# Preface

At the end of 2017, after my first monograph was published by the Juliusz Schauder Center for Nonlinear Studies, the director of the center, Wojciech Kryszewski, encouraged me to write a “comprehensive book for formal analysis,” the first such book for this special area of mathematics. I accepted the challenge.

The earliest power series is believed to be the binomial series invented by young Isaac Newton in 1669. Formal power series attracted mathematicians as early as 1871, when Ernst Schröder introduced them by investigating iterated functions. During this period (1850), Victor Puiseux introduced the so-called Puiseux series, which are generalization of power series that allow for negative and fractional exponents of the indeterminate. The space of formal power series, $\mathbb{X}$, endowed with linear operation, Cauchy product, and composition, was systematically introduced by Ivan Niven in 1969. This volume presents $\mathbb{X}$ as a mathematical system.

A regular power series has a strong relationship with its coefficients for the variable inside the convergence region if it has a positive radius of convergence. A formal power series is totally determined by its coefficients, if it is denoted by a *form* of power series, although its variable does not take any value.

As sequences, formal power series could be useful for the computer sciences and today’s digital world. We know that $\ell^p$ is a popular space of sequences with linear operation and dot product. The sets $\mathbb{X}$ and $\mathbb{L}$ are the spaces of formal power series and formal Laurent series, respectively, and they are the spaces of sequences with linear operation and dot product, plus Cauchy product and composition. The formal power series space $\mathbb{X}$ has more operations and applications than the space $\ell^p$.

Mathematicians distinguish between *formal* and *classical* analysis. *Formal analysis* has two characters: formality and analyticity. Formal theory usually uses algebraic approaches to tackle mathematical objects without using the limit or convergence; formal analysis keeps this tradition and, in addition, it does not reject certain kinds of limit or convergence. This volume lays forth the *principle of formal analysis* that a formal power series $f$ over $\mathbb{R}$ or $\mathbb{C}$ should maintain all properties when $f$ has a positive radius of convergence. In order to fulfill this principle, we generalize several existing formal series such as formal logarithms. We also introduce some developments of classical analysis, such as the Lagrange inversion problem and boundary convergence problem. The power of formal power series with real exponent is another development of formal analysis that improves the completeness of the space $\mathbb{X}$.

In response to my book proposal, I received a 7-page, handwritten review that proved indispensable to me in the development of this work. I was deeply moved by the reviewer’s insights and suggestions, which were as valuable to me as the guidance of my Ph. D. advisor, Karl Stromberg, almost 30 years ago. I am grateful for this reviewer’s recommendations, which drove the research and writing of this book.

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This volume contains the evolution of our work in formal analysis since 2000 and collects the comprehensive works related to formal power series since 1871.

Chapter 1 presents the basic algebraic structures of formal power series, including addition, multiplication (Cauchy product), and composition with nonunit formal power series. Here, we also introduce the matrix representations of a formal power series: one matrix representation for the Cauchy product and one matrix representation for the composition of almost units.

Formal differentiation has nothing to do with the *limit of difference quotient*, but it is a very useful tool in studying formal analysis. Chapter 2 introduces many applications of formal differentiation and certain metrics on $\mathbb{X}$. Formal differentiation returns in late chapters.

The Riordan group or Riordan array, a hot topic in recent research, is introduced in Chapter 3. This chapter also collects some well-known classical theorems that were reproved using formal power series. Chapter 4 features several other applications of formal power series, including real analysis, functional analysis, differential equations, and numerical analysis. The Lagrange inversion problem and other functional analysis problems are introduced in this chapter.

Chapter 5 transforms the studies of formal power series into formal analysis, using the general composition theorem. This theorem not only provides a necessary and sufficient condition for composition but also releases the composition of formal power series from the environment of so-called *admitting addition*. This chapter uses our recently published analytical approach to prove this theorem. The matrix representation for the general composition is also our construction.

Chapter 6 further discusses the relationship between classical analysis and formal analysis. First, we tackle a problem that has mystified mathematicians for more than a century: the boundary convergence behavior of power series. A *formal analytic point* is introduced. We prove that if a power series converges at a boundary point of the convergent region, and this point is formal analytic, then this power series converges at all boundary points. The special Banach spaces $\mathcal{H}^p(\beta)$ and $\mathcal{L}^p(\beta)$ are introduced in this chapter. The general formal logarithm Ln works on all formal power series in $\mathbb{X}^+(\mathbb{R})$ with positive constant term, similar to the logarithmic function $\ln(x)$ working on all positive real numbers in calculus. This generalization produces many interesting properties for Ln.

Several kinds of formal Laurent series existed before the space $\mathbb{L}$ of formal Laurent series was systematically invented in 2012 by Dariusz Bugajewski and the author. The semiformal Laurent series space $\mathcal{L}_s$ and semireversed formal Laurent series space $\mathcal{L}_r$ are introduced in Chapter 7. We construct $\mathcal{L}_s$ using all results provided in the previous chapters although it is known that $\mathcal{L}_s$ can be generated by the quotients of formal power series in certain sense. We propose a *canonical mapping* that connects formal Laurent series with the well-known Lebesgue measurable functions and the Lebesgue integral. The linkage of these two concepts sheds light on the rather complicated multiplication of formal Laurent series and facilitates further investigation of $\mathbb{L}$. The composition of formal Laurent series remains a challenge for mathematicians. This chapter introduces a composition of formal Laurent series with formal power series.

Iteration and iterative roots are closely related to functional equations, and the studies on this subject have a long history. The space of almost unit formal power series forms a group under composition, which yields many interesting results about iteration in this field. Those developments are presented in Chapter 8.

The last chapter introduces the power of formal power series with all real exponents, from the root series $g^{1/n}$ to rational exponents $g^r$, $r \in \mathbb{Q}$, and finally to the real exponents $g^s$, $s \in \mathbb{R}$, where $g$ is a formal power series on $\mathbb{R}$ with positive constant term. The formal exponent series is a new subject for the theory of formal power series. In this chapter, we introduce the exponent algorithm for computing formal real exponents, which is a convenient mathematical tool in analysis and in combinatorics.

Higher-dimensional formal power series warrant further study. But they are outside the scope of this volume and the author's expertise.

As the first comprehensive book on formal analysis, this volume presents to the mathematical world the most complete system of formal power series and formal Laurent series yet.

I would like to express my heartfelt thanks to Wojciech Kryszewski, without whom this book would not exist. His mathematical philosophy and vision inspired me to do this work.

I would also like to thank my friend, Dariusz Bugajewski. He gave me advice and encouragement throughout this process, and co-authored several papers on formal series with me. Whenever I discuss formal analysis with him and his team at Adam Mickiewicz University in Poznań, Poland, I refresh my mathematical skills and see new mathematical structures.

Finally, I extend my gratitude to De Gruyter for giving me this opportunity to contribute to the mathematical world.