Preface

Aristotle held that human intellectual activity or philosophical (in a broad sense) knowledge can be seen as a threefold research program. This program contains metaphysics, the most fundamental branch, which tries to find the right way to deal with “Being” as such; mathematics, an exact science studying calculable – at least, in principle – abstract objects and formal relations between them; and, finally, physics, the science working with changeable things and the causes of the changes. Therefore, physics is the science of evolution – in the first place the evolution in time. To put it in more ‘contemporary’ terms, at any energy scale there are things which a physicist has to accept as being ‘given from above’ and then try to formulate a theory of how do these things, whatever they are, change. Of course, by increasing the energy and, therefore, by improving the resolution of experimental facility, one discovers that those things emerge, in fact, as a result of evolution of other things, which should now be considered as ‘given from above’.

The very possibility that the evolution of material things, whatever they are, can be studied quantitatively is highly non-trivial. First of all, to introduce changes of something, one has to secure the existence of something that does not change. Indeed, changes can be observed only with respect to something permanent. Kant proposed that what is permanent in all changes of phenomena is substance. Although phenomena occur in time and time is the substratum, wherein co-existence or succession of phenomena can take place, time as such cannot be perceived. Relations of time are only possible on the background of the permanent. Given that changes ‘really’ take place, one derives the necessity of the existence of a representation of time as the substratum and defines it as substance. Substance is, therefore, the permanent thing only with respect to which all time relations of phenomena can be identified.

Kant gave then a proof that all changes occur according to the law of the connection between cause and effect, that is, the law of causality. Given that the requirement of causality is fulfilled, at least locally, we are able to use the language of differential equations to describe quantitatively the physical evolution of things. There is, however, a hierarchy of levels of causality. For example, Newton’s theory of gravitation is causal only if we do not ask how the gravitational force gets transported from one massive body to another. The concept of a field as an omnipresent mediator of all interactions allows us to step up to a higher level of causality. The field approach to the description of the natural forces culminated in the creation in the 20th century of the quantum field theoretic approach as an (almost) universal framework to study the physical phenomena at the level of the most elementary constituents of matter.

1 It is worth noticing that this scheme is one of the most consistent ways to introduce the concept of the renormalization group, which is crucial in a quantum field theoretical approach to describe the three fundamental interactions.

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To be more precise, the quantitative picture of the three fundamental interactions is provided by the Standard Model, the quantum field theory of the strong, weak and electromagnetic forces. The aesthetic attractivity and unprecedented predictive power of this theory is due to the most successful, and nowadays commonly accepted, way to introduce the interactions by adopting the principle of local (gauge) symmetry. This principle allows us to make use of the local field functions, which depend on the choice of the specific gauge and, as such, do not represent any observables, to construct a mathematically consistent and phenomenologically useful theory. In any gauge field theory we need, therefore, gauge-invariant objects, which are supposed to be the fundamental ingredients of the Lagrangian of the theory, and which can be consistently related, at least, in principle, to physical observables.

The most straightforward implementation of the idea of a scalar gauge invariant object is provided by the traced product of field strength tensors

$$\text{Tr} \left[ F_{\mu\nu}(x) F^{\mu\nu}(x) \right],$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \pm ig [A_\mu(x), A_\nu(x)],$$

$A_\mu(x)$ being the local gauge potentials belonging to the adjoint representation of the $N$-parametric group of local transformations $U(x)$, and $g$ a coupling constant. Field strength tensors are also local quantities, which change covariantly under the gauge transformations:

$$F_{\mu\nu}(x) \rightarrow U(x) F_{\mu\nu}(x) U^\dagger(x).$$

Interesting non-local realizations of gauge-invariant objects emerge from Wilson lines defined as path-ordered ($\mathcal{P}$) exponentials of contour (path, loop, line) integrals of the local gauge fields $A_\mu(z)$

$$U_y[y, x] = \mathcal{P} \exp \left[ \pm ig \int_x^y dz^\mu A_\mu(z) \right].$$

The integration goes along an arbitrary path $y$:

$$z \in y$$

from the initial point $x$ to the end point $y$. The notion of a path will be one of the crucial issues throughout the book.

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2 The terminology and the choice of the signature $\pm$ will be explained below.
The Wilson line (4) is \textit{gauge covariant}, but, in contrast to the field strength, the transformation law reads

\[ U_y[y, x] \rightarrow U(y)U_y[y, x]U^\dagger(x), \]

so that the transformation operators \( U, U^\dagger \) are defined in different space-time points.

For closed paths \( x = y \), so that we speak about the \textit{Wilson loop}:

\[ U_y \equiv U_y[x, x] = \mathcal{P} \exp \left[ \pm ig \oint dz^\mu A_\mu(z) \right], \]

which transforms similarly to the field strength

\[ U_y = U_y[x, x] \rightarrow U(x)U_yU^\dagger(x). \]

The simplest scalar \textit{gauge invariant} objects made from Wilson loops are, therefore, the traced Wilson loops

\[ \mathcal{W}_y = \text{Tr} U_y. \]

From the mathematical point of view, one can construct a \textit{loop space} whose elements are the Wilson loops defined on an infinite set of contours. The recast of a quantum gauge field theory in loop space is supposed to enable one to utilize the scalar gauge-invariant field functionals as the fundamental degrees of freedom, instead of the traditional gauge-dependent boson and fermion fields. Physical observables are then supposed to be expressed in terms of the vacuum averages of the products of Wilson loops

\[ \mathcal{W}^{(n)}_{\{y\}} = \langle 0 | \text{Tr} U_{y_1} \text{Tr} U_{y_2} \cdots \text{Tr} U_{y_n} |0\rangle. \]

The concept of Wilson lines finds an enormously wide range of applications in a variety of branches of modern quantum field theory, from condensed matter and lattice simulations, to quantum chromodynamics, high-energy effective theories, and gravity. However, there exist surprisingly few reviews or textbooks which contain a more or less comprehensive pedagogical introduction into the subject. Even the basics of the Wilson lines theory may put students and non-experts in significant trouble. In contrast to generic quantum field theory, which can be taught with the help of plenty of excellent textbooks and lecture courses, the theory of Wilson lines and loops still lacks such a support. The objective of the present book is, therefore, to collect, overview and present in the appropriate form the most important results available in literature, with the aim to familiarize the reader with the theoretical and mathematical foundations of the concept of Wilson lines and loops. We also intend to give an introductory idea of how to implement elementary calculations utilizing Wilson lines within the context of modern quantum field theory, in particular, in Quantum Chromodynamics.
The target audience of our book consists of graduate and postgraduate students working in various areas of quantum field theory, as well as curious researchers from other fields. Our *lettore modello* is assumed to have already followed standard university courses in advanced quantum mechanics, theoretical mechanics, classical fields and the basics of quantum field theory, elements of differential geometry, etc. However, we give all necessary information about those subjects to keep with the logical structure of the exposition. Chapters 2, 3, and 4 were written by T. Mertens, Chapter 5 by F. F. Van der Veken. Preface, Introduction and general editing are due to I. O. Cherednikov. In our exposition we used extensively the results, theorems, proofs and definitions, given in many excellent books and original research papers. For the sake of uniformity, we usually refrain from citing the original works in the main text. We hope that the dedicated literature guide in Appendix D will do this job better.

Besides this, we have benefited from presentations made by our colleagues at conferences and workshops and informal discussions with a number of experts. Unfortunately, it is not possible to mention everybody without the risk of missing many others who deserve mentioning as well. However, we are happy to thank our current and former collaborators, from whom we have learned a lot: I. V. Anikin, E. N. Antonov, U. D’Alesio, A. E. Dorokhov, E. Iancu, A. I. Karanikas, N. I. Kochelev, E. A. Kuraev, J. Lauwers, L. N. Lipatov, O. V. Teryaev, F. Murgia, N. G. Stefanis, and P. Taels. Our special thanks go to I. V. Anikin, M. Khalo, and P. Taels for reading parts of the manuscript and making valuable critical remarks on its content. We greatly appreciate the inspiring atmosphere created by our colleagues from the Elementary Particle Physics group in University of Antwerp, where this book was written. We are grateful to M. Efroimsky and L. Gamberg for their invitation to write this book, and to the staff of *De Gruyter* for their professional assistance in the course of the preparation of the manuscript.

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