Preface

Nonlinearity arises when one wishes to describe the complicated structure of a problem in a compact form, reducing the number of unknown parameters via conservative quantities, multiscaled simplifications, dualities, and so on. Then semilinear elliptic equations arise often when the reduced system is stationary. Since this model is independent of time, one may feel a sort of mildness to its solution, which, however, is far from my impression got from mathematical experience. Needless to say, an elliptic equation describes a transient process. Its nonlinearity inside is spatial, but it casts a time conductor to the outside. It is a silent driver, or writer of the unstable stationary state in the field variable.

This book is a revised version of my old book published in 1994 [332]. As it was, I have not developed any general theory. On the contrary, this book is restricted to several special phenomena hidden in an event, which has an unexpected vertical hierarchy, gradually spreading to the horizon. In this new version, Part I is devoted to the fundamental theory, composed of four chapters. Chapter 1 is devoted to the calculus of variations to describe the existence and nonexistence of the solution for the power nonlinearity. It deals with Keller–Segel model as an introduction, Pohozaev identity [280], Lagrangian multiplier, critical exponent, and mountain pass lemma. The proof of Brezis–Kato’s theorem is included for completeness. Chapter 2 is concerned with symmetry and its breaking. It deals with an ignition model as an introduction, then the structure of its stationary state is revealed by several analytical methods. Here we added further results on the blowup of the nonstationary solution. The ODE approach is then justified by the symmetry result of Gidas, Ni, and Nirenberg theorem. The maximum principle is then applied to ensure the log-concavity [126] of the solution to the heat equation. Finally, symmetry breaking is approached by calculus of variations. Chapter 3 is a description of curvature equations in accordance with the surface theory [256]. Classical singular limit analysis is performed using complex variables. Then, Chapter 4 is concerned with classical and nonclassical results of the isoperimetric inequality, based on the method of rearrangement. Potential theory for spherically harmonic functions, capacity estimate of blowup set, extremal case of the Trudinger–Moser inequality are biproducts of these methods and results, and an analytic proof of Hamilton’s result on the normalized Ricci flow [181, 182] is provided. Part II is concerned with the applications oriented towards extremal structures of several specific models. First, Chapter 5 is concerned with the supplementary topics related to Part I, that is, Nehari principle, applications of moving planes, method of scaling, new comparison theorem associated with a differential inequality, Trudinger–Moser inequality in the original form, and topological arguments on the level set of variational functionals. Then Chapters 6 and 7 are devoted to the Gel’fand equation on an annulus and Hardy-BMO duality, respectively.

The original version was composed of five chapters. The first two chapters were fundamental, concerning calculus of variations and maximum principles. All of them

are included in this new edition in Part I. The following two chapters are mostly concerned with the Gel'fand equation in two space dimensions, that is, classification of singular limits of the classical solution in accordance with the Green function and a uniqueness theorem. These topics are also kept in this edition in Part I, including later developments in Part II. Concerning the former, a paradigm shift is done from complex analysis to real analysis. Benefits are observed in the localization, refinement, extension, and the reverse theory – construction of the solution in accordance with the topological and variational methods. Some of the details are described in Part II. Progress on the uniqueness in Part I is seen for multiply-connected domains. Basically, a change of the study is derived with the movement from the Gel'fand equation to the Boltzmann–Poisson equation. Then the Hamiltonian control is reformulated as a recursive hierarchy. This static theory is presented in Part II of this book. A recent study, however, reveals a kinetic theory in the context of Liouville theory, the details of which will be presented in another monograph. Finally, Chapter 5 of the original version, concerned with the Euler–Lagrange equation of the Trudinger–Moser functional, is put into a section in Chapter 5. The associated Euler equation is $-\Delta u = \lambda u e^{u^2}$ in two-space dimensions, where a quantized blowup mechanism was observed recently [105].

The original sections, Nehari Principle and Symmetric Criticality, are thus moved to the last chapter of Part I and the last section in Chapter 2, respectively. Symmetric Criticality, Gel'fand Equation on Annulus (2D and Higher Dimensional Cases), and Bifurcation from Symmetry are moved to Part II. The paragraphs on sinh-Poisson equation (quaternion, harmonic map), Neumann problem (of Boltzmann–Poisson equation), convex domains (for classification of singular limits), area functional (in singular perturbation), capacity estimates of the singular set (including parabolic case), Trudinger–Moser inequality, and normalized Ricci flow are added. New topics include an application of the moving plane method, new comparison theorem associated with differential inequalities, topological methods in the flavor of the Morse theory, regularity of the weak solution, blowup analysis inducing recursive hierarchy, Hardy spaces and BMO, bubble towers of harmonic map, and general form of Wente's inequality. This book thus comes back to the system of chemotaxis, suggesting a quantized blowup mechanism and its recursive control of the Hamiltonian at the end.

I have left some contents oriented for specialists, but most of the material is for beginners, graduate students, and general researchers interested in nonlinear mathematical analysis. As in the original version, each chapter is provided with introduction, description of the problem, methods, and conclusion. Thus Sections 7.1.1 and 7.1.2 may be used as an introduction to Chapter 1, dealing with calculus of variations. I have tried to make the description self-contained except for some cases. They are several fundamental results on the rearrangement (Propositions 4.18–4.22), those on the geometric measure theory, De Giorgi's isoperimetric inequality (4.34) and Fleming–Rishel's coarea formula (4.35), the asymptotics of a zero point of ODE equation for the proof of Carleson–Chang' theorem, Lemma 5.64, quantization of collapsed masses

in the sinh-Poisson equation emerged with the use of quaternion; the last part of Lemma 5.71, and several topological results used in §5.6.

I conclude this preface with gratitude to Nikos Kavallaris for suggesting me to write this monograph. Thanks are also due to Yoshihisa Miyanishi for providing me with guidelines to the conformal geometry used in Sections 3.1.1, 3.1.2, and Futoshi Takahashi for permission of including a part of the previous article [342] in §7.4.

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