)$ and $f(\cdot)$ are defined by (5.89) and (5.90). Therefore, the conditions in the Lax–Milgram theorem are still fulfilled, so the approximate discrete problem (5.98) is also well-posed, having a solution that is unique and is continuously dependent on data. This is yet another advantage of the weak reframing of the field problems.

The finite-dimensional space $\mathcal{H}_h$ is generated by a finite set of basis functions also named *shape functions* or *test functions*; $\mathcal{H}_h$ consists of the set of linear combinations of these basis functions, so to identify the numerical solution in the case of the null Dirichlet boundary conditions, it is enough to find a finite number of degrees of freedom, coordinates of the solution in this space $\mathcal{H}_h$. In FEM, the basis functions are defined starting with a disjoint (nonoverlapping) partitioning of the computing domain in a mesh of cells having simple geometrical shapes (the most frequent are triangles or quadrilaterals in two dimensions and tetrahedrons or hexahedrons in three dimensions), called finite elements, with maximum size h . The basis functions are complete polynomials of degree p , i. e., they contain all terms $x^i y^j (z^k)$ with respect to the Cartesian coordinates x and y (and z in three dimensions, compared to two dimensions, where $k = 0$), with degree $i + j + k < p + 1$. In each cell, the polynomial interpolates the values of the potential in a minimum number of nodes, points located on the element boundary. This choice ensures the continuity of the potential at the transition through interfaces between elements, this being an essential FEM condition, called *inter-element compatibility*.

The floating nodes are located in the interior of the computing domain and on the Neumann boundary of the domain, while those located on the Dirichlet boundary are not floating, since they have an imposed, fixed known potential. The degrees of freedom resulting from solving a system of linear algebraic equations are those potential values in the floating nodes which ensure the best global approximation of the exact solution. The number of floating nodes N is the number of degrees of freedom of the problem, as each floating node defines a basis function which is a polynomial of degree p for every element, having a unitary value for that node and zero value for all

Figure 5.17: Basis function in linear two-dimensional FEM.

Figure 5.18: Example of a three-dimensional mesh (Comsol).

other floating nodes. For the simplest case, when $p = 1$, called *linear (first-order) elements*, the nodes are located in the vertices of the triangles/tetrahedrons. The basis function φ_k associated with the node k is null over the entire computing domain except for the elements which have that node in common. For the two-dimensional case with $p = 1$, the k basis function has a pyramidal graph with unity height for node k and the base formed by the assembly of triangles which have this node in common (Figure 5.17).

The maximum size for all edges, denoted by h , gives the mesh norm, and describes the level of mesh refining. The degree p of basis polynomials gives the finite element order. The most frequent orders are linear ($p = 1$), quadratic ($p = 2$), and cubic ($p = 3$). We say that the shape functions constitute a *nodal base* if the degrees of freedom are the values of the numerical solution in the nodes, i. e., the basis functions have the values $\varphi_k \in \mathcal{H}_h$, $\varphi_k(n_j) = \delta_{kj}$, $j = 1, \dots, N$, in triangulation nodes $n_1, n_2, \dots, n_N$. For each cell, the number of polynomial coefficients is equal to the number of nodes. For example, in two dimensions, the linear triangular finite elements have three nodes placed in vertices, and the quadratic elements have six nodes, three nodes being added in the middle of the edges. The coefficients are associated to $(1, x, y)$ in the linear two-dimensional case and to $(1, x, y, x^2, y^2, xy)$ in the quadratic two-dimensional case. In three dimensions (Figure 5.18), the linear finite elements have four nodes placed in tetrahedron vertices, the polynomial being a linear combination of $(1, x, y, z)$. The quadratic tetrahedral elements have 10 nodes, three nodes in vertices, and six in the middle of the edges, the polynomial being generated by $(1, x, y, z, x^2, y^2, z^2, yz, xz, xy)$. In practice quadrilateral (hexahedral) meshes are used, having only nodes on the boundaries of elements, internal nodes being removed, to-

Figure 5.19: Typical elements and their nodes.

gether with highest-order terms of the shape functions. They are called “serendipity elements” and have only eight (20) nodes (Figure 5.19).

In FEM a frequently used concept is that of *barycentric coordinates*, which in triangles and tetrahedrons give indication about the distances to the vertices. In the case of a triangle, there are three barycentric coordinates: λ_i , associated with vertices $i = 1, 2, 3$, which have affine variation inside the triangle. The coordinate λ_i has unitary value in the vertex i , and it is zero in the other two nodes. They identify the position of any point inside the triangle, although actually only two of them is enough, since the sum of all barycentric coordinates is unitary ($\lambda_1 + \lambda_2 + \lambda_3 = 1$). In the same manner the four barycentric coordinates of a tetrahedron are defined. The nodal shape function of first order associated with the node i has in each adjacent triangle/tetrahedron just the value $\varphi_i = \lambda_i$. With these selections, any element of the solutions’ space with null Dirichlet conditions has the following expression:

$$u_h = \sum_{j=1}^N u_j^h \varphi_j \quad \forall u_h \in \mathcal{H}_h \subset \mathcal{H}_0. \quad (5.99)$$

By replacing (5.99) in (5.98), a system of N linear algebraic equations is obtained:

$$\mathbf{A} \mathbf{u} = \mathbf{b}, \quad (5.100)$$

where

$$\mathbf{A} = [a(\varphi_i, \varphi_j)]; \quad \mathbf{b} = [f(\varphi_i)]; \quad \mathbf{u} = [u_j^h]. \quad (5.101)$$

The solution of this system is the vector of potential values in the floating nodes. The matrix $\mathbf{A}$, called the *rigidity (stiffness) matrix*, and the right-hand side term $\mathbf{b}$ have the

following elements:

$$\begin{aligned} a_{ij} &= a(\varphi_i, \varphi_j) = \int_{\mathcal{D}} \varepsilon(\nabla \varphi_i) \cdot (\nabla \varphi_j) \, dx, \\ b_{ki} &= f(\varphi_i) = \int_{\mathcal{D}} \rho \varphi_i \, dx + \int_{S_N} \varepsilon f_N \varphi_i \, dA. \end{aligned} \quad (5.102)$$

The matrix $\mathbf{A}$ is symmetrical, positive definite, and sparse due to the limited support of the basis functions. The integrals in (5.102) are estimated analytically (Holland–Bell) or, more frequently, numerically with Gaussian quadrature [58]. Since this quadrature is exact for low-degree polynomials, it gives exact values for the integrals (5.102) if appropriate quadrature nodes are used.

Considering the properties of matrix $\mathbf{A}$, the system of linear equations (5.100) can be solved with direct methods (such as Cholesky factorization type LLT, LDLT, or SVD) or iterative methods (such as conjugate gradient, without or with various preconditioning). Regardless of the solving method, the condition number of the rigidity matrix dictates the accuracy of the solution and the efficiency of computations. There are many preconditioning techniques, which avoid the ill-conditioning. In particular, multimesh preconditioning (called also multigrid) is an extremely efficient one, giving a strong acceleration of the iterative methods [10]. The essential part of the rigidity matrix $\mathbf{A}$ can be identified by truncating SVD factorization of that matrix. It is an efficient technique for the order reduction of the FEM model (see also Chapter 2 of [4]).

The linear system (5.100) is just the nodal equation of a capacitive circuit, connected according to the FEM mesh. In the EC case, the equivalent circuit contains resistances. These circuits may be reduced by star-polygon transforms which remove the internal nodes. This process corresponds to the Gaussian elimination applied to the internal nodes in (5.100). The resulting circuit has fewer nodes, but usually, the number of branches is increased, because the matrix is no longer a sparse one. To further reduce the resulted circuit, several techniques to sparsify the matrix can be applied, by clustering the nodes which have equal or almost equal potentials.

The nodes on the Dirichlet boundary have known potentials, which in the general case may be nonnull, and need to be carefully addressed in the numerical approach, i. e., to preserve the symmetry of the matrix. If we separate the nodes on the Dirichlet boundary, we obtain for floating nodes a system of linear equations with a form similar to that of the problem with null Dirichlet conditions:

$$u_h = \sum_{k \in S_D} f_{Dk} \varphi_k + \sum_{j=1}^N u_j^h \varphi_j \quad \forall \varphi_k \in \mathcal{H}_h \subset \mathcal{H}; f_{Dk} = f_D(P_k), P_k \in S_D, \quad (5.103)$$

$$a(u_h, \varphi_i) = f(\varphi_i) \quad \Rightarrow \quad a\left(\sum_{j=1}^N u_j^h \varphi_j, \varphi_i\right) = f(\varphi_i) - a\left(\sum_{k \in S_D} f_{Dk} \varphi_k, \varphi_i\right), \quad (5.104)$$

$$\mathbf{A}\mathbf{u} = \mathbf{b}, \quad (5.105)$$

$$\mathbf{A} = [a(\varphi_i, \varphi_j)], \quad \mathbf{b} = [f(\varphi_i) + f_{Di}], \quad \mathbf{u} = [u_j^h], \quad (5.106)$$

$$a_{ij} = a(\varphi_i, \varphi_j) = \int_{\mathcal{D}} \varepsilon (\nabla \varphi_i) \cdot (\nabla \varphi_j) \, dx, \quad (5.107)$$

$$b_i = f(\varphi_i) + f_{Di} = \int_{\mathcal{D}} \rho \varphi_i \, dx + \int_{S_N} \varepsilon f_N \varphi_i \, dA - \sum_{k \in S_D} a_{ik} f_{Dk}. \quad (5.108)$$

This time, the right-hand side term has three contributions: internal field sources (charge density inside the cells); boundary natural conditions (type Neumann, addressed similarly with a superficial charge density); and the boundary essential condition (type Dirichlet, addressed similarly with the floating nodes potentials, but imposed forcefully). In this manner, the numerical solution will satisfy exactly the essential Dirichlet boundary conditions, while the Neumann, natural boundary conditions are satisfied as exactly as possible. Consequently, these results conclude the step of numerical modeling.

FEM, from theory to code

If we suppose that the problem was previously detailed and well formulated, knowing the computational domain, the solved equation, values of the material constant, and the field sources, including the boundary conditions, the general FEM has the following steps of the computational modeling stage:

1. *Domain discretization* selects the shape of the finite elements and generates the mesh which covers the computational domain, determining the mesh norm h .
2. *Selection of the element type* consists of choosing the shape functions, in particular the elements order p and node placement.
3. The *weak formulation step* identifies the bilinear and linear functionals of the weak form, considering the equation to be solved and the boundary condition.
4. The *derivation of element matrices* step identifies the coordinate transformation which maps the local coordinates of a common reference element to global ones, and thus its Jacobian matrix, and selects the variant of the Gaussian quadrature method used to compute element contributions to both functionals.
5. *Assemblage of element equations* computes and localizes the contribution of each element to the system matrix and to the right-hand side term, evaluating both bilinear and linear functionals, respectively, treating with a different technique in particular the Dirichlet boundary conditions. The use of sparse matrix techniques is highly recommended since it produces superior performances in execution. In most software packages, steps 4 and 5 are encapsulated, being hidden from the user.
6. *Solution of equations* is a step in which the linear system generated previously is solved with a direct or an iterative method. As will be seen, in the case of transient

problems, the discretized equations to be solved are of ODE type, which require a numerical integration in time. A different approach is used to find the eigenvalues in the case of modal analysis, when the problem is reduced to finding the eigenvectors and eigenvalues of a matrix.

7. *Order reduction* may be realized by iterative cycling of this procedure and/or by processing simulation results or the state equations generated with the FEM.

Examples of simple FEM codes illustrating these steps can be found in [20].

In FEM programs, most often, coefficients (5.102) are not computed directly, but through a bijective transformation from triangular, local coordinates to physical, global coordinates. This transformation maps a standard *reference element* (e. g., a right angle triangle with unity legs, a unitary square, a unitary tetrahedron with orthogonal, unity legs, or a unitary cube, with *natural, local coordinates*), to elements in the physical space (called *global coordinates*), being determined by the correspondence of the nodes in these two spaces. These elements are called *isoparametric*. As the main advantage, this approach allows the encapsulation and separate development of the reference element library. When implemented on a computer, a difference is acknowledged between the basis functions, defined on the reference cell, and the shape functions, defined in the real physical space, such that each shape function is associated to a degree of freedom from the global grid.

The derivation of element matrices and the assemblage of element equations are the core steps of any FEM program. The combination of two ideas, isoparametric elements and Gaussian numerical quadrature, allows a simple and uniform treatment of the elemental assembly procedure, even for higher-order elements and elements with curved boundaries [18].

Convergence of the finite element method

The convergence of the FEM guarantees that the numerical solutions will tend toward the exact one when the discretization mesh is uniformly refined and thus the number of degrees of freedom tends to infinity.

The convergence towards the exact solution assumes that the discrete, finite-dimensional space $\mathcal{H}_h$ tends to an infinite-dimensional space, dense in $\mathcal{H}$. This convergence condition, named *completeness*, is obtained if the shape functions are continuous (ensuring the inter-element compatibility) and they have the unity partition property. Under the conditions of the Lax–Milgram theorem, the completeness guarantees the convergence [18].

A theorem which provides the rate of convergence of the numerical solution v_h in FEM assumes that the shape functions are polynomials of p degree over each element of a mesh having the norm h , satisfying the inequality [55]

$$\|v - v_h\|_{\mathcal{H}^m} < C h^{p+1-m} \|v\|_{\mathcal{H}^{p+1}} \quad (5.109)$$

if the exact solution v , continuous on the entire domain, is sufficiently smooth.

This relation known as *a priori error estimators* guarantees the global convergence of the FEM, showing that if the norm h of the discretization mesh tends to zero (which implies that the number of degrees of freedom N tends to infinity), then the numerical solution v_h tends to the exact one v . We observe that (5.109) ensures a global, “average” convergence of the numerical solution and its m derivatives towards the exact solution on the entire domain, as opposed to a local, pointwise convergence, characteristic of strong formulations of problems. If $m = 0$, it refers to potential, and if $m = 1$, it refers to the field strength.

The numerical solution is trusted if it is relatively independent of the mesh, i. e., h is sufficiently low, so that the error (5.109) has an acceptable level. The mesh independence may be checked by refining the current mesh, e. g., by halving h . If the numerical solution does not change too much, the mesh is adequate; otherwise the refining process continues. This approach is named uniform mesh (or “ h ”) refining. If new nodes are added in the middle of each edge, face, or cell, the number of degrees of freedom increases just as when passing from the first-order to the second-order FEM. Although the rigidity matrices have the same size, in the first-order refined case it has fewer nonzero elements. The additional computational effort to generate and solve the system in the case of the quadratic FEM order is rewarded by a more accurate and smoother numerical solution, since the order of convergence for the field is just the order p of the elements.

Figure 5.20 (left) illustrates the convergence of the FEM in the first- and second-order cases. The faster convergence of the second-order elements ($p = 2$), related to that of the first order ($p = 1$), is obvious. According to this figure, we observe that in practice, the numerical error cannot be made lower than a limit, which is about 10^{-8} to 10^{-10} . If the individual mesh elements start to get very small, we run into the limits of numerical precision. That is, the numbers in our model are smaller than can be accurately represented on a computer. This is an inherent problem with all computational

Figure 5.20: Left: Relative error vs. number of degrees of freedom (<https://www.comsol.com/>). Right: Relative AMR error vs. number of nodes [6].

methods, not just the FEM; computers cannot represent all real numbers accurately. Moreover, the condition number of the FEM matrix grows with mesh refinement at least linearly (or faster, depending on the problem). Therefore, it is important for the users to be aware of this rapidly growing accumulated round-off error as they refine meshes, especially with very large models. One million degrees of freedom in three dimensions is a usual limit for mesh refinement.

Adaptive mesh refinement

The nonsmooth problems have a lower convergence rate. One method to recover the depreciation of convergence caused by singularities is to use a self-adaptive algorithm to refine the discretization mesh. This technique is called *adaptive mesh refinement* (AMR) or the *adaptive FEM* [6]. This iterative algorithm begins from a coarse mesh and solves the problem on a series of meshes, adding successively new nodes, only when and where necessary. Consequently, the number of elements and the number N of degrees of freedom are progressively increased, but not so fast as in the uniform refinement. At each iteration, certain elements, specifically those for which the solution has an unacceptable error, are refined through segmentation. For example, in two dimensions, a triangle can be divided in two, three, or four subtriangles, depending on how supplemental nodes (zero or one) are appended in the middle of each edge.

Figure 5.20 (right) shows the mode in which the relative error decreases with respect to the number of nodes N (equal to the number of degrees of freedom), in the case of a uniform refinement of the mesh and in the case of an adaptive refinement. In the first case the convergence rate is approximately 0.7, two times lower than $q = 1.4 \approx 4/3 < 2$ (which is the ideal value in the absence of the singularity). These results were obtained for the example shown in Figure 5.21, where the discretization mesh is generated by AMR. It can be seen that the mesh is selectively refined, especially near the vertex that generates the singularity.

Figure 5.21: Multimesh FEM with adaptive refinement [6].

Figure 5.22: FEM convergence h for $p = 1$ (blue), h for $p = 2$ (red), and hp (green) [22].

Finite element with hp -refinement

As previously observed, another method for error reduction of numerical solutions is to increase the degree of the polynomials, which define the shape functions. The refinement by successively increasing of the degree p provides very good results for the case of extremely smooth solutions (which are analytical functions, indefinitely differentiable functions, harmonic functions, solutions to the Laplace equation, etc.), for which the error diminishes exponentially with the increase in the degree p . However, it is useless in the case of nonsmooth problems. Combination of h - and p -adaptive refinements regains the accelerated convergence for problems with singularities. Its justification is based on the observation that in zones where the solution is smooth, it is more efficient to use high values for the degree p , while in zones adjacent to the singularities, the spatial refinement through reduction of the h parameter is more efficient [41]. The hp algorithms are super-convergent, ensuring an exponential decrease of the error, which cannot be achieved with h or p strategies alone. As can be seen in Figure 5.22, the L-shape problem is solved through h -refinement at a convergence rate of $2/3$; through p -refinement at a rate of $5/4$, and through hp -refinement with an exponential convergence, for which real numbers C , c , and δ exist so that

$$\|v - v_h\| < C e^{-cN^\delta}. \quad (5.110)$$

Through hp -refinement field relative errors of the order of 10^{-6} are obtained, i. e., numerical solutions with six correct significant digits, with fewer than 10^4 degrees of freedom. Since uniform h -refinement requires a much larger number of degrees of freedom to achieve a similar accuracy, the hp -refinement may be considered as being the most efficient method for complexity reduction of numerical FEM solutions.

AMR and hp -refinement are techniques which allow the reduction of the number of degrees of freedom, for a given accuracy. Therefore, they are improvements of the numerical model, which provides *on-the-fly* MOR, the latter being the most efficient known technique of this category.

The weak form of curl-curl equations

Static and stationary magnetic regimes (MS, EC) are similar to the ES regime for non-charged bodies, and their scalar potential satisfies the same generalized Poisson div-grad equation, which can be solved as discussed above. The magnetostationary (MG) regime is an exception, which needs a magnetic vector potential, since in this case a native scalar potential does not exist. Essentially, a PDE of curl-curl type (5.57) is solved to determine the magnetic field, whereas for all other static and stationary regimes, a PDE equation of div-grad type (5.46) is solved.

The weak form of MG regime equations with mixed boundary conditions is obtained by projecting the residual on the space of test functions, which are now vectors. This set is actually the Sobolev space of the vector square integrable functions, which have a curl of integrable square:

$$\mathcal{H}_0(\text{curl}, \mathcal{D}) = \{\vec{u} \in (\mathcal{L}^2(\mathcal{D}))^3 \mid \nabla \times \vec{u} \in (\mathcal{L}^2(\mathcal{D}))^3, \vec{n} \times \vec{u} = \vec{0} \text{ on } S_B\}. \quad (5.111)$$

In this functional framework, the variational formulation of the problem searches for the solution $\vec{v}$, such that

$$a(\vec{u}, \vec{v}) = f(\vec{u}) \quad \forall \vec{u} \in \mathcal{H}_0(\text{curl}, \mathcal{D}), \quad (5.112)$$

where

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{D}} \nu (\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) \, dx, \quad (5.113)$$

$$f(\vec{u}) = \int_{\mathcal{D}} \vec{J} \cdot \vec{u} \, dx + \int_{S_H} \vec{J}_s \cdot \vec{u} \, dA. \quad (5.114)$$

We will proceed as in the scalar case, where it was found that the solution v with null essential boundary conditions was sought in the linear space $\mathcal{H}_0$, as a function with such essential boundary conditions. If the boundary condition was nonnull, the solution v was sought in the affine space resulting from translating $\mathcal{H}_0$, with elements which meet that nonnull condition. This time, according to (5.114) the essential boundary condition $\vec{A}_t = (\vec{n} \times \vec{A}) \times \vec{n}$ is that on $S_B \subset \partial\mathcal{D}$, while on $S_H = \partial\mathcal{D} - S_B$ natural boundary conditions $\vec{H}_t = (\vec{n} \times \nu \nabla \times \vec{A}) \times \vec{n} = \vec{J}_s$ are imposed, and the vector solution must be sought in a curl-conform subspace, in which the vector fields conserve their tangential components when crossing discontinuity interfaces.

However, regrettably in this instance of three-dimensional MG, the functional $a(\vec{u}, \vec{v})$ is not coercive, and the Lax–Milgram theorem cannot be directly applied. Consequently, the nongauged magnetic potential vector, solution of equation (5.112), is not unique. An exception is the two-dimensional plan-parallel case, with two magnetic field components in the problem's plane and a sole component for current, and vector potential (orthogonal to that plane), which is automatically gauged, having a null divergence. Moreover, the sole component of vector $\vec{A}$ satisfies a curl-curl-type

equation which devolves in one of type div-grad. To surmount this standoff, there are several solutions:

- **Impose gauge.** For the magnetic potential vector, this is most simply achieved with the Coulomb condition ($\nabla \cdot \vec{A} = 0$) and by adding the boundary condition $A_n = 0$ on S_B .
- **Do regularization.** Another approach, called regularization, consists in adding a coercive term to the bilinear functional, meaning that a *mass matrix* is added to the stiffness matrix, weighted with a parameter k :

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{D}} \nu (\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) \, dx + k \int_{\mathcal{D}} \vec{v} \cdot \vec{u} \, dx. \quad (5.115)$$

Due to this term, the coercivity condition is met again, and the problem is well formulated, no matter how small is the parameter k . Then, the solution is moved towards the limit for $k \rightarrow 0$.

- **Use Lagrange multipliers.** In the case of the weak form of curl-curl type, another possible approach consists of extending the Lax–Milgram theorem such that it can be applied to variational problems with Lagrange multipliers, which do not have minimum points, but have critical points of saddle type. Such an extension is achieved by the Brezzi theorem, which highlights the conditions to have a well-posed weak formulation of the equation for vector potential [55].
- **Use reduced scalar potential.** The idea behind this method is to extract a field with a curl from the solution, which is precisely the given current density, such that what is left is a nonrotating field, solution of an MS problem. Thus, the problem accepts a scalar magnetic potential, called *reduced*. After its numerical determination by solving a div-grad equation, this field is added to the field extracted initially, resulting in the numerical solution of the MG problem. Other expressions with different potentials of the magnetic field in MG and MQS regimes are proposed in [37].
- **No gauging.** This is a very interesting approach in which the potential vector $\vec{A}$ is left without gauge and its indeterminate equation is solved numerically with an iterative method of Krylov type, which has native minimization properties. In addition to surprisingly reaching a unique magnetic field, the iterations are faster convergent than in the gauging case [26].

Analysis of the MG field with edge elements

When computing resonance frequencies for cavities, false resonant modes, called *spurious modes*, emerge when nodal finite elements are used. This phenomenon, called *spectral pollution*, disappears when the vector solution is sought in a curl-conform space. The reason is that in $\mathcal{H}_1(\mathcal{D})$ the kernel of the curl-curl operator is not properly represented [6]. The response is to use shape functions appropriate for the vector problem, called *edge elements* [8]. In three dimensions a vector field is represented by the

three coupled scalar fields, components of the vector field:

$$\vec{T} = T_x \vec{e}_x + T_y \vec{e}_y + T_z \vec{e}_z, \quad (5.116)$$

where $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are the unit vectors of the Cartesian system of coordinates. In principle, each component T_x , T_y , and T_z could be represented by combinations of nodal shape functions, but as mentioned before, this approach should be avoided. It does not guarantee the conservation of the normal component of the field flux density (or equivalently, the tangential component of the magnetic vector) either when passing from an element to another, or on the discontinuity surfaces. An alternative is to seek numerical solutions resulting from the expression

$$\vec{T}(x, y, z) = \sum_{i=1}^E t_i \vec{w}_i(x, y, z), \quad (5.117)$$

for which the degree of freedom t_i is the integral of the vector field $\vec{T}$ along the edge l_i from the assessed triangulation with E floating edges. Therefore, *vector shape functions* $\vec{w}_i$ are defined by

$$\int_{l_i} \vec{w}_i(x, y, z) ds = \delta_{ij}. \quad (5.118)$$

Consequently, the vector shape function $\vec{w}_i$ is tangentially oriented along the edge l_i and is normally oriented along the other edges of element K which includes l_i . For this reason, such basis functions are called *edge elements*. Furthermore, in these circumstances the vector shape function $\vec{w}_i$ has in three dimensions the tangential component null on any face which does not include the edge i . The encountered restrictions cause two triangles in two dimensions with a common edge to have on the interface through this edge the same tangential component of the shape functions, but a different normal component. In three dimensions the vector shape functions conserve the tangential component where passing from a tetrahedron to another by way of their common face. As a result, the tangential component of these “edge” vector shape functions is continuous, in all computing domains, for which reason they are also called *vector shape functions with continuous tangent*. This curl-conformance makes them appropriate for representing the electric or magnetic field strength, but also for the magnetic potential vector (in which case the conservation of the normal component of magnetic induction is automatically ensured, where passing through any surface, hence the solenoidal character of this quantity).

The vector functions in barycentric coordinates are

$$\vec{w}_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i, \quad (5.119)$$

valid both in two dimensions, where $i, j = 1, 2, 3$, and in three dimensions, where $i, j = 1, 2, 3, 4$, and where ij is the edge connecting the nodes i and j . These functions,

called Whitney elements [55], are appropriate to be selected as shape functions for the magnetic potential vector, since they have a constant tangential component of $1/l_{ij}$ on the ij edge and null on the other edges, divergence zero on the entire K element, and constant curl for that element: $\nabla \times \vec{w}_{ij} = 2\nabla\lambda_j \times \nabla\lambda_i$. The degrees of freedom associated with these functions are the integral values on characteristic edges [8].

FEM of MG field

The numerical solution $\vec{v}_h$ of the MG problem, described by the vector $\mathbf{t} = [t_i]$, is found by solving the following system of linear equations:

$$\mathbf{A}\mathbf{t} = \mathbf{b}, \quad (5.120)$$

where

$$\mathbf{A} = [a(\vec{w}_i, \vec{w}_j)]; \quad \mathbf{b} = [f(\vec{w}_i)]; \quad \mathbf{t} = [t_i^h], \quad (5.121)$$

with

$$\begin{aligned} a_{ij} &= \int_{\mathcal{D}} \nu (\nabla \times \vec{w}_i) \cdot (\nabla \times \vec{w}_j) \, d\mathbf{x}, \\ b_i &= \int_{\mathcal{D}} \vec{J} \cdot \vec{w}_i \, d\mathbf{x} + \int_{S_H} \vec{J}_s \cdot \vec{w}_i \, dA, \end{aligned} \quad (5.122)$$

where $\nu = 1/\mu$, $\vec{J}$ is the current density, and $\vec{J}_s$ is related to $\vec{H}_t$ on the boundary.

Again, the central part of the FEM algorithm consists of assembling the matrix $\mathbf{A}$ of the linear system and of the vector $\mathbf{b}$ of the right-hand terms. To do this, elements of triangulation are sequentially traversed and the contribution (5.122) of each element K is added to the rigidity matrix and to the right-hand side term. Then, this linear system is solved with direct or iterative methods and the degrees of freedom t_j are determined, one for each floating edge. The numerical solution is the linear combination of the basis functions (of the edge), with these degrees of freedom as coefficients

$$\vec{v}_h = \vec{A}(x, y, z) = \sum_{j=1}^E t_j \vec{w}_j(x, y, z) = \mathbf{t}^T \vec{\mathbf{w}}. \quad (5.123)$$

The linear system (5.120) is similar to the mesh equations of the system of partial inductances, connected according the FEM mesh. Its robust reduction is discussed in [17].

FEM of the MQS field in the time domain

The established functional framework finds its application in the MQS regime as well, in which there are two approaches: in the time domain and in the frequency domain.

The weak form of the MQS equation in the time domain with interface conditions and boundary conditions identical with those in the MG regime and with the initial condition for the transient potential vector $\vec{A}$ is obtained by the Galerkin projection

$$a(\vec{u}, \vec{v}) = f(\vec{u}) \quad \forall \vec{u} \in \mathcal{H}_0(\text{curl}, \mathcal{D}), \quad (5.124)$$

where

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{D}} \nu (\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) \, dx, \quad (5.125)$$

$$f(\vec{u}) = \int_{\mathcal{D}} \left(\vec{J}_t - \sigma \left(\frac{\partial \vec{v}}{\partial t} + \nabla V \right) \right) \cdot \vec{u} \, dx + \int_{S_H} \vec{J}_s \cdot \vec{u} \, dA. \quad (5.126)$$

Actually, the difference in MQS is that, to the current density, the field source in MG, is added the induced current density. In additions, the electroconduction (EC) equation $\nabla \cdot (\sigma \nabla V) = 0$ needs to be solved for the scalar potential V , with Dirichlet boundary conditions (the potential value $V = f_D$) or Neumann boundary conditions (the value of the normal derivative of the potential $dV/dn = f_N$, given by the normal component of the current density J_n of the injected in the surface). In the weak form, the scalar V solution is sought in $\mathcal{H}_D^1$ given by (5.93) such that

$$b(u, v) = g(u), \quad \forall u \in \mathcal{H}_0^1, \quad (5.127)$$

where $\mathcal{H}_0^1$ is given by (5.92) and

$$b(u, v) = \int_{\mathcal{D}} \sigma (\nabla v) \cdot (\nabla u) \, dx, \quad (5.128)$$

$$g(u) = \int_{S_N} \sigma u f_N \, dA. \quad (5.129)$$

The dependency between the scalar potential V and the input signals (boundary conditions – boundary potential and the currents injected in the boundary) is instantaneous, without any delay, since V is the solution of a scalar-elliptical problem without time variable. Using a discretization with edge elements, the vector potential is expressed as $\vec{A} = \mathbf{t}^T \vec{\mathbf{w}}$, where $\mathbf{t}$ is the vector of degrees of freedom and $\vec{\mathbf{w}}$ is the vector of shape (edge) functions. A system of first-order ODEs is obtained by performing the Galerkin projection on this finite-dimensional space of test functions:

$$\mathbf{E} \frac{d\mathbf{t}}{dt} = \mathbf{S} \mathbf{t} + \mathbf{f}, \quad (5.130)$$

where

$$\mathbf{S} = [s_{ij}] = \int_{\mathcal{D}} \nu (\nabla \times \vec{w}_i) \cdot (\nabla \times \vec{w}_j) \, dx, \quad (5.131)$$

$$\mathbf{f} = [f_i] = \int_{\mathcal{D}} (\vec{J}_t - \sigma \nabla V) \cdot \vec{w}_i \, dx + \int_{S_H} \vec{J}_s \cdot \vec{w}_i \, dA, \quad (5.132)$$

$$\mathbf{E} = [e_{ij}] = \int_{\mathcal{D}} \sigma (\nabla \vec{w}_i) \cdot (\nabla \vec{w}_j) \, dx. \quad (5.133)$$

Thus, if ECE type boundary conditions are used, an I/O linear time-invariant dynamic system is defined, which has as state vector the degrees of freedom of the edge elements (circulations of the vector potential $\vec{A}$ along the floating edges of the discretization mesh). Its state matrix is actually the rigidity matrix $\mathbf{S}$, $\mathbf{E}$ (representing damping) is the descriptor matrix, and the input signals are the injected currents or the applied potentials. Due to reciprocity and passivity, matrices $\mathbf{S}$ and $\mathbf{E}$ are symmetrical and positive definite, and the real and negative eigenvalues λ_k of the matrix $\mathbf{E}^{-1}\mathbf{S}$ give the system's time constants $\tau_k = -1/\lambda_k$. The system (5.130) is similar to the system of state equations in descriptor form of the RL circuits, which have the topology described by the FEM mesh. The reduction of such circuits is treated in [19].

Now it becomes apparent that the application of the *hp*-refinement, which ensures a minimal number of degrees of freedom for an acceptable accuracy, is an *on-the-fly* MOR approach. However, there is a supplementary restriction: in order to ensure a good accuracy, the cell size h must be smaller than the field penetration depth ($h < \delta = \sqrt{2/(\omega\mu\sigma)}$). By applying *a posteriori* MOR classic methods (e. g., Krylov, balanced truncation, modal analysis of data, or circuit reduction) to the system (5.130), a model with a more reduced order is obtained eventually. A *posteriori* MOR is a feature encountered in all important FEM software packages (see Chapter 13 of this volume [5]), such as ANSYS or COMSOL [46].

FEM of the ED field in the time domain and MOR

In the case of the ED regime, the magnetic vector potential satisfies the hyperbolic PDE:

$$\nabla \times (\nu \nabla \times \vec{A}) + \sigma \frac{\partial \vec{A}}{\partial t} + \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} + \sigma \nabla V + \varepsilon \nabla \left(\frac{\partial V}{\partial t} \right) = \vec{J}_i, \quad (5.134)$$

with boundary conditions (5.21) and (5.22) and initial conditions for $\vec{A}$, $\partial \vec{A}/\partial t$, and V to have a unique solution if $\nabla \cdot \vec{A} = 0$. The weak form of (5.134) is obtained by Galerkin projection:

$$a(\vec{u}, \vec{v}) = f(\vec{u}) \quad \forall \vec{u} \in \mathcal{H}_0(\text{curl}, \mathcal{D}), \quad (5.135)$$

where

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{D}} \nu (\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) \, dx, \quad (5.136)$$

$$f(\vec{u}) = \int_{\mathcal{D}} \left(\vec{J}_t - \sigma \left(\frac{\partial \vec{v}}{\partial t} + \nabla V \right) - \varepsilon \left(\frac{\partial^2 V}{\partial t^2} + \nabla \left(\frac{\partial V}{\partial t} \right) \right) \right) \cdot \vec{u} \, dx + \int_{S_H} \vec{J}_s \cdot \vec{u} \, dA, \quad (5.137)$$

to which the weak form of the EC equation for the scalar potential V is added, as in the MQS case.

By using a discretization with edge elements, the potential vector is $\vec{A} = \mathbf{t}^T \vec{\mathbf{w}}$, where $\mathbf{t}$ is the vector of degrees of freedom and $\vec{\mathbf{w}}$ is the vector of shape-edge functions. The weak form becomes the system of second-order ODEs

$$\mathbf{M} \frac{d^2 \mathbf{t}}{dt^2} + \mathbf{E} \frac{d\mathbf{t}}{dt} - \mathbf{S} \mathbf{t} = \mathbf{f}, \quad (5.138)$$

where $\mathbf{S}$ is given by (5.131), $\mathbf{E}$ is given by (5.133), and

$$\mathbf{f} = [f_i] = \int_{\mathcal{D}} \left(\vec{J}_t - \sigma \nabla V - \varepsilon \frac{\partial \nabla V}{\partial t} \right) \cdot \vec{w}_i \, dx + \int_{S_H} \vec{J}_s \cdot \vec{w}_i \, dA, \quad (5.139)$$

$$\mathbf{M} = [m_{ij}] = \int_{\mathcal{D}} \varepsilon (\nabla \vec{w}_i) \cdot (\nabla \vec{w}_j) \, dx. \quad (5.140)$$

This time, the system is characterized by three matrices – of mass $\mathbf{M}$, of damping $\mathbf{E}$, and of stiffness $\mathbf{S}$. The elements of these matrices are the inner products of shape functions (or their derivatives) weighted with several material constants. For instance, in $\mathbf{M}$ (characteristic to the inertial phenomena in mechanical systems), the dielectric permittivity describes the capacitive effects; in $\mathbf{E}$ (characteristic to the damping phenomena), the conductivity describes the conduction effect; and in $\mathbf{S}$ the magnetic permeability describes the inductive effects, similar to elastic stiffness/rigidity.

The system (5.138) is similar to the equations of the RLC circuits which have the topology described by the FEM mesh. The reduction of such circuits is treated extensively in the literature [40, 48, 49]. Several methods for order reduction of second-order ODE systems of (5.138) type are also proposed in the literature [2]. The similarity to mechanical systems (see also Chapter 2 of this volume [5]), also suggested by the names of the three matrices, allows the use of reduction methods from structural analysis in electrodynamics, including the static condensation of matrices $\mathbf{S}$ and $\mathbf{M}$, called Guyan reduction. This reduction consists of the selection of a reduced number of master degrees of freedom and the removal of the other state variables from the stationary regime equation, in which the field sources associated with the omitted variables are ignored. Retaining for the dynamic regime as well only the most important, master variables, the state equation reduces correspondingly its dimensions. It is expected that the first eigenvalues will not be changed substantially and, therefore, its dynamic behavior will not be fundamentally affected. The *hp*-refinement ensures *on-the-fly* MOR. However, this time, for a good accuracy, the cells dimension needs to be much smaller than the wavelength of the field $h \ll \lambda$. The numerical integration

over time for this system can be made also by discretizing the time derivative with finite differences. The Courant–Friedrichs–Lewy (CFL) condition compels the time step to be smaller than the time necessary for the EM wave to travel through one cell. It ensures the numerical stability of the computation, limiting the size of the time step used in the numerical integration over time ($\Delta t < h/c$). It is observed that, as for the explicit scheme, the factorization of the mass matrix $\mathbf{M}$ becomes necessary. An approach which removes this effort consists of approximating the matrix $\mathbf{M}$ with a diagonal one, an operation called mass lumping, which has sufficient precision only for rectangular cells (squares, hexahedrons), but not for simplex cells (triangles, tetrahedrons). For the case of triangles/tetrahedrons it is preferable to use the implicit Newmark scheme:

$$\mathbf{M} \frac{\mathbf{t}_{k+1} - 2\mathbf{t}_k + \mathbf{t}_{k-1}}{h^2} = -\mathbf{E} \frac{\mathbf{t}_k - \mathbf{t}_{k-1}}{h} + \mathbf{S}(\theta \mathbf{t}_{k+1} + (1 - 2\theta)\mathbf{t}_k + \theta \mathbf{t}_{k-1}) + \mathbf{f}(\theta \mathbf{t}_{k+1} + (1 - 2\theta)\mathbf{t}_k + \theta \mathbf{t}_{k-1}). \quad (5.141)$$

This scheme is stable for any time step if $\theta \geq 1/4$. However, the solution loses accuracy if the time step exceeds the limit imposed by the CFL condition [6]. If $\theta = 0$, the scheme becomes that of explicit integration over time with finite differences.

Both simulation and order reduction are more difficult in the ED regime than in the MQS and EQS regimes, since now the system eigenvalues are complex. For this case there is a large collection of methods for a posteriori MOR of the system generated by FEM [11, 56, 50, 57, 45, 36, 53, 7, 1, 51]. They are based on truncated balanced models (see Chapter 2 of [3]), moment matching (see Chapter 3 of [3]), or data-driven and interpolation (see Chapter 6 of [3]). The research in the area of MOR of EM devices functioning with ED or MQS fields continues, since finding the best technique to solve this problem is still an open problem. The passivity enforcement of the reduced model is discussed in Chapter 5 of [3].

FEM of the ED field in the frequency domain and MOR

In the time-harmonic ED regime, the complex (operational) magnetic potential satisfies a complex equation of Helmholtz type, obtained after applying the Laplace transform to (5.134) with zero initial conditions:

$$\begin{aligned} \nabla \times (\nu \nabla \times \vec{A}(s)) + (\sigma + \varepsilon s)(s\vec{A}(s) + \nabla V(s)) &= \vec{J}_i(s), \\ \nabla \cdot (\sigma \nabla V(s)) &= sp. \end{aligned} \quad (5.142)$$

The weak forms of these equation are obtained by Galerkin projection:

$$a(\vec{u}, \vec{v}) = f(\vec{u}) \quad \forall \vec{u} \in \mathcal{H}_0(\text{curl}, \mathcal{D}), \quad (5.143)$$

$$b(u, v) = g(u) \quad \forall u \in \mathcal{H}_0^1(\mathcal{D}), \quad (5.144)$$

where

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{D}} \nu (\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) \, dx + s \int_{\mathcal{D}} (\sigma + \varepsilon s) \vec{v} \cdot \vec{u} \, dx, \quad (5.145)$$

$$f(\vec{u}) = \int_{\mathcal{D}} (\vec{J}_t - (\sigma + \varepsilon s) \nabla V) \cdot \vec{u} \, dx + \int_{S_H} \vec{J}_s \cdot \vec{u} \, dA, \quad (5.146)$$

$b(u, v)$ is given by (5.128), and

$$g(u) = \int_{\mathcal{D}} \sigma u \, dx + \int_{S_N} \sigma u f_N \, dA. \quad (5.147)$$

Here the vector potential $\vec{A}$ is labeled as $\vec{v}$ and the scalar potential V is labeled as v . In the case of the ECE, with null initial conditions, S_B means $A_t = 0$ on the entire boundary, whereas S_D means that the scalar potential is given on the terminals. In this case

$$f(\vec{u}) = - \int_{\mathcal{D}} (\sigma + \varepsilon s) (\nabla V) \cdot \vec{u} \, dx, \quad (5.148)$$

$$g(u) = \int_{S_N} \sigma u f_N \, dA = 0. \quad (5.149)$$

We need to note that, as opposed to the case of elliptical equations, for which the Galerkin projection formulation is equivalent to the minimizing Ritz formulation, in the case of hyperbolic equations this equivalence disappears. Now the solution in the variational formulation corresponds to a critical point which, however, is not anymore a minimum of the energy functional, since the energy functional is not a convex one, having a saddle shape in the critical point, of the solution.

As for the MQS case, the numerical ED solution is sought with nodal elements for the scalar electric potential and with edge elements for the magnetic vector potential, by solving the following systems of linear complex equations:

$$a(\vec{u}_h, \vec{v}_h) = f(\vec{u}_h) \quad \forall \vec{u}_h \in \mathcal{H}_{0h}(\text{curl}, \mathcal{D}), \quad (5.150)$$

$$b(u_h, v_h) = g(u_h) \quad \forall u_h \in \mathcal{H}_{0h}^1(\mathcal{D}), \quad (5.151)$$

where f and g are linear complex functionals, which generate the right-hand side term of the system of equations, while a and b are bilinear functionals (a is complex and b is real), which generate after the discretization the system matrix. It is the complex transform of (5.138):

$$(s^2 \mathbf{M} + s \mathbf{E} - \mathbf{S}) \mathbf{t}(s) = \mathbf{f}(s) \quad \Rightarrow \quad \mathbf{t}(s), \quad (5.152)$$

$$\vec{A}(x, y, z, s) = \mathbf{t}^T(s) \vec{\mathbf{w}}(x, y, z). \quad (5.153)$$

The total number of degrees of freedom is equal to the number of floating edges, but before solving a linear system with size equal to the number of floating nodes, to find the scalar potential. These complex systems may be solved with direct or iterative methods. Unfortunately, this time, the multimesh approach is not applicable because

it is not convergent, but the methods of Krylov type are efficient, especially with pre-conditioning.

The solution of system (5.152) depends linearly on the right-hand side term and, therefore, in the case of the ECE, with null initial conditions, in the absence of internal sources, the solution vector and, implicitly, the output signals depend linearly on the excitations of the terminals. The linearity allows the correct definition of the admittance $\mathbf{Y}$ and, implicitly, computation of the scattering, complex operational matrix $\mathbf{S}$. Again, in this instance the numerical solution to the problem for a series of frequencies allows the determination of the frequency characteristics of the EM systems $\underline{Y}(j\omega)$. On this base, applying MOR techniques based on data, such as vector fitting (see Chapter 8 of [3]) very efficiently reduced models of these systems can be extracted [13].

FEM of the ED field in the frequency domain with ECE boundary conditions

The previous formulation used the pair $(\vec{A}, V)$ in every point of the computational domain. Alternatively, a formulation in $(\vec{E}, V)$ can be used, as described in [21], where V is defined also inside the domain, or a formulation in $(\vec{E}, V)$, where the electric field $\vec{E}$ is defined strictly inside the domain and the electric scalar potential V is described solely on $\partial\mathcal{D}$. The following weak equation for $\vec{E}$ is obtained [15]:

$$\int_{\mathcal{D}} [(\nu \nabla \times \vec{E}) \cdot (\nabla \times \vec{E}') + j\omega(\sigma + j\omega\epsilon)\vec{E} \cdot \vec{E}'] dx = j\omega \sum_{k \in \mathcal{I}_c} V'_k I_k, \quad (5.154)$$

where $\mathcal{I}_c$ is the set of indices of current excited terminals. The equation for the electric potential on the boundary is obtained by projecting the normal component of the total current density $\underline{J}_n^{\text{tot}} \equiv (\nabla \times \underline{\mathbf{H}}) \cdot \mathbf{n}$ onto a set of scalar test functions $\underline{V}'$:

$$\oint_{\partial\mathcal{D}} (\nabla \times \vec{H}) \cdot \vec{n} V' ds = \oint_{\partial\mathcal{D}} J_n V' ds = \sum_{k=1}^m \int_{S_k} J_n V' ds = \sum_{k \in \mathcal{I}_c} V'_k I_k,$$

where $\vec{E} \in \mathcal{H}_E$, $V \in \mathcal{H}_V$, $\vec{E}' \in \mathcal{H}_{E,0}$, $V' \in \mathcal{H}_{V,0}$, and

$$\mathcal{H}_E = \left\{ \mathbf{u} \in \mathcal{H}(\text{curl}, \Omega) \mid \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = -\nabla_2 \underline{V}' \text{ on } \partial\Omega, \underline{V}' \in \mathcal{H}_V, \right.$$

$$\left. \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{0} \text{ on } \bigcup_{k=1}^m S_k \right\},$$

$$\mathcal{H}_{E,0} = \left\{ \mathbf{u} \in \mathcal{H}(\text{curl}, \Omega) \mid \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = -\nabla_2 \underline{V}' \text{ on } \partial\Omega, \underline{V}' \in \mathcal{H}_{V,0}, \right.$$

$$\left. \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{0} \text{ on } \bigcup_{k=1}^m S_k \right\},$$

$$\mathcal{H}_V = \{u \in \mathcal{H}(\text{grad}, \partial\Omega) \mid u = \underline{V}_k \text{ on } S_k, k \in \mathcal{I}_v,$$

$$u = \text{constant(unknown)} \text{ on } S_k, k \in \mathcal{I}_c\},$$

$$\mathcal{H}_{V,0} = \{u \in \mathcal{H}(\text{grad}, \partial\Omega) \mid u = 0 \text{ on } S_k, k \in \mathcal{I}_v, \\ u = \text{constant(unknown)} \text{ on } S_k, k \in \mathcal{I}_c\}.$$

The model complexity is reduced since the degrees of freedom are scalar quantities associated to edges strictly inside the domain and nodes on the boundary.

5.8 Conclusions

The optimal design of EM field devices requires the solving of Maxwell PDEs. Consequently, the state of such a device is completely described by local and instantaneous vector quantities, i. e., by a system with an infinite number of degrees of freedom depending on both space and time. Classical MOR approaches can be applied only to finite-dimensional systems, and that is why complexity reduction of Maxwell-based models is needed. Complexity reduction is obtained by discretization of field quantities and conduces to a model described by a finite number of degrees of freedom, expressed as a system of ODEs. The accuracy and order of the reduced EM model depend not only on the chosen *a posteriori* MOR method, but also on the *a priori* and *on-the-fly* complexity reduction approaches. A posteriori reduction, i. e., the reduction after the discretization, can be carried out with one or several methods described in [3, 4], but the complexity reduction can be carried out only with methods specific to electromagnetism, which were described in this chapter: ECE boundary conditions, an appropriate EM field regime, geometrical simplification, and discretization methods and meshes adapted to the problem. This chapter described FEM discretization, which is the most popular, but other approaches are also used in practice, such as the boundary element method and the finite integration technique/finite difference method, which generate systems of ODEs that can be reduced with the same methods as the ones used for the FEM.

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