Preface

“Has not every ... variational problem a solution, provided ... if need be that the notion of a solution shall be suitably extended?” [386, p.470]

DAVID HILBERT (1862-1943)

“This sentence¹ had an immense influence on analysis: it led ultimately to ‘weak’ or generalised solutions, to my generalised curves and to Schwartz distributions, ... , and thence to chattering controls, to mixed strategies in Game Theory, all things that we find essential today.” [809, p.241]

LAURENCE CHISHOLM YOUNG (1905-2000)

Let us begin with a piece of history. In his 20th problem, David Hilbert started a fascinating development going through the whole 20th century, namely an effort to generalize more and more the notion of solutions to various applied problems that mathematics met. This includes in particular the calculus of variations, ordinary and partial differential equations, problems from optimization theory and game theory. The generalization is always based on a natural (i.e. continuous²) extension of the original problems; such continuous extension is often addressed as *relaxation*.³ The first success was a generalization of the classical solution to differential equations to the weak solution⁴, which admits less smoothness than necessary to evaluate the original differential equation in the usual sense. This is basically related with the theory of distributions invented later by L. Schwartz [723]. The weak formulation of boundary and/or initial value problems for differential equations is nowadays accepted so generally that the original classical solution is often reckoned as less natural. Later it was found that in some nonlinear problems one must handle beside the loss of

¹ It refers to a sentence by D. Hilbert with a similar meaning as that one cited above.

² The continuity always refers to some topologies and, in fact, may mean also lower semicontinuity, cf. Remark 2.38.

³ Actually, the word “relaxation” is used also in quite very many different occasions with completely different meanings, ranging from time-evolving processes in various rheological materials or in nuclear physics or chemistry aiming towards some “stress-free states” or “rest states” or, in a certain parallel of decreasing some stress whatever “stress” may mean, to relaxation in physiotherapy or recreology or psychology (e.g. before visiting dentist or so). Even in mathematics itself, the word relaxation is also used as iterative techniques for solving systems of equations or as specific handling of constraints in mathematical programming, or for “relaxation oscillations” generated by nonlinear oscillators.

⁴ First appearance of this philosophy is probably in the work by J. Leray [488], where essentially the spaces later named as Sobolev had been introduced. The huge development took place since fifties since the works by S.L. Sobolev, J.-L. Lions, and many other.

smoothness also two other phenomena: oscillations and concentrations of the solutions. The former phenomenon was for the first time treated in the pioneering works by L.C. Young [805–807], followed also by E.J. McShane [526, 527]. The latter one was investigated typically in connection with the famous Plateau’s minimal surface problem by a lot of authors⁵.

The renaissance of Young’s idea was in connection with generalization of solutions to some optimal control problems⁶ and much later to some nonlinear partial differential equations and nonconvex variational problems arising in continuum mechanics⁷. Just recently, DiPerna and Majda in their pioneering work [266] made the first attempt to couple with concentration and oscillations effects simultaneously.

The common feature of this ever going generalization is to solve more and more general problems and to ensure existence of their solutions (in a reasonable sense) in larger classes than the original ones where the existence can actually fail. Typical property that makes it possible is compactness. The enlarged sets, where the solutions are sought, represent thus certain compactifications of the original sets where the problems are “classically” formulated, and the finer the compactifications, the “more generalized” the solutions we thus obtain. This can, in principle, yield eventually very fine and abstract compactifications which are not easy to imagine, nor to use for more detailed investigations.

A natural restriction of a generality is to require the existence of some “auxiliary” algebraic structure which could be also used for a more detailed analysis. It appears useful to require the investigated compactification to be a convex subset of some linear topological space; then we will speak about a *convex compactification*.⁸

Typical usage of the convex structure is for optimality conditions. The development of the research of optimality conditions has been also encouraged by David Hilbert [386], namely in his 23th problem in connection with the calculus of varia-

⁵ See Giusti [356] for detailed references.

⁶ The first contributions appeared from sixties by Gamkrelidze [343, 344], Ghouila-Houri [352], McShane [528], Medhin [531], Rishel [644] and Warga [786–790]. Recently many other authors dealt with these so-called relaxed controls, e.g. Ahmed [5, 6, 8], Balder [50–53], Berliocchi and Lasry [114], Carlson [170], Chrysosoverghi [209, 210], Goh and Teo [358, 758], Halanay [373], Fattorini [294–299], Papa-georgiou [588, 589, 591], Rosenblueth [651–653], Schwarzkopf [724], etc. Cf. also McShane [529] for a survey. Another relaxation approach has been invented by J.E. Rubio, see Remark 4.49.

⁷ This was initialized by Tartar [747], followed by a lot of other authors, especially by Ball [61], Ball and James [63, 64], Chipot and Kinderlehrer [204], Dacorogna [241, 242], DiPerna [265], Evans [287], Kinderlehrer and Pedregal [424, 426], Murat [564], Schonbek [721], etc.

⁸ In fact, my first attempts in late 80tieth were to deal with problems formulated on Banach space (as is indeed a typical case) and then rather to extent as much as possible (in an as much continuous way as possible) the linear-space algebraic structure on the compactification, see [656]. Later, another attempt of such a sort has been done by J. Perán [607]. Such constructions occur however very cumbersome and I thus developed the theory of convex compactifications not relying on the linear structure of original spaces, if any.

tions. This led to the weak formulation of the Euler-Lagrange equation, and later also to an appropriate optimality condition for problems involving oscillation phenomena, namely the so-called Euler-Weierstrass condition for variational problems⁹ and the Pontryagin maximum principle for optimal control problems.¹⁰ The essential advantage of the convex-compactification method is that the convex geometry is used to derive the optimality conditions by a quite conventional differential calculus in various problems, not only in optimal control.¹¹

During the whole 20th century, we can also observe a parallel, intensive development of the supporting branches of mathematics, in particular of general topology, abstract functional analysis, and later also nonlinear analysis and optimization theory. The purpose of this book is to reflect these achievements and give a fairly abstract-analysis viewpoint to the concrete problems mentioned above. Also it can be said that the presented viewpoint represents a properly nonlinear approach because it forgets (to more or less extent) the original linear structure (if any) and imposes a new one for the relaxed problem. I believe this reflects genuinely the fact that original nonlinear problems themselves violate (to more or less extent) the linear structure of the original spaces.

Let us now go briefly through the content of the book.

After Chapter 1, which only summarizes very briefly and mostly without proofs some more or less standard needed mathematical background, the general theory of convex compactification is introduced in Chapter 2. This represents an abstract framework of our relaxation method. Then, in Chapter 3, this general convex-compactification theory is applied to get (σ - or also locally) compact convex envelopes of Lebesgue spaces, which represents a basic tool for relaxation of concrete problems appearing in variational calculus and optimization of systems described by differential equations. In particular cases, such envelopes can be sequentially locally compact convex subsets, imitating thus basic attributes of finite-dimensional (Euclidean) spaces and facilitating usage of conventional analytical methods.

9 For one-dimensional relaxed problems it was first formulated by Young [806] and McShane [527], and generalized for special two-dimensional cases in [807].

10 First it was formulated only for ordinary controls for system governed by ordinary differential equations by Russian school around L.S. Pontryagin, involving also V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko in [125, 127, 344, 616, 617], following some earlier ideas of D.W. Bushaw [158] and M. Hestenes [382]. Origin of these ideas can be found, however, already in works by C. Carathéodory [168], F.A. Valentine [775] and K. Weierstraß [796]; cf. also Pesch et al. [608–611] for historical surveys. The extension of maximum principle for relaxed controls is due to Gamkrelidze [345] and Rishel [644] and, in a more general form by Avakov and Magaril-Il'yaev [39, 40], Fattorini [294, 298], Halanay [373], Kaşkosz [420], Schwarzkopf [724], Warga [791], and many others.

11 This generalizes R.V. Gamkrelidze's proof which used chattering controls in contrast to the original derivation of maximum principles which exploited quite technical, so-called needle (sometimes called also spike) variations invented by V.G. Boltyanskii and E.J. McShane, cf. [610] for a historical reminiscence. Cf. [509] for a certain simplification of such arguments.

Having the tools prepared, we will be able to treat concrete problems. In Chapters 4–6 they will have typically the abstract structure

$$(P) \quad \begin{cases} \text{Minimize} & J(u, y) \\ \text{subject to} & \Pi(u, y) = 0, \quad B(u, y) \leq 0, \\ & u \in U, \quad y \in Y, \end{cases}$$

where $J : U \times Y \rightarrow \mathbb{R}$ is a cost function, Y a Banach space of “states”, U is a set of admissible “controls”, and $\Pi : Y \times U \rightarrow \Lambda$ and $B : U \times Y \rightarrow \Lambda_1$ are mappings forming respectively the “state equation” and “state constraints”, Λ and Λ_1 are Banach spaces, the latter one being ordered. The extended (relaxed) problems will then have the structure:

$$(RP) \quad \begin{cases} \text{Minimize} & \bar{J}(z, y) \\ \text{subject to} & \bar{\Pi}(z, y) = 0, \quad \bar{B}(z, y) \leq 0, \quad z \in K, \\ & z \in Z, \quad y \in Y, \end{cases}$$

where $\bar{J} : Z \times Y \rightarrow \mathbb{R}$, K is a convex set in a locally convex linear topological space Z , and $\bar{\Pi} : Z \times Y \rightarrow \Lambda$ and $\bar{B} : Z \times Y \rightarrow \Lambda_1$ are continuous mappings. The original set U is to be considered as densely embedded into K and $\bar{J}$, $\bar{\Pi}$, and $\bar{B}$ as extensions of J , Π , and B , respectively.

The following questions will be pursued both on an abstract level and in particular cases:¹²

- Relation between (P) and (RP); in particular a so-called correctness of the relaxation scheme.
- Relations between various relaxation schemes (RP) for a given (P).
- Existence and stability of solutions to (RP); well-posedness of (RP).
- First-order optimality conditions for (RP); Pontryagin's or Weierstrass' maximum principles.
- Impacts of results for (RP) to the original problem (P).
- Approximation theory for the relaxed problem (RP).
- Numerical implementation of approximate relaxed problems.

Going from simpler tasks to more complicated ones, we begin in Chapter 4 with the optimal control problems, which certainly represents the simplest variant of (P), at least if the state constraints have a reasonable structure. A typical example is an optimal control problem for a nonlinear dynamical system ($t \in (0, T)$ represents a time variable):

¹² As the emphasis is put to relaxation method itself, a lot of other aspects will remain untouched; this includes higher-order optimality conditions, problems yielding a nonsmooth relaxed problems, sensitivity analysis, etc.

$$(P_1) \quad \left\{ \begin{array}{ll} \text{Minimize} & \int_0^T \varphi(t, y, u) dt \quad (\text{cost functional}) \\ \text{subject to} & \begin{array}{ll} \frac{dy}{dt} = f(t, y, u) \text{ for a.a. } t \in (0, T), & (\text{state equation}) \\ y(0) = y_0, & (\text{initial condition}) \\ u(t) \in S(t) \text{ for a.a. } t \in (0, T), & (\text{control constraints}) \\ \psi(t, y(t)) \leq 0 \text{ for all } t \in [0, T], & (\text{state constraints}) \\ y \in W^{1,q}(0, T; \mathbb{R}^n), \quad u \in L^p(0, T; \mathbb{R}^m), \end{array} \end{array} \right.$$

where $\varphi : (0, T) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $f : (0, T) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\psi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^k$, $y_0 \in \mathbb{R}^n$, and $S(t) \subset \mathbb{R}^m$ are subjected to certain data qualification, $W^{1,q}$ and L^p denote respectively the Sobolev and the Lebesgue spaces. Quite equally, the state equation can (and will) be a nonlinear partial differential equation, say of an elliptic or a parabolic type, or a nonlinear integral equation. In every case, the resulted relaxed problem (RP) has the property that the extended state equation $\tilde{I}(z, y) = 0$ admits, for any $z \in K$, precisely one solution $y = \tilde{\pi}(u)$ and the mapping $\tilde{\pi} : K \rightarrow Y$, called an (extended) state operator, is continuous and even smooth.

Chapter 5 is devoted to scalar-variational-calculus problems of the type

$$(P_2) \quad \left\{ \begin{array}{ll} \text{Minimize} & \int_{\Omega} \varphi(x, y(x), \nabla y(x)) dx \\ \text{subject to} & y \in W^{1,p}(\Omega), \end{array} \right.$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and the “energy density” $\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies certain data qualification but $\varphi(x, r, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is allowed to be nonconvex. Then the relaxed variational problem has the form (RP) with $\tilde{I}$ linear, K convex σ -compact but noncompact, and $\tilde{B} = 0$.

The vectorial-variational-calculus problems of the type

$$(P_3) \quad \left\{ \begin{array}{ll} \text{Minimize} & \int_{\Omega} \varphi(x, y(x), \nabla y(x)) dx \\ \text{subject to} & y \in W^{1,p}(\Omega; \mathbb{R}^m), \end{array} \right.$$

with $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ are handled in Chapter 6; the adjective “vectorial” refers to that y is $\mathbb{R}^m$ -valued, $m > 1$. Although at the first sight (P_3) has the same form as (P_2) , (P_3) is much more difficult than (P_2) when simultaneously $n \geq 2$ and $m \geq 2$ and there are several essential differences between Chapters 5 and 6. One of them is that either K is nonconvex or $\tilde{I}$ is nonlinear and also $\tilde{B} \neq 0$ in general. Contrary to (P_2) whose understanding is fairly complete, there are still many open essential questions as far as the relaxation of (P_3) concerns. This is basically connected with our poor understanding of quasiconvexity and related questions.

In Chapters 4–6, two fundamental concepts (i.e. compactness and convexity) have been used rather separately – the former one ensured existence and stability of solutions while the latter one enabled us to make a more detailed analysis, e.g. to pose optimality conditions.

However, there are applications with much more intimate connections between convexity and compactness. We have in mind the noncooperative game theory, or more generally the underlying Schauder-type fixed-point theory, where typically compactness and convexity are required simultaneously to ensure mere existence of solutions. This will be the topic of Chapter 7, though it represents rather a sample of the wide area of potential applications. Let me conjecture here that every abstract problem where compactness and convexity plays a certain role can reasonably be interpreted as a relaxed problem to a certain “original” problem. Another usage of compactness together with convexity is for evolutionary relaxed problems where chosen convexity serves for definition of some time derivative. On top of it, the mentioned Schauder-type fixed point technique can be used here for existence of time-periodic solutions. This is exposed in Chapter 8.¹³

It should be emphasized that the relaxation method has, beside its purely mathematical aspects, also an essential interpretation aspect. For example, the fact $\inf(P) > \min(RP)$ must be in some situations prevented while in other situations it is welcome. The former case is typical for variational problems where this fact means that optimal relaxed solution cannot be attained by a minimizing sequence for the original problem. This is related with a necessity to hold the variational constraint $u = \nabla y$ exactly.¹⁴ The latter case appears typically in state-constrained optimal control problems where only the state equation is to be held exactly while the state constraints may be held only approximately, with a certain tolerance. Then the gap $\inf(P) > \min(RP)$ means that relaxed controls can achieve a lower “cost” than the original ones, which is naturally welcome.¹⁵

The reader is asked for tolerating occasionally a bit unusual notation, created as a compromise by unifying the standard notation from various fairly diverse fields.¹⁶ Sometimes, the very standard notation appearing in fairly different occasions was kept hopefully without any confusion.¹⁷ Bibliographical notes are mostly mentioned as footnotes; anyhow, because of the wideness of the subject, only basic references are provided either from the historical purposes or just as a source of other references for a more detailed study.

¹³ The evolution on convex compactifications and in particular periodic-solution existence theory (i.e. the whole Chapter 8) has expanded only the 2nd edition in 2019.

¹⁴ Cf. also Remark 5.38.

¹⁵ This aspect can be reflected by introducing suitable “tolerances” in optimization problems; cf. [658] for a systematic pursuit of the “tolerance approach”.

¹⁶ A typical dilemma was, e.g., that “ u ” normally stands for the control variable in optimal control theory while in the variational calculus it denotes usually the state variable, “ p ” denotes the polynomial growth in L^p -spaces while in optimal control it stands for the adjoint state, etc.

¹⁷ E.g., δ stands, beside a small positive real number, also for the Dirac distribution as well as for the indicator function.

Finally, I would like to mention a plenty of colleagues with whom I had a lot of fruitful discussions, among them especially Professors K. Bhattacharya, N.D. Botkin, B. Dacorogna, H.O. Fattorini, J. Haslinger, J. Malý, S. Müller, I. Netuka, J.V. Outrata, W.H. Schmidt, J. Souček, and V. Šverák. Within writing this monograph, I also benefited from the courses I hold during the academical years 1993/94 and 1995/96 at Charles University in Prague for under- and graduate students. Moreover, M. Kružík and M. Mátlová contributed, beside careful reading and commenting the whole manuscript, by computer implementation of proposed algorithms and by calculation of the examples. It is my duty and pleasure to express my deep thanks to all of them. Last but not least, special gratitude is to Professor Karl-Heinz Hoffmann, Professor Jindřich Nečas and Dr. Jiří Jarušek, who influenced very essentially both my intellectual live and professional career and thus, directly or not, the theme of this book.

I also warmly acknowledge the hospitality of Institut für Angewandte Mathematik und Statistik (TU München), where a great deal of the book has been accomplished. Besides, shorter stays at ENS (Lyon), IMA (Minneapolis), EPF (Lausanne), and Ernst-Moritz-Arndt-Universität (Greifswald) were inspiring.

Praha / München, January 1996 ¹⁸

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¹⁸ Actually, in 2020, few updates and completions were added also into this 1996-preface to make it more relevant regarding this 2nd edition.