Preface

As a generalization of classical calculus, fractional calculus has become an important branch of mathematics. It is popularly believed that this concept is stemmed from a letter by G. W. Leibniz (1646–1716) in the year 1695, where the one-half order of derivative was discussed. During the development of the past more than three centuries, numerous mathematicians made outstanding contributions on this field.

Now the fractional differential equations (FDEs) have become one of the important tools to model complex mechanics and physical behaviors and widespread applications have been found in anomalous diffusion, viscoelasticity, fluid flow, boundary layer effect of pipeline, electromagnetism, signal processing and control, quantum economy, fractal theory, etc., whereas, it is difficult to get the analytical solutions to the FDEs, even for the linear FDEs. Hence it becomes an important task to find some effective numerical simulations in current researches.

This book aims to make a systematic introduction to the finite difference method of FDEs. There are six chapters in this book.

Chapter 1 serves as a mathematical introduction to fractional calculus. It commences with four basic definitions of fractional derivatives. The analytical solutions to two kinds of fractional ordinary differential equations (FODEs) are given, from which, readers can have a general idea on the behaviors of solutions to FODEs. Several numerical approximation ways to fractional derivatives are introduced together with their numerical accuracy analysis. The applications of these formulae are also illustrated by solving the FODEs. This part is the important foundation of the following numerical solutions to fractional partial differential equations (FPDEs).

In Chapter 2, we study the finite difference methods for solving time-fractional subdiffusion equations. The time-fractional derivatives are approached by the G-L formula, the L1 approximation, the L2-1 _{σ} approximation, the fast L1 approximation and the fast L2-1 _{σ} approximation, respectively; The spatial derivatives are discretized by using the second-order central difference quotient or the compact approximation. For the 2D problem, several ADI difference schemes are derived. The unique solvability, stability and convergence for each scheme are proved.

Chapter 3 shows the finite difference methods for solving time-fractional wave equations. The time-fractional derivatives are discretized by the L1 approximation, the fast L1 approximation, the L2-1 _{σ} approximation and the fast L2-1 _{σ} approximation, respectively. For the 1D problem, two kinds of difference schemes are developed, among which one is of order two in space and the other is of order four in space. For the 2D problem, the ADI scheme and compact ADI scheme are both mentioned. The unique solvability, stability and convergence for each scheme are proved.

In Chapter 4, we introduce the finite difference methods for solving the space-fractional partial differential equations. For the 1D problem, the first-order method based on the shifted G-L formula, the second-order method based on the weighted-shifted G-L (WSGL) formula and the fourth-order method based on the WSGL formula are developed in turn. For the 2D problem, a fourth-order ADI method based on the WSGL formula is presented. The unique solvability, stability and convergence for each scheme are shown.

Chapter 5 considers the finite difference methods for solving a class of the time-space fractional differential equations. The time Caputo derivative is treated by the $L2-1_\sigma$ approximation and the spatial Riesz derivatives are discretized by the second-order fractional central difference quotient and the fourth-order weighted fractional central difference quotient formula, respectively. The second-order and the fourth-order difference schemes in space are established, respectively. The unique solvability, stability and convergence for each scheme are proved.

In Chapter 6, the finite difference methods for solving a class of time distributed-order subdiffusion equations are concerned. The distributed integral is discretized using the composite trapezoid formula or composite Simpson formula and the Caputo time-fractional derivatives are approximated using the second-order WSGL formula. The second-order scheme in both time and distributed order, and another fourth-order scheme in both time and distributed order are constructed, respectively. In addition, for the 2D problem, a second-order ADI difference scheme and another fourth-order ADI difference scheme are developed, respectively. The unique solvability, stability and convergence for each scheme are analyzed.

There are abundant results on the numerical method for FDEs in recent 20 years. In the last section of each chapter, we give a brief overview and only a limit part among them is listed in the references of this book, which are the resource or the referred materials of this book.

The main part of this book is based on the research results from the authors and their research group. The authors express their heartfelt thanks to all the collaborators.

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As the authors' limited scientific research work experience, they sincerely hope that scholars and colleagues will not hesitate to correct the shortcomings and omissions in the book. Thank you for sending emails to us.

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