Preface

The study of nonlocal operators of fractional type possesses a long tradition, motivated both by mathematical curiosity and by real-world applications. Though this line of research presents some similarities and analogies with the study of operators of integer order, it also presents a number of remarkable differences, one of the greatest being the recently discovered phenomenon that *all functions are (locally) fractionally harmonic (up to a small error)*. This feature is quite surprising, since it is in sharp contrast with the case of classical harmonic functions, and it reveals a genuinely nonlocal peculiarity.

More precisely, it has been proved in [25] that given any C^k-function f in a bounded domain Ω and given any $\epsilon > 0$, there exists a function f_ϵ which is fractionally harmonic in Ω such that the C^k-distance in Ω between f and f_ϵ is less than ϵ.

Interestingly, this kind of results can be also applied at any scale, as shown in Figures 1, 2, and 3. Roughly speaking, given *any* function, without any special geometric prescription, in a given bounded domain (as in Figure 1), one can "complete" the function outside the domain in such a way that the resulting object is fractionally harmonic. That is, one can endow the function given in the bounded domain with a number of suitable oscillations outside the domain in order to make an integro-differential operator of fractional type vanish. This idea is depicted in Figure 2. As a matter of fact, Figure 2 must be considered just a "qualitative" picture of this method, and should not be regarded "realistic." However, even if Figure 2 does not provide a correct fractional harmonic extension of the given function outside the given domain, the result can be repeated at a larger scale, as in Figure 3, adding further remote oscillations in order to obtain a fractional harmonic function.

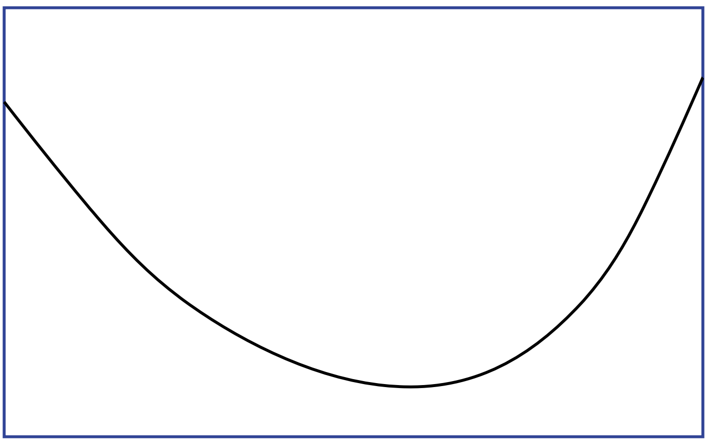

Figure 1: All functions are fractional harmonic, at different scales (scale of the original function).

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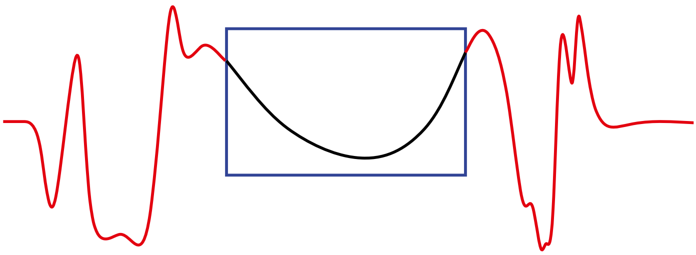

Figure 2: All functions are fractional harmonic, at different scales ("first" scale of exterior oscillations).

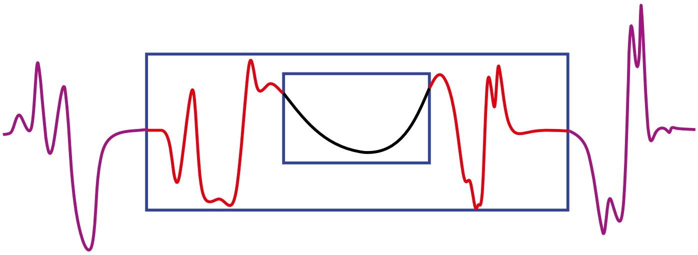

Figure 3: All functions are fractional harmonic, at different scales ("second" scale of exterior oscillations).

In this sense, this type of results really says that whatever graph we draw on a sheet of paper, it is fractionally harmonic (more rigorously, it can be shadowed with an arbitrary precision by another graph, which can be appropriately continued outside the sheet of paper in a way which makes it fractionally harmonic).

This book contains a *new result* in this line of investigation, stating that *every function lies in the kernel of every linear equation involving some fractional operator, up to a small error*. That is, *any given function can be smoothly approximated by functions lying in the kernel of a linear operator involving at least one fractional component*. The setting in which this result holds is very general, since it takes into account anomalous diffusion, with possible fractional components in both space and time. The operators taken into account comprise the case of the sum of classical and fractional Laplacians, possibly of different orders, in the space variables, and classical or fractional derivatives in the time variables. Namely, the equation can be of any order, it does not need any structure (it needs no ellipticity or parabolicity conditions), and the fractional behavior is in time, space, or both.

In a sense, this type of approximation results reveals the true power of fractional equations, independently of the structural "details" of the single equation under consideration, and shows that *space-fractional and time-fractional equations exhibit a variety of solutions which is much richer and more abundant than in the case of classical diffusion*.

Though space- and time-fractional diffusions can be seen as related aspects of nonlocal phenomena, they arise in different contexts and present important structural differences. The paradigmatic example of space-fractional diffusion is embodied by the fractional Laplacian, that is, a fractional root of the classical Laplace operator. This setting often surfaces from stochastic processes presenting jumps and it exhibits the classical spatial symmetries such as invariance under translations and rotations, plus a scale invariance of the integral kernel defining the operator. Differently from this, time-fractional diffusion is typically related to memory effects, and therefore it distinguishes very strongly between the "past" and the "future," and the arrow of time plays a major role (in particular, since the past influences the future, but not viceversa, time-fractional diffusion does not possess the same type of symmetries of the space-fractional one). In these pages, we will be able to consider operators which arise as superpositions of both space- and time-fractional diffusion, possibly taking into account classical derivatives as well (the cases of diffusion which is fractional just in either space or time are comprised as special situations of our general framework). Interestingly, we will also consider fractional operators of any order, showing, in a sense, that some properties related to fractional diffusion persist also when higher order operators come into play, differently from what happens in the classical case, in which the theory available for the Laplacian operator presents significant differences with respect to the case of polyharmonic operators.

To achieve the original result presented here, we develop a broad theory of some fundamental facts about space- and time-fractional equations. Some of these additional results are known from the literature, at least in some particular cases, but some other are new and interesting in themselves, and, in developing these auxiliary theories, this monograph presents a completely self-contained approach to a number of basic questions, such as:

- boundary behavior for the time-fractional eigenfunctions;
- boundary behavior for the time-fractional harmonic functions;
- Green representation formulas;
- existence and regularity for the first eigenfunction of the (possibly higher order) fractional Laplacian;
- boundary asymptotics of the first eigenfunctions of the (possibly higher order) fractional Laplacian;
- boundary behavior of (possibly higher order) fractional harmonic functions.

We now dive into the technical details of this matter.