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4 Intuitionistic fuzzy trigonometric distance and similarity measure and their properties

Abstract: In this chapter, we introduce the concept of intuitionistic fuzzy sets (IFS). Similarity and distance measures between IFS are explained and also extended these measures to IFSs. Some new trigonometric similarity and distance measures of IFSs are studied, and it is shown that intuitionistic fuzzy distance measures satisfy the required identities of intuitionistic fuzzy similarity measures. Also, basic properties of trigonometric similarity and distance measures of IFS (IF_s) are described.

Keywords: Fuzzy set (F_s), intuitionistic fuzzy set (IF_s), distance measures (DM), similarity measures (SM), intuitionistic fuzzy distance measures (IFDM), intuitionistic fuzzy similarity measures (IFSM)

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4.1 Introduction

The concept of fuzzy set was proposed by Zadeh [1]. Fuzzy sets conjecture has to transact business with uncertainty and dreaminess anywhere the notch of belongingness of any part to a backdrop which is represented by a attachment function. However, in the frank mean condition, it is not constantly doable that the scale of nonmembership meeting of a facet to the fuzzy set is barely peer to 1 minus the level of membership. It implies that there is some kind of degree of ambiguity and has been called hesitation degree.

The classical concept of fuzzy sets [1] is a strong tool to deal with uncertainty, vagueness, and imprecise information. There are various generalizations of higher order fuzzy sets for different specialized purposes. Intuitionistic fuzzy sets (IFS) introduced by Atanassov [2] are quite useful and applicable, and fuzzy sets are described by two functions: a membership function and a nonmembership function. Atanassov [2] introduced the concept of IFS as simplification of fuzzy sets, which is competent of capturing the order that includes several grades of timid and has been practical in countless return of research.

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Thus, another degree of freedom has been introduced to describe IFSs. Such a generalization of fuzzy sets equips us with an additional possibility to represent incomplete knowledge, which leads to describe many real problems in a more adequate way. Further, Atanassov and Gargov [3] also introduced the concept of interval-valued IFS and studied their properties.

Atanassov [4] provided an example, where fuzzy sets are IFS; however, converse is not automatically rightful. Burillo and Bustince [5] suggested an algorithm for IFS from fuzzy sets with the help of Atanassov's [6] operator. Each of the issues that manage fuzzy set hypothesis can also manage IFS hypothesis; however, there are circumstances where fuzzy set hypothesis cannot be appropriate to make a reasonable investigation. The membership values cannot be always determined, due to inefficiency of available information, besides the presence of vagueness in the information. Similarly, while determining the nonmembership, the same situation arises. A part of such estimation remains of indeterministic nature. This indeterministic part has not been considered in the case of fuzzy set theory, so we have IFS. For instance, suppose that a human being is to express his/her opinion about the beauty of a flower. In classical theory, he/she has only two choices 0 or 1, that is, a flower is either beautiful or ugly. In fuzzy sets, he/she has the liberty to express his/her view that the flower is beautiful with a grade of membership 0.8, which determines that the grade of nonmembership is 0.2, but in real life, a flower may have the beauty with a grade of membership 0.8 but not ugly at all, that is, its grade of nonmembership has the value 0. Therefore, one needs a most appropriate approach, which would help to deal with such a problem, whose situation depends on human perception. Therefore, IFS is a very useful mathematical tool in many situations having imprecise, vague, and inexact data.

It is noted that IFSs can describe the fuzzy characters of the things more comprehensively and thus are powerful and effective tools in dealing with fuzzy information.

An intuitionistic fuzzy conventional is characterized by two functions. Former is the point of link and subsequently is the level of nonmembership function. Their addition is each time fewer than one and the same to one.

Definition 4.1 (Fuzzy Set).

Fuzzy set E in a universal set Y is defined as

$$E = \{ \langle y, \alpha_E(y) \rangle : y \in E \} \quad (4.1)$$

where $\alpha_E(y_i)$ is a membership function defined as follows:

$$\alpha_E(y) = \begin{cases} 0, & y \text{ does not belong to } E \text{ and there is no ambiguity} \\ 1, & y \text{ belongs to } E \text{ and there is no ambiguity} \\ 0.5, & \text{there is maximum ambiguity whether } y \text{ belongs to } E \text{ or not} \end{cases} \quad (4.2)$$

In fact $\alpha_E(y)$ associates with each y , a grade of membership, to the set E .

Definition 4.2 (Intuitionistic Fuzzy Set).

An IFS E in a universal set Y is defined as

$$E = \{ \langle y, \alpha_E(y), \beta_E(y) \rangle : y \in Y \} \quad (4.3)$$

where the functions

$$E = \{ \langle y, \alpha_E(y), \beta_E(y) \rangle : y \in Y \} \quad (4.4)$$

are the degree of membership to the element and the degree of nonmembership to the element, respectively, with the condition

$$0 \leq \alpha_E(y) + \beta_E(y) \leq 1 \quad (4.5)$$

the intuitionistic index of an element $y \in Y$ is defined as follows:

$$\pi_E(y) = 1 - \alpha_E(y) - \beta_E(y) \quad (4.6)$$

$\pi_A(x)$ is called a degree of membership of an element in E and it is evident that $0 \leq \pi_E(y) \leq 1$.

Note that for an IFS E , if $\alpha_E(y) = 0$ then $\alpha_E(y) + \beta_E(y)$ and if $\alpha_E(y) = 1$ then $\pi_E(y) = 0$.

We present the following basic operations on IFS, which is necessary for our discussion:

Definition 4.3 (Basic Operations of Intuitionistic Fuzzy Set). If E and F are two IFS of the universal set Y , then

- (i) iff $E \subset F$ then $\alpha_E(y) \leq \alpha_F(y)$ and $\beta_E(y) \geq \beta_F(y) \quad \forall y \in Y$
- (ii) iff $E = F$ then $\alpha_E(y) = \alpha_F(y)$ and $\beta_E(y) = \beta_F(y)$
- (iii) $E^C = \{ \langle y, \beta_E(y), \alpha_E(y) \rangle : y \in Y \}$, where E^C is complement of E
- (iv) $E \cap F = \{ \langle y, \min(\alpha_E(y), \alpha_F(y)), \max(\beta_E(y), \beta_F(y)) \rangle : y \in Y \}$
- (v) $E \cup F = \{ \langle y, \max(\alpha_E(y), \alpha_F(y)), \min(\beta_E(y), \beta_F(y)) \rangle : y \in Y \}$

Burillo and Bustince [5] have also given some results on IFS. In the subsequent discussion, the following notations are used:

$R = [0, \infty)$, $Y = \{y_1, y_2, \dots, y_n\}$ is the universal set; $F_s(Y)$ is the class of all fuzzy sets of Y ; IFS(Y) is the class of all IFSs of Y .

Liu [7] given new measures without saying definitions of aloofness actions and similarity events of fuzzy sets and examined the connections between them. These axioms are spontaneous and possess laboring in the fuzzy literature.

Definition 4.4 (Fuzzy Distance Measure). Let D be a real-valued function defined as $D: F_s \times F_s \rightarrow R^+$. A distance measure D is called fuzzy distance measure if it satisfies the following conditions:

- (d1) $D(E, F) = D(F, E) \quad \forall E, F \in F_s$
- (d2) $D(E, E) = 0 \quad \forall E \in F_s$
- (d3) $D(d, d^c) = \max_{E, F \in F_s} D(E, F)$, if d is the crisp set
- (d4) $\forall E, F, G \in F_s$, if $E \subset F \subset G$ then $D(E, F) \leq D(E, G)$ and $D(E, F) \leq D(F, G)$

Definition 4.5 (Fuzzy Similarity Measure). Suppose that S be a real-valued function $T: F_s \times F_s \rightarrow R^+$. A similarity measure T is called fuzzy similarity measure if it satisfies the following conditions:

- (S1) $T(E, F) = T(F, E) \quad \forall E, F \in F_s$
- (S2) $T(d, d^c) = 0$, if d is the crisp set
- (S3) $T(K, K) = \max_{E, F \in F_s} D(E, F) \quad \forall K \in F_s$
- (S4) $\forall E, F, G \in F_s$, if $E \subset F \subset G$ then $T(E, F) \geq T(E, G)$ and $T(E, F) \geq T(F, G)$

If we standardize d and T in Definitions 4.3 and 4.4, we get $0 \leq D(E, F) \leq 1$ and $0 \leq T(E, F) \leq 1$ for $E, F \in F_s$. The relation between standardized d and S is $D = 1 - T$. Hence, from this relation, it can be seen that the distance and similarity measures are dual concepts, and we can give the distance and similarity measure axiomatic definitions for IFS.

Definition 4.6 (Intuitionistic Fuzzy Distance Measure). Suppose D is a real-valued function $D: IFS \times IFS \rightarrow R^+$. Then distance measure D is called intuitionistic fuzzy distance measure if it satisfies the following conditions:

- (IS1) $T(E, F) = T(F, E) \quad \forall E, F \in IFS$
- (IS2) $T(d, d^c) = 0$, if d is the crisp set
- (IS3) $T(K, K) = \max_{E, F \in IFS} D(E, F) \quad \forall K \in IFS$
- (IS4) $\forall E, F, G \in IFS$, if $E \subset F \subset G$ then $T(E, F) \geq T(E, G)$ and $T(E, F) \geq T(F, G)$

Definition 4.7 (Intuitionistic Fuzzy Similarity Measure). Let T be a real-valued function $T: IFS \times IFS \rightarrow R^+$. Then similarity measure T is called fuzzy similarity measure if it satisfies the following conditions:

$$(IS1) \ T(E, F) = T(F, E) \quad \forall E, F \in \text{IFS}$$

$$(IS2) \ T(d, d^c) = 0, \text{ if } d \text{ is the crisp set}$$

$$(IS3) \ T(K, K) = \max_{E, F \in \text{IFS}} D(E, F) \quad \forall K \in \text{IFS}$$

$$(IS4) \ \forall E, F, G \in \text{IFS}, \text{ if } E \subset F \subset G \text{ then } T(E, F) \geq T(E, G) \text{ and } T(E, F) \geq T(F, G)$$

4.2 Sine trigonometric intuitionistic fuzzy distance measures

Definition 4.8. Let E and F be two IFS on universal set Y as $Y = \{y_1, y_2, \dots, y_n\}$, whose membership values $\alpha_E(y_i)$, $i = 1, 2, \dots, n$ and $\alpha_F(y_i)$, $i = 1, 2, \dots, n$, respectively, are the measures of sine intuitionistic fuzzy distance $D_{\text{IFS}}(E, F)$ between $D_{\text{IFS}}(E, F)$ fuzzy sets E and F , as

$$D_{\text{IFS1}}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left(\sin \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right) \quad (4.7)$$

$$D_{\text{IFS2}}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2} \right) \pi + \sin \left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2} \right) \pi \right] \quad (4.8)$$

Theorem 4.1. Identities (4.7) and (4.8) prove all the conditions of intuitionistic fuzzy distance measures.

Proof. Given that

$$D_{\text{IFS1}}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right]$$

Here, we prove all four conditions of intuitionistic fuzzy distance measures that are mentioned in Definition 4.6:

$$\begin{aligned} (\text{Id1}) \ D_{\text{IFS1}}(E, F) &= \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_F(y_i) - \alpha_E(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_F(y_i) - \beta_E(y_i)|}{2} \right) \pi \right] \\ &= D_{\text{IFS1}}(F, E) \end{aligned}$$

$$\begin{aligned}
 (\text{Id2}) \ D_{\text{IFS}_1}(E, E) &= \frac{1}{2n} \sum_{i=1}^n \left[\sin\left(\frac{|\alpha_E(y_i) - \alpha_E(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_E(y_i) - \beta_E(y_i)|}{2}\right) \pi \right] \\
 &= 0 \quad \forall E, F \in \text{IFS}
 \end{aligned}$$

(Id3) Let

$$\begin{aligned}
 E = d \text{ and } F = d^C, \text{ i.e., } \alpha_E(y_i) = \alpha_d(y_i), \alpha_F(y_i) = \alpha_{d^C}(y_i) \text{ and } \beta_E(y_i) = \beta_d(y_i), \\
 \beta_F(y_i) = \beta_{d^C}(y_i)
 \end{aligned}$$

Then we get

$$D_{\text{IFS}_1}(d, d^C) = 1 \quad \forall E, F \in \text{IFS}$$

that is

$$D_{\text{IFS}_1}(d, d^C) = \max_{E, F \in \text{IFS}} D_{\text{IFS}_1}(E, F) \quad \forall E, F \in \text{IFS}.$$

(Id4) For all $E, F, G \in \text{IFS}(Y)$, if $E \subset F \subset G$, then $\alpha_E(y_i) \leq \alpha_F(y_i) \leq \alpha_G(y_i)$

$$\text{or } \beta_E(y_i) \geq \beta_F(y_i) \geq \beta_G(y_i)$$

$$\begin{aligned}
 &\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \leq \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right) \pi \\
 &\quad + \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right) \pi
 \end{aligned}$$

and the nature of sine function, we get

$$\begin{aligned}
 \sin\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi &\leq \sin\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right) \pi \\
 &+ \sin\left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right) \pi
 \end{aligned}$$

$$\Rightarrow D_{\text{IFS}_1}(E, F) \leq D_{\text{IFS}_1}(E, G)$$

Similarly,

$$\begin{aligned}
 &\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \leq \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2}\right) \pi \\
 &\quad + \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2}\right) \pi
 \end{aligned}$$

$$\sin\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right)\pi + \sin\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right)\pi \leq \sin\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right)\pi \\ + \sin\left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right)\pi$$

$$\Rightarrow D_{\text{IFS}}(E, F) \leq D_{\text{IFS}}(F, G)$$

Now we will prove all the required conditions using identity given in eq. (4.8):

$$\begin{aligned} (\text{Id1}) D_{\text{IFS}_2}(E, F) &= \frac{1}{2n} \sum_{i=1}^n \left[\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2} \right) \pi + \sin\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2}\right) \pi \right] \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\sin\left(\frac{|\sqrt{\alpha_F(y_i)} - \sqrt{\alpha_E(y_i)}|}{2}\right) \pi + \sin\left(\frac{|\sqrt{\beta_F(y_i)} - \sqrt{\beta_E(y_i)}|}{2}\right) \pi \right] \\ &= D_{\text{IFS}_2}(F, E) \end{aligned}$$

$$\begin{aligned} (\text{Id2}) D_{\text{IFS}_2}(E, E) &= \frac{1}{2n} \sum_{i=1}^n \left[\sin\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_E(y_i)}|}{2}\right) \pi + \sin\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_E(y_i)}|}{2}\right) \pi \right] \\ &= 0 \quad \forall E \in \text{IFS} \end{aligned}$$

(Id3) Let

$E = d$ and $F = d^c$, i.e., $\alpha_E(y_i) = \alpha_d(y_i)$, $\alpha_F(y_i) = \alpha_{d^c}(y_i)$ and $\beta_E(y_i) = \beta_d(y_i)$, $\beta_F(y_i) = \beta_{d^c}(y_i)$

Then we get,

$$D_{\text{IFS}_2}(d, d^c) = 1 \quad \forall E, F \in \text{IFS}$$

that is, $D_{\text{IFS}_2}(d, d^c) = \max_{E, F \in \text{IFS}} D_{\text{IFS}_2}(E, F) \quad \forall E, F \in \text{IFS}$.

(Id4) For all $E, F, G \in \text{IFS}(Y)$, if $E \subset F \subset G$, then $\alpha_E(y_i) \leq \alpha_F(y_i) \leq \alpha_G(y_i)$ or

$$\beta_E(y_i) \geq \beta_F(y_i) \geq \beta_G(y_i)$$

$$\begin{aligned} \left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2} \right) \pi + \left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2} \right) \pi &\leq \left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_G(y_i)}|}{2} \right) \pi \\ &+ \left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_G(y_i)}|}{2} \right) \pi \end{aligned}$$

and the nature of sine function must be defined as

$$\begin{aligned} & \sin\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2}\right)\pi + \sin\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2}\right)\pi \\ & \leq \sin\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_G(y_i)}|}{2}\right)\pi + \sin\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_G(y_i)}|}{2}\right)\pi. \end{aligned}$$

$$\Rightarrow D_{\text{IFS}_2}(E, F) \leq D_{\text{IFS}_2}(E, G)$$

Similarly,

$$\begin{aligned} & \left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2}\right)\pi + \left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2}\right)\pi \leq \left(\frac{|\sqrt{\alpha_F(y_i)} - \sqrt{\alpha_G(y_i)}|}{2}\right)\pi \\ & \quad + \left(\frac{|\sqrt{\beta_F(y_i)} - \sqrt{\beta_G(y_i)}|}{2}\right)\pi \\ & \Rightarrow \sin\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2}\right)\pi + \sin\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2}\right)\pi \\ & \leq \sin\left(\frac{|\sqrt{\alpha_F(y_i)} - \sqrt{\alpha_G(y_i)}|}{2}\right)\pi + \sin\left(\frac{|\sqrt{\beta_F(y_i)} - \sqrt{\beta_G(y_i)}|}{2}\right)\pi. \\ & \Rightarrow D_{\text{IFS}_2}(E, F) \leq D_{\text{IFS}_2}(F, G) \end{aligned}$$

4.3 Properties of sine trigonometric intuitionistic fuzzy distance measures

Let D be a distance measure on IFS and for $E, F \in \text{IFS}$. If they satisfy that for all $y_i \in Y$, either $E(y_i) \subseteq F(y_i)$ or $E(y_i) \supseteq F(y_i)$, then

$$(i) D_{\text{IFS}_1}(E \cup F, E \cap F) = D_{\text{IFS}_1}(E, F) \quad (4.9)$$

and

$$D_{\text{IFS}_2}(E \cup F, E \cap F) = D_{\text{IFS}_2}(E, F) \quad \forall E, F, G \in \text{IFS} \quad (4.10)$$

$$(ii) D_{\text{IFS}_1}(E \cup F, G) \leq D_{\text{IFS}_1}(E, G) + D_{\text{IFS}_1}(F, G) \quad (4.11)$$

and

$$D_{\text{IFS}_2}(E \cup F, G) \leq D_{\text{IFS}_2}(E, G) + D_{\text{IFS}_2}(F, G) \quad \forall E, F, G \in \text{IFS} \quad (4.12)$$

$$(iii) D_{\text{IFS}_1}(E \cap F, G) \leq D_{\text{IFS}_1}(E, G) + D_{\text{IFS}_1}(F, G) \quad (4.13)$$

$$\text{and } D_{\text{IFS}_2}(E \cap F, G) \leq D_{\text{IFS}_2}(E, G) + D_{\text{IFS}_2}(F, G) \quad \forall E, F, G \in \text{IFS} \quad (4.14)$$

then the above equations satisfy the properties.

Proof.

(i) Using Definition 4.6, we get

$$\begin{aligned} D_{\text{IFS}_1}(E \cup F, E \cap F) &= \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_{E \cup F}(y_i) - \alpha_{E \cap F}(y_i)|}{2} \right) \pi \right. \\ &\quad \left. + \sin \left(\frac{|\beta_{E \cup F}(y_i) - \beta_{E \cap F}(y_i)|}{2} \right) \pi \right] \\ &= \frac{1}{2n} \left[\sum_{y_i \in Y_1} \left[\sin \left(\frac{|\alpha_F(y_i) - \alpha_E(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_F(y_i) - \beta_E(y_i)|}{2} \right) \pi \right] \right. \\ &\quad \left. + \sum_{y_i \in Y_2} \left[\sin \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] \right] \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] = D_{\text{IFS}_1}(E, F) \end{aligned}$$

Similarly, we can prove for eq. (4.10).

(ii) Let us consider for

$$\begin{aligned} &D_{\text{IFS}_1}(E, G) + D_{\text{IFS}_1}(F, G) - D_{\text{IFS}_1}(E \cup F, G) \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ &\quad + \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ &\quad - \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_{E \cup F}(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_{E \cup F}(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \sin \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ &\quad + \frac{1}{2n} \sum_{i=1}^n \left[\sin \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2n} \left[\sum_{y_1 \in Y_1} \sin\left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2}\right) \pi \right] \\
& + \sum_{y_1 \in Y_2} \left[\sin\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right) \pi \right] \\
& = \frac{1}{2n} \left[\sum_{y_1 \in Y_1} \sin\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right) \pi \right] \\
& + \frac{1}{2n} \sum_{y_1 \in Y_2} \left[\sin\left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2}\right) \pi + \sin\left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2}\right) \pi \right] \geq 0 \\
& \Rightarrow D_{\text{IFS}_1}(E, G) + D_{\text{IFS}_1}(F, G) - D_{\text{IFS}_1}(E \cup F, G) \geq 0
\end{aligned}$$

Similarly, we can prove

$$D_{\text{IFS}_2}(E, G) + D_{\text{IFS}_2}(F, G) - D_{\text{IFS}_2}(E \cup F, G) \geq 0$$

(iii)

$$D_{\text{IFS}_1}(E, G) + D_{\text{IFS}_1}(F, G) - D_{\text{IFS}_1}(E \cap F, G) \geq 0$$

and

$$D_{\text{IFS}_2}(E, G) + D_{\text{IFS}_2}(F, G) - D_{\text{IFS}_2}(E \cap F, G) \geq 0 \quad \forall E, F, G \in \text{IFS}$$

4.4 Cosine trigonometric intuitionistic fuzzy similarity measures

Definition 4.9. Let E and F be two IFS on universal set Y as $Y = \{y_1, y_2, \dots, y_n\}$, whose membership values are $\alpha_E(y_i)$, $i = 1, 2, \dots, n$ and $\alpha_F(y_i)$, $i = 1, 2, \dots, n$, respectively. Then the measures of cosine intuitionistic fuzzy similarity $S_{\text{IFS}_M}(E, F)$ between $S_{\text{IFS}}(E, F)$ fuzzy sets E and F are defined as

$$S_{\text{IFS}_1}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left[\cos\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \cos\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \right] \quad (4.15)$$

$$S_{\text{IFS}_2}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left[\cos\left(\frac{|\sqrt{\alpha_E(y_i)} - \sqrt{\alpha_F(y_i)}|}{2}\right) \pi + \cos\left(\frac{|\sqrt{\beta_E(y_i)} - \sqrt{\beta_F(y_i)}|}{2}\right) \pi \right] \quad (4.16)$$

Theorem 4.2. *Identities (4.15) and (4.16) prove all the conditions of intuitionistic fuzzy similarity measures.*

Proof. Given

$$S_{\text{IFS}_1}(E, F) = \frac{1}{2n} \sum_{i=1}^n \left[\cos\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \cos\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \right]$$

Here we prove all four conditions of intuitionistic fuzzy distance measures that are mentioned in Definition 4.7:

$$\begin{aligned} (\text{IS1}) \quad S_{\text{IFS}_1}(E, F) &= \frac{1}{2n} \left[\sum_{i=1}^n \cos\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \cos\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \right] \\ &= \frac{1}{2n} \left[\sum_{i=1}^n \cos\left(\frac{|-(\alpha_F(y_i) - \beta_E(y_i))|}{2}\right) \pi + \cos\left(\frac{|-(\beta_F(y_i) - \beta_E(y_i))|}{2}\right) \pi \right] \end{aligned}$$

(IS2) $S(d, d^c) = 0$, if d is the crisp set

(IS3) $S(K, K) = \max_{E, F \in \text{IFS}} D(E, F) \quad \forall K \in \text{IFS}$

(IS4) $\forall E, F, G \in \text{IFS}$, if $E \subset F \subset G$ then $\alpha_E(y_i) \leq \alpha_F(y_i) \leq \alpha_G(y_i)$ and

$$\begin{aligned} \beta_E(y_i) \leq \beta_F(y_i) \leq \beta_G(y_i) &= \frac{1}{2n} \left[\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] \\ &\geq \frac{1}{2n} \left[\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \end{aligned}$$

and the nature of cosine function is given by

$$\begin{aligned} &\frac{1}{2n} \left[\sum_{i=1}^n \cos\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2}\right) \pi + \cos\left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2}\right) \pi \right] \\ &\leq \frac{1}{2n} \left[\sum_{i=1}^n \cos\left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2}\right) \pi + \cos\left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2}\right) \pi \right] \\ S_{\text{IFS}_1}(E, F) &\geq S_{\text{IFS}_1}(E, G) \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{1}{2n} \left[\left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ &\geq \frac{1}{2n} \left[\left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \end{aligned}$$

and the nature of cosine function is given by

$$\begin{aligned} & \frac{1}{2n} \left[\sum_{i=1}^n \cos \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ & \leq \frac{1}{2n} \left[\sum_{i=1}^n \cos \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\ & S_{\text{IFS1}}(E, F) \geq S_{\text{IFS1}}(F, G) \end{aligned}$$

Similarly, we can prove the identity given by eq. (4.16).

4.5 Properties of cosine trigonometric intuitionistic fuzzy similarity measures

Let T be a similarity measure on IFS and for $E, F \in \text{IFS}$. If they satisfy that for all $y_i \in Y$, either $E(y_i) \subseteq F(y_i)$ or $E(y_i) \supseteq F(y_i)$, then

$$(i) \quad T_{\text{IFS1}}(E \cup F, E \cap F) = T_{\text{IFS1}}(E, F) \quad (4.17)$$

and

$$T_{\text{IFS2}}(E \cup F, E \cap F) = T_{\text{IFS2}}(E, F) \quad \forall E, F, G \in \text{IFS} \quad (4.18)$$

and

$$(ii) \quad T_{\text{IFS1}}(E \cup F, G) \leq T_{\text{IFS1}}(E, G) + T_{\text{IFS1}}(F, G) \quad (4.19)$$

$$T_{\text{IFS2}}(E \cup F, G) \leq T_{\text{IFS2}}(E, G) + T_{\text{IFS2}}(F, G) \quad \forall E, F, G \in \text{IFS} \quad (4.20)$$

$$(iii) \quad T_{\text{IFS1}}(E \cap F, G) \leq T_{\text{IFS1}}(E, G) + T_{\text{IFS1}}(F, G) \quad (4.21)$$

and

$$T_{\text{IFS2}}(E \cap F, G) \leq T_{\text{IFS1}}(E, G) + T_{\text{IFS2}}(F, G) \quad \forall E, F, G \in \text{IFS} \quad (4.22)$$

Proof.

(i) Using Definition 4.7

$$\begin{aligned} T_{\text{IFS1}}(E \cup F, E \cap F) &= \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_{E \cup F}(y_i) - \alpha_{E \cap F}(y_i)|}{2} \right) \pi \right. \\ &\quad \left. + \cos \left(\frac{|\beta_{E \cup F}(y_i) - \beta_{E \cap F}(y_i)|}{2} \right) \pi \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2n} \left[\sum_{y_i \in Y_1} \left[\cos \left(\frac{|\alpha_F(y_i) - \alpha_E(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_F(y_i) - \beta_E(y_i)|}{2} \right) \pi \right] \right. \\
&\quad \left. + \sum_{y_i \in Y_2} \left[\cos \left(\frac{|\alpha_E(y_i) - \alpha_F(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] \right] \\
&= \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_E(y_i) - \beta_F(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_F(y_i)|}{2} \right) \pi \right] \\
&= T_{\text{IFS}_1}(E \cup F, E \cap F)
\end{aligned}$$

Similarly we can prove eq. (4.18).

(ii) Let us consider for

$$\begin{aligned}
&T_{\text{IFS}_1}(E, G) + T_{\text{IFS}_1}(F, G) - T_{\text{IFS}_1}(E \cup F, G) \\
&= \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&\quad + \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&\quad - \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_{E \cup F}(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_{E \cup F}(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&= \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&\quad + \frac{1}{2n} \sum_{i=1}^n \left[\cos \left(\frac{|\alpha_F(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_F(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&= \frac{1}{2n} \left[\sum_{y_1 \in Y_1} \cos \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right] \\
&\quad + \sum_{y_1 \in Y_2} \left[\cos \left(\frac{|\alpha_E(y_i) - \alpha_G(y_i)|}{2} \right) \pi + \cos \left(\frac{|\beta_E(y_i) - \beta_G(y_i)|}{2} \right) \pi \right]
\end{aligned}$$

Similarly, we can prove $T_{\text{IFS}_2}(E \cup F, G) \leq T_{\text{IFS}_2}(E, F) + T_{\text{IFS}_2}(F, G) \quad \forall E, F, G \in \text{IFS}$

(iii) Similarly, we can prove

$$T_{\text{IFS}_1}(E \cap F, G) \leq T_{\text{IFS}_1}(E, G) + T_{\text{IFS}_1}(F, G)$$

and

$$T_{\text{IFS}_1}(E \cap F, G) \leq T_{\text{IFS}_1}(E, G) + T_{\text{IFS}_1}(F, G) \quad \forall E, F, G \in \text{IFS}$$

4.6 Conclusion

First, we defined IFS. Then, we introduced a general intuitionistic inclusion on a universe of discourse X and intuitionistic similarity measure between two IFS on X . Definitions have been given to illustrate the concept of IFS, intuitionistic similarity between two IFS, and intuitionistic fuzzy relation based on intuitionistic similarity between two IFS, intuitionistic fuzzy distance measure, sine and cosine trigonometric intuitionistic fuzzy distance measures. Properties of sine and cosine trigonometric intuitionistic fuzzy distance measures are also given.

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