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4 Reduced basis methods

Abstract: In this chapter we describe the reduced basis (RB) method for parameterized partial differential equations (PDEs). We first describe the motivation for RB methods in the many-query and real-time contexts and the associated offline-online computational paradigm. We next introduce the framework for parameterized PDEs and the associated theoretical rationale for reduction. We then turn to projection techniques: formulation, a priori and a posteriori error estimation, and offline-online computational strategies. We next discuss techniques for identification of optimal approximation spaces, in particular the weak greedy approach. We emphasize linear elliptic PDEs, but we also consider nonlinear elliptic PDEs as well as linear parabolic PDEs.

Keywords: weak greedy sampling, empirical interpolation method, Galerkin projection, a posteriori error estimation, offline-online procedure

MSC 2010: 65M60, 65N30

4.1 Motivation

Parameterized partial differential equations (PDEs) are important in many scientific and engineering applications. The parameters typically characterize the spatial domain, the boundary conditions and initial conditions and sources, and the coefficients associated with the underlying constitutive relations. In general the solution of our (say, elliptic) PDE shall be a parameterized field: For given parameter value $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_p) \in \mathcal{P}$, $u(\boldsymbol{\mu}) \in V$; here $\mathcal{P} \in \mathbb{R}^p$ is a compact parameter domain, and V is the appropriate function space associated with our PDE. In what follows, we assume that V is a Hilbert space. In the forward context we prescribe $\boldsymbol{\mu}$ to deduce $u(\boldsymbol{\mu})$; in the inverse context, such as parameter estimation, classification, and optimization, we deduce $\boldsymbol{\mu}$ from functionals applied to $u(\boldsymbol{\mu})$.

We introduce the parametric manifold $\mathcal{M} \equiv \{u(\boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \mathcal{P}\}$. The premise for parameterized model order reduction is well established [28, 1]: For the approximation of the solution $u(\boldsymbol{\mu})$ for many $\boldsymbol{\mu} \in \mathcal{P}$, we need not necessarily consider a finite element (FE) approximation space $V_h \subset V$ which can well represent *any* function in V; we need only

Acknowledgement: YM gratefully acknowledges the financial support provided by Institut Universitaire de France. ATP gratefully acknowledges the financial support provided by ONR Grant N00014-17-1-2077.

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https://doi.org/10.1515/9783110671490-004

consider a reduced basis (RB) approximation space $V_N \subset V_h$ which can well represent any function in $\mathcal{M} \subset V$ in the sense that $\text{dist}(\mathcal{M}, V_N)$ is small. We may thus anticipate that the dimension of V_N, N, will be much smaller than the dimension of V_h, N_h, with attendant reductions in computational cost: A query to the RB approximation, $\boldsymbol{\mu} \in \mathcal{P} \mapsto u_N(\boldsymbol{\mu}) \in V_N$, will be much less expensive than a query to the FE approximation, $\boldsymbol{\mu} \in \mathcal{P} \mapsto u_h(\boldsymbol{\mu}) \in V_h$. In actual fact, not all manifolds are "reducible," where reducible here is defined as a sufficiently rapid decrease of the Kolmogorov N-width [30, 23]; we shall provide some a priori and a posteriori tests to confirm the latter. Furthermore, given a reducible manifold, the identification of a good RB space V_N – such that $\text{dist}(\mathcal{M}, V_N)$ also decreases rapidly with N – requires considerable computational effort, and in particular many appeals to the FE approximation.

Parameterized model order reduction thus proceeds in two stages: in the offline stage, given our parameter domain $\mathcal{P}$ and parameterized PDE, we construct a sequence of parameter-independent spaces $\{V_N\}_{N=1,\ldots,N_{\max}}$; in the online stage, for given N, we query the RB approximation, $\boldsymbol{\mu} \in \mathcal{P} \mapsto u_N(\boldsymbol{\mu}) \in V_N$. This offline-online paradigm is computationally relevant if (i) the offline effort to construct V_N can be justified either by a real-time or many-query context, and (ii) the online effort to evaluate $\boldsymbol{\mu} \mapsto u_N(\boldsymbol{\mu})$ is indeed much less than the online effort to evaluate $\boldsymbol{\mu} \mapsto u_h(\boldsymbol{\mu})$. We elaborate on (i) and (ii).

(i) In the real-time context, we simply choose to "write off" the offline effort given the stated premium on rapid response in the online (deployed) stage. In the many-query context, we explicitly amortize the offline effort over many online RB queries.

(ii) We will typically require not only $N_{\max} \ll N_h$ but also special structural properties of the parameterized PDE and associated solution procedures: This special structure is often realized through an empirical interpolation method (EIM) [5] which introduces additional "variational-crime" errors; we shall denote the resulting FE and RB approximations by $\tilde{u}_h(\boldsymbol{\mu}) \in V_h$ and $\tilde{u}_N(\boldsymbol{\mu}) \in V_N$, respectively.

Note from an applications perspective we proceed not from model order reduction to context, but from context to model order reduction: A real-time or many-query application justifies an offline-online computational strategy which in turn can be realized through (among other strategies) model order reduction.

We briefly discuss the choice of the parameter dimensionality, p, and parameter domain, $\mathcal{P}$. We may consider as a first proposal all parameters of possible interest, p_0 large, and a parameter domain $\mathcal{P}_0$ which contains all values of $\boldsymbol{\mu}_0$ for which our PDE is well-posed (in the sense to be described below). However, in the context of model order reduction, the offline and online computational cost will depend on the number of parameters and also the extent and "shape" of the parameter domain, and hence we must typically accept $p < p_0$ and hence $\mathcal{P} \subset \mathcal{P}_0$: the parameterization and parameter domain must be chosen to anticipate the parameter values of ultimate interest in the online applications to be considered.

We provide a roadmap of the chapter. In general, we emphasize general (second-order) linear elliptic PDEs, but we also consider a nonlinear elliptic PDE as well as general linear parabolic PDEs; extension to nonlinear parabolic PDEs is then immediate. In Section 4.2 we formulate our parameterized PDEs and summarize the associated theoretical foundation for dimension reduction: conditions, or at least guidelines, under which a manifold is reducible. In Section 4.3 we develop the projection method, in fact simple Galerkin projection, by which we determine $\tilde{u}_N(\boldsymbol{\mu}) \in V_N$; we also provide a priori error estimates and a posteriori error estimators, and we describe the associated offline-online computational procedures. In Section 4.4 we describe methods for construction of the RB approximation space, V_N, with emphasis on the weak greedy procedure: The weak greedy procedure efficiently identifies a parameter sample $S_{N_{\max}} \equiv \{\boldsymbol{\mu}^j\}_{j=1,\ldots,N_{\max}}$ (from a rich train set $\Xi_{\mathrm{RB}} \subset \mathcal{P}$) to form hierarchical RB spaces $V_N \equiv \mathrm{span}\{\tilde{u}_h(\boldsymbol{\mu}^j)\}_{j=1,\ldots,N}$, $1 \le N \le N_{\max}$, which well represent the parametric manifold $\mathcal{M}$; the weak greedy procedure, and in particular the rate of decrease of $\mathrm{dist}(\mathcal{M}, V_N)$ with N, is a constructive test of reducibility.

The prerequisite for this chapter is experience in the formulation, elementary theory, and implementation of FE methods for PDEs, as well as some exposure to associated functional analysis. The intended audience is graduate students and professionals who wish to consider RB methods in their research or design efforts. The chapter emphasizes (i) the conditions and hypotheses under which RB methods may prove fruitful, (ii) the fundamental ingredients and procedures which must be incorporated in any RB formulation, and (iii) the underlying error analysis, a priori and a posteriori, which informs successful RB practice. We focus on the "inputs" – related to the particular PDE, parameter domain, and context – which must be provided by the prospective user, and on methods which can be generally and easily implemented given a standard FE foundation.

Finally, for readers who seek further details, a broader range of alternative techniques, more general classes of problems, and deeper coverage of both theory and implementation, we recommend two recent research monographs on RB methods [21, 32]. We hope our chapter here can serve as a portal to further study.

4.2 Parameterized PDEs

In Section 4.2.1 and Section 4.2.2 we consider linear elliptic PDEs. In Section 4.2.3 we consider parabolic PDEs as well as nonlinear elliptic PDEs.

4.2.1 Weak form

4.2.1.1 Formulation

We introduce a spatial domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$; we denote a point in Ω as $x \equiv (x_1, \ldots, x_d)$. We then define the space V as $V \equiv \{v \in H^1(\Omega) \,|\, v|_{\Gamma_D} = 0\}$ for Γ_D a nonempty portion of the boundary $\partial\Omega$; we denote the inner product and induced norm associated to V as $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, respectively. Unless otherwise noted, we shall take for our inner product

$$(w, v)_V \equiv \int_\Omega \nabla w \cdot \nabla v + c_{L^2}\, w\, v, \tag{4.1}$$

for c_{L^2} a nonnegative real number. We further introduce the dual space to V, V', of linear functionals continuous with respect to $\|\cdot\|_V$; we equip V' with the usual dual norm,

$$\|g\|_{V'} = \sup_{v \in V} \frac{|g(v)|}{\|v\|_V}, \quad \forall g \in V'. \tag{4.2}$$

We also define the Riesz representation of any g in V', $\mathcal{R}g \in V$, by

$$(\mathcal{R}g, v)_V = g(v), \quad \forall v \in V. \tag{4.3}$$

Finally, we recall that

$$\|g\|_{V'} = \|\mathcal{R}g\|_V, \quad \forall g \in V', \tag{4.4}$$

which follows directly from (4.2), (4.3), and the Cauchy–Schwarz inequality.

We now introduce the parameterized linear forms $\boldsymbol{\mu} \in \mathcal{P} \mapsto f(\cdot;\boldsymbol{\mu})$ and $\boldsymbol{\mu} \in \mathcal{P} \mapsto \ell(\cdot;\boldsymbol{\mu})$. In fact, for simplicity of exposition, we shall assume that

$$f(v;\boldsymbol{\mu}) = \int_\Omega f_\Omega(\cdot;\boldsymbol{\mu})\, v + \int_{\Gamma_{N,R}} f_{\Gamma_{N,R}}(\cdot;\boldsymbol{\mu})\, v, \;\; \ell(v;\boldsymbol{\mu}) = \int_\Omega \ell_\Omega(\cdot;\boldsymbol{\mu})\, v + \int_{\Gamma_{N,R}} \ell_{\Gamma_{N,R}}(\cdot;\boldsymbol{\mu})\, v, \tag{4.5}$$

where $f_\Omega(\boldsymbol{\mu}) \in L^2(\Omega)$, $\ell_\Omega(\boldsymbol{\mu}) \in L^2(\Omega)$, $f_{\Gamma_{N,R}}(\boldsymbol{\mu}) \in L^2(\Gamma_{N,R})$, and $\ell_{\Gamma_{N,R}}(\boldsymbol{\mu}) \in L^2(\Gamma_{N,R})$. Here $\Gamma_{N,R} \equiv \partial\Omega \setminus \bar{\Gamma}_D$ is the portion of the boundary on which non-Dirichlet (Neumann or Robin) boundary conditions are applied. It follows from our assumptions that $f(\cdot;\boldsymbol{\mu})$ and $\ell(\cdot;\boldsymbol{\mu})$ are continuous for all $\boldsymbol{\mu} \in \mathcal{P}$.

We further introduce the parameterized bilinear form $\boldsymbol{\mu} \in \mathcal{P} \mapsto a(\cdot,\cdot;\boldsymbol{\mu}) : V \times V \to \mathbb{R}$. In general, a may take the form

$$a(w, v;\boldsymbol{\mu}) = \sum_{i,j=0}^{d} \int_\Omega \Upsilon_{ij}(\cdot;\boldsymbol{\mu}) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j}, \tag{4.6}$$

for $\Upsilon_{ij} \in L^\infty(\mathcal{P}; L^\infty(\Omega))$ and (for convenience) $\partial./\partial x_0 \equiv \mathrm{Id}$ (the identity operator); note that a can also include a contribution such as $\int_{\Gamma_{N,R}} \overline{\Upsilon}(\cdot;\boldsymbol{\mu})\, wv$ with $\overline{\Upsilon} \in L^\infty(\mathcal{P}; L^\infty(\Gamma_{N,R}))$. We shall need several constants to characterize our bilinear form a: For any $\boldsymbol{\mu} \in \mathcal{P}$, the coercivity constant, $\alpha(\boldsymbol{\mu})$, continuity constant, $\gamma(\boldsymbol{\mu})$, and inf-sup constant, $\beta(\boldsymbol{\mu})$, are given respectively by

$$\alpha(\boldsymbol{\mu}) \equiv \inf_{w\in V} \frac{|a(w,w;\boldsymbol{\mu})|}{(w,w)_V}, \quad \gamma(\boldsymbol{\mu}) \equiv \sup_{w\in V}\sup_{v\in V} \frac{|a(w,v;\boldsymbol{\mu})|}{\|w\|_V \|v\|_V}, \quad \beta(\boldsymbol{\mu}) \equiv \inf_{w\in V}\sup_{v\in V} \frac{|a(w,v;\boldsymbol{\mu})|}{\|w\|_V \|v\|_V}; \tag{4.7}$$

we denote by $\alpha = \min_{\boldsymbol{\mu}\in\mathcal{P}} \alpha(\boldsymbol{\mu})$, $\gamma = \max_{\boldsymbol{\mu}\in\mathcal{P}} \gamma(\boldsymbol{\mu})$, and $\beta = \min_{\boldsymbol{\mu}\in\mathcal{P}} \beta(\boldsymbol{\mu})$ the corresponding worst case quantities over the entire parameter domain. For economy of presentation we define

$$c^{\mathrm{s}}(\boldsymbol{\mu}) \equiv \begin{cases} \alpha(\boldsymbol{\mu}) & \text{for the coercive case,} \\ \beta(\boldsymbol{\mu}) & \text{for the noncoercive case;} \end{cases} \tag{4.8}$$

we may also write $c^{\mathrm{s}}(\boldsymbol{\mu}) = \max(\alpha(\boldsymbol{\mu}), \beta(\boldsymbol{\mu}))$, however for computational purposes we prefer the more explicit definition (4.8). We can then state our hypotheses on a: in the coercive case, α is positive and γ is finite, and in the noncoercive case, β is positive and γ is finite; more succinctly, we require c^{s} positive and γ finite.

We now define the weak form of our parameterized PDE: Given $\boldsymbol{\mu} \in \mathcal{P}$, find a (or the) field $u(\boldsymbol{\mu}) \in V$ such that

$$a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v;\boldsymbol{\mu}), \quad \forall v \in V, \tag{4.9}$$

and evaluate the scalar output $s(\boldsymbol{\mu}) \in \mathbb{R}$ as $s(\boldsymbol{\mu}) = \ell(u(\boldsymbol{\mu});\boldsymbol{\mu})$.[1] (We implicitly assume that all inhomogeneous essential boundary conditions $u_D(\boldsymbol{\mu}) \in H^{1/2}(\Gamma_D)$ are lifted and hence implicitly incorporated in $f(\cdot;\boldsymbol{\mu})$.) It follows from our hypotheses on a and f and the Lions–Lax–Milgram–Babuška theorem that (4.9) admits a unique solution for all $\boldsymbol{\mu} \in \mathcal{P}$; furthermore, from our hypothesis on ℓ, $s(\boldsymbol{\mu})$ is finite for all $\boldsymbol{\mu} \in \mathcal{P}$. (In fact, for the noncoercive case, we require a third condition on a, typically satisfied in our context.)

We provide a simple illustration, which we denote Example 1.0. We consider a connected open domain $\Omega \subset \mathbb{R}^{d=3}$ and further assume that Ω is decomposed as the union of two nonoverlapping open subdomains, $\Omega_{(1)}$ and $\Omega_{(2)}$: $\overline{\Omega} = \overline{\Omega}_{(1)} \cup \overline{\Omega}_{(2)}$ and $\Omega_{(1)} \cap \Omega_{(2)} = \emptyset$. We set $p = 3$ and introduce parameter $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \mu_3) \in \mathcal{P} \subset \{v \in \mathbb{R}^3 \mid v_1 > 0, v_2 \geq 0\}$. We then define

$$a(w,v;\boldsymbol{\mu}) = \int_{\Omega_{(1)}} \mu_1 \nabla w \cdot \nabla v + \int_{\Omega_{(2)}} \nabla w \cdot \nabla v + \int_{\Omega} \mu_2 wv, \quad \forall w, v \in V^2, \tag{4.10}$$

1 We may associate to our output functional a dual problem and corresponding adjoint; the latter can serve (for example) in the development of improved RB output approximation and error estimation [25]. In the interest of space, we consider only simpler primal-only approximation.

as well as

$$f(v;\mu) = \int_{\Omega} (1+\mu_3 x_1)v \quad \text{and} \quad \ell(v;\mu) = \int_{\Gamma_{N,R}} v. \tag{4.11}$$

We can readily demonstrate that a is continuous and coercive, and furthermore symmetric, and that f and ℓ are continuous. (We can also include a convection term: For a divergence-free convection velocity which is furthermore outward on $\Gamma_{N,R}$, the bilinear form a remains continuous and coercive but will no longer be symmetric.) In the case in which Ω is polyhedral we would expect $u(\boldsymbol{\mu}) \in H^{1+\sigma_{\text{regularity}}}$ for $\sigma_{\text{regularity}} > 0$; the latter would limit the convergence rate of the FE approximation, but need not limit the convergence rate of the RB approximation.

We comment briefly on the treatment of parameter-dependent geometry. In our exposition we shall consider only the case in which Ω is independent of $\boldsymbol{\mu}$. In actual practice, we may treat problems in parameter-dependent geometry, $\boldsymbol{\mu} \in \mathcal{P} \mapsto \Omega_{\text{orig}}(\boldsymbol{\mu})$. In that case we introduce $V_{\text{orig}}(\boldsymbol{\mu}) = H^1(\Omega_{\text{orig}}(\boldsymbol{\mu}))$ and bilinear and linear form $\boldsymbol{\mu} \in \mathcal{P} \mapsto a_{\text{orig}}(\cdot,\cdot;\boldsymbol{\mu}) : V_{\text{orig}}(\boldsymbol{\mu}) \times V_{\text{orig}}(\boldsymbol{\mu}) \to \mathbb{R}$ and $\boldsymbol{\mu} \in \mathcal{P} \mapsto f_{\text{orig}}(\cdot;\boldsymbol{\mu}) : V_{\text{orig}}(\boldsymbol{\mu}) \to \mathbb{R}$, respectively; we then seek $u_{\text{orig}}(\mu) \in V_{\text{orig}}(\mu)$ solution of $a_{\text{orig}}(u_{\text{orig}}(\boldsymbol{\mu}), v;\boldsymbol{\mu}) = f_{\text{orig}}(v;\boldsymbol{\mu}), \forall v \in V_{\text{orig}}(\boldsymbol{\mu})$. RB methods will rely on some similarity of solutions on the parametric manifold: We thus map Ω_{orig} to a parameter-independent reference domain Ω, $\boldsymbol{\mu} \in \mathcal{P} \mapsto \mathcal{T}(\cdot;\boldsymbol{\mu}) : \Omega \to \Omega_{\text{orig}}(\boldsymbol{\mu})$; a variety of RB-relevant mapping procedures are described in Chapter 1 of this volume (Volume II) of this handbook. We thereby arrive at the statement (4.9) for $u(\boldsymbol{\mu}) = u_{\text{orig}}(\cdot;\boldsymbol{\mu}) \circ \mathcal{T}(\cdot;\boldsymbol{\mu})$, which is then the point of departure for the RB formulation. Note for the case of parameter-dependent geometry the functions $\Upsilon_{ij}(\cdot;\boldsymbol{\mu}), 0 \le i,j \le d$, of (4.6) will include the usual transformation terms associated with the Jacobian of the mapping function $\mathcal{T}(\cdot;\boldsymbol{\mu})$.

4.2.1.2 FE approximation

In general, $\tilde{u}_N(\boldsymbol{\mu})$ approximates $u(\boldsymbol{\mu})$, but typically we must construct the RB approximation through a computable intermediary or "surrogate"; in the context of this chapter, and quite often in practice, the latter takes the form of an underlying finite element (FE) approximation. Towards that end, we introduce a conforming FE space $V_h \subset V$ of dimension N_h. We shall choose V_h such that, for any $\boldsymbol{\mu} \in \mathcal{P}$ (say), $\|u(\boldsymbol{\mu}) - u_h(\boldsymbol{\mu})\|_V \le \text{tol}_V/2$, where tol_V is the prescribed error tolerance; we will then subsequently require $\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V \le \text{tol}_V/2$ to ensure $\|u(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V \le \text{tol}_V$.

We shall require for purposes of our subsequent RB approximation that V_h is independent of $\boldsymbol{\mu}$. As we shall see, the RB online cost is largely independent of N_h, however the offline cost will indeed depend on N_h, and hence we may be conservative but not profligate in the design of the FE approximation space. We might construct V_h as follows: Consider a representative sequence of parameter values $\Xi_{\text{FE}} \equiv \{\boldsymbol{\mu}^i_{\text{FE}} \in \mathcal{P}\}_{i=1,\ldots,K_{\text{FE}}}$;

initialize the FE approximation for parameter value $\boldsymbol{\mu}^i_{\rm FE}$, V^i_h, as V^{i-1}_h (and for V^1_h choose an initial uniform coarse mesh) and adaptively refine to the desired error tolerance $\text{tol}_V/2$; set $V_h = V^{K_{\rm FE}}_h$. We further introduce the dual space to V_h, V'_h, of linear functionals continuous with respect to $\|\cdot\|_V$ for all functions in V_h; we equip V'_h with dual norm

$$\|g_h\|_{V'_h} = \sup_{v\in V_h} \frac{|g_h(v)|}{\|v\|_{V_h}}. \tag{4.12}$$

We may then define the Riesz representation of any g_h in V'_h, $\mathcal{R}_h g_h \in V_h$, by

$$(\mathcal{R}_h g_h, v)_V = g_h(v), \quad \forall v \in V_h, \tag{4.13}$$

in terms of which we can evaluate the dual norm from

$$\|g_h\|_{V'_h} = \|\mathcal{R}_h g_h\|_V. \tag{4.14}$$

We note that (4.13) is a finite-dimensional problem.

We now define the (continuous) Galerkin-FE approximation: Given $\boldsymbol{\mu} \in \mathcal{P}$, find $u_h(\boldsymbol{\mu}) \in V_h$ such that

$$a(u_h(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}), \quad \forall v \in V_h, \tag{4.15}$$

and evaluate the scalar output $s_h(\boldsymbol{\mu}) \in \mathbb{R}$ as $s_h(\boldsymbol{\mu}) = \ell(u_h(\boldsymbol{\mu}); \boldsymbol{\mu})$. We denote the FE version of the constants in (4.7) – with V replaced by V_h – by subscript h: for all $\mu \in \mathcal{P}$, $\alpha(\boldsymbol{\mu}) \le \alpha_h(\boldsymbol{\mu}) \le \gamma_h(\boldsymbol{\mu}) \le \gamma(\boldsymbol{\mu})$; however, in general $\beta_h(\boldsymbol{\mu}) \not> \beta(\boldsymbol{\mu})$, and thus in the noncoercive case we include the additional hypothesis (on V_h) $\beta_h(\boldsymbol{\mu}) > 0$, in order to ensure well-posedness of the FE approximation. We may then also define the corresponding worst case constants (over $\mathcal{P}$) as α_h, γ_h, and β_h. Finally, we introduce $c^{\rm s}_h(\boldsymbol{\mu}) \equiv \max(\alpha_h(\boldsymbol{\mu}), \beta_h(\boldsymbol{\mu}))$ and corresponding worst case (minimum over $\mathcal{P}$) constant $c^{\rm s}_h$.

We next represent V_h by a nodal basis $\{\varphi^j\}_{j=1}^{N_h}$ associated to nodes $\{x^{\rm node}_j \in \mathbb{R}^d\}_{j=1,\ldots,N_h}$; given any function $v_h \in V_h$, we shall denote by $\mathbf{v}_h \in \mathbb{R}^{N_h}$ the corresponding vector of (nodal) basis coefficients. The FE discrete equations now directly follow from (4.15) and our basis for V_h: Given $\boldsymbol{\mu} \in \mathcal{P}$, find $\mathbf{u}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h}$ such that

$$\mathbb{A}_h(\boldsymbol{\mu})\mathbf{u}_h(\boldsymbol{\mu}) = \mathbf{f}_h(\boldsymbol{\mu}), \tag{4.16}$$

and evaluate the scalar output $s_h(\boldsymbol{\mu}) \in \mathbb{R}$ as $s_h(\boldsymbol{\mu}) = \boldsymbol{\ell}^{\rm T}_h(\boldsymbol{\mu})\mathbf{u}_h(\boldsymbol{\mu})$ (for $^{\rm T}$ the transpose operator). Here $\mathbb{A}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h \times N_h}$, $\mathbf{f}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h}$, and $\boldsymbol{\ell}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h}$ are given by

$$(\mathbb{A}_h(\boldsymbol{\mu}))_{ij} = a(\varphi^j, \varphi^i; \boldsymbol{\mu}), \quad (\mathbf{f}_h(\boldsymbol{\mu}))_i = f(\varphi^i; \boldsymbol{\mu}), \quad (\boldsymbol{\ell}_h(\boldsymbol{\mu}))_i = \ell(\varphi^i; \boldsymbol{\mu}), \quad 1 \le i, j \le N_h; \tag{4.17}$$

note that $\mathbb{A}_h$ and $\mathbf{f}_h$ are the FE stiffness and load vector, respectively.

For future reference we also introduce the parameter-independent inner-product matrix $\mathbb{X}_h \in \mathbb{R}^{N_h \times N_h}$,

$$(\mathbb{X}_h)_{ij} = (\varphi^j, \varphi^i)_V, \quad 1 \le i, j \le N_h. \tag{4.18}$$

We note that for any $w_h \in V_h$, $v_h \in V_h$, $(w_h, v_h)_V = \mathbf{w}_h^{\mathrm{T}} \mathbb{X}_h \mathbf{v}_\mathbf{h}$. Furthermore, it follows from (4.13) that for any $g_h \in V_h'$ the FE basis coefficients of the Riesz representation $\mathcal{R}_h g_h$ are given by $\mathbb{X}_h^{-1} \mathbf{g}_h$ for $(\mathbf{g}_h)_i = g_h(\varphi^i)$, $1 \le i \le N_h$. It follows from (4.14) that the dual norm $\|g_h\|_{V_h'}$ may be evaluated as

$$\left(\mathbf{g}_h^{\mathrm{T}} \mathbb{X}_h^{-1} \mathbf{g}_h\right)^{1/2}; \tag{4.19}$$

note that we require only the action of $\mathbb{X}_h^{-1}$, which might be effected (in the direct context) through Cholesky decomposition and subsequent forward/back substitution.

In actual practice the FE matrices and vectors are formed by numerical quadrature: The integral of (4.6) is replaced by a corresponding sum with quadrature points and weights

$$x_j^{\mathrm{quad},\Omega} \in \Omega, \quad \rho_j^{\mathrm{quad},\Omega} \in \mathbb{R}_+, \quad j = 1, \ldots, N_h^{\mathrm{quad},\Omega}, \tag{4.20}$$

and the integrals of (4.5) are replaced by corresponding sums with quadrature points and weights

$$x_j^{\mathrm{quad},\Gamma_{N,R}} \in \Gamma_{N,R}, \quad \rho_j^{\mathrm{quad},\Gamma_{N,R}} \in \mathbb{R}_+, \quad j = 1, \ldots, N_h^{\mathrm{quad},\Gamma_{N,R}}; \tag{4.21}$$

for simplicity of exposition, we shall presume that the error induced by quadrature is negligible relative to tol_V. The latter is plausible if h is sufficiently small and furthermore $\Upsilon_{i,j}$, $0 \le i, j \le d$, f_Ω, $f_{\Gamma_{N,R}}$, ℓ_Ω, and $\ell_{\Gamma_{N,R}}$ are sufficiently smooth. Note that in practice the FE quadrature is effected as a sum of elemental quadratures.

4.2.1.3 Affine parameter dependence

Affine dependence of the forms $\{a, f, \ell\}$ on the parameter $\boldsymbol{\mu}$ greatly reduces the computational complexity of the online stage of the RB method. We note that parameterized model order reduction can proceed without (appeal to) affine parameter dependence – but less effectively than if we can and do take advantage of affine parameter dependence. We thus wish either to confirm affine parameter dependence or alternatively, and more generally, to impose affine parameter dependence though approximate forms $\{\tilde{a}, \tilde{f}, \tilde{\ell}\} \approx \{a, f, \ell\}$; we pursue here the latter, which includes the former as a special case.

In particular, we introduce $\tilde{a}$, $\tilde{f}$, and $\tilde{\ell}$ which can be expressed, for all $\boldsymbol{\mu} \in \mathcal{P}$, as

$$\tilde{a}(w, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) a^q(w, v), \quad \tilde{f}(v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) f^q(v), \quad \tilde{\ell}(v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_\ell} \Theta_\ell^q(\boldsymbol{\mu}) \ell^q(v), \tag{4.22}$$

where $\Theta_a^q : \mathcal{P} \to \mathbb{R}$, $1 \le q \le Q_a$, $\Theta_f^q : \mathcal{P} \to \mathbb{R}$, $1 \le q \le Q_f$, and $\Theta_\ell^q : \mathcal{P} \to \mathbb{R}$, $1 \le q \le Q_\ell$, are suitably smooth functions, $a^q : V \times V \to \mathbb{R}$, $1 \le q \le Q_a$, are parameter-independent continuous bilinear forms, and $f^q : V \to \mathbb{R}$, $1 \le q \le Q_f$, $\ell^q : V \to \mathbb{R}$, $1 \le q \le Q_\ell$, are parameter-independent continuous linear forms; we assume the forms are linearly independent.

Given $\boldsymbol{\mu} \in \mathcal{P}$, we now seek $\tilde{u}(\boldsymbol{\mu})$ such that

$$\tilde{a}(\tilde{u}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tilde{f}(v; \boldsymbol{\mu}), \quad \forall v \in V, \tag{4.23}$$

and evaluate the scalar output $\tilde{s}(\boldsymbol{\mu}) \in \mathbb{R}$ as $\tilde{s}(\boldsymbol{\mu}) = \tilde{\ell}(\tilde{u}(\boldsymbol{\mu}); \boldsymbol{\mu})$. We also define the corresponding FE approximation: Find $\tilde{u}_h(\boldsymbol{\mu}) \in V_h$ such that

$$\tilde{a}(\tilde{u}_h(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tilde{f}(v; \boldsymbol{\mu}), \quad \forall v \in V_h, \tag{4.24}$$

and evaluate the scalar output $\tilde{s}_h(\boldsymbol{\mu}) \in \mathbb{R}$ as $\tilde{s}_h(\boldsymbol{\mu}) = \tilde{\ell}(\tilde{u}_h(\boldsymbol{\mu}); \boldsymbol{\mu})$. We denote the FE stability and continuity constants (hence over V_h) associated to $\tilde{a}$ by $\tilde{\alpha}_h(\boldsymbol{\mu})$, $\tilde{\gamma}_h(\boldsymbol{\mu})$, and $\tilde{\beta}_h(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathcal{P}$; we may then also define the corresponding worst case constants (over $\mathcal{P}$) as $\tilde{\alpha}_h$, $\tilde{\gamma}_h$, and $\tilde{\beta}_h$. Finally, we introduce $\tilde{c}_h^{\mathrm{s}}(\boldsymbol{\mu}) \equiv \max(\tilde{\alpha}_h(\boldsymbol{\mu}), \tilde{\beta}_h(\boldsymbol{\mu}))$ and the corresponding worst case (minimum over $\mathcal{P}$) constant $\tilde{c}_h^{\mathrm{s}}$. We shall shortly provide a perturbation result for $\tilde{c}_h^{\mathrm{s}}$.

The discrete FE equations now read

$$\tilde{\mathbb{A}}_h(\boldsymbol{\mu}) \tilde{\boldsymbol{u}}_h(\boldsymbol{\mu}) = \tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}) \tag{4.25}$$

and $\tilde{s}_h(\boldsymbol{\mu}) = \tilde{\boldsymbol{\ell}}_h^{\mathrm{T}}(\boldsymbol{\mu}) \tilde{\boldsymbol{u}}_h(\boldsymbol{\mu})$, where $\tilde{\mathbb{A}}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h \times N_h}$, $\tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h}$, and $\tilde{\boldsymbol{\ell}}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h}$ are given by

$$(\tilde{\mathbb{A}}_h(\boldsymbol{\mu}))_{ij} = \tilde{a}(\varphi^j, \varphi^i; \boldsymbol{\mu}), \quad (\tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}))_i = \tilde{f}(\varphi^i; \boldsymbol{\mu}), \quad (\tilde{\boldsymbol{\ell}}_h(\boldsymbol{\mu}))_i = \tilde{\ell}(\varphi^i; \boldsymbol{\mu}), \quad 1 \le i, j \le N_h. \tag{4.26}$$

We further note from (4.22) and (4.26) that

$$\tilde{\mathbb{A}}_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) \mathbb{A}_h^q, \quad \tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) \boldsymbol{f}_h^q, \quad \tilde{\boldsymbol{\ell}}_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_\ell} \Theta_\ell^q(\boldsymbol{\mu}) \boldsymbol{\ell}_h^q, \tag{4.27}$$

for parameter-independent $\mathbb{A}_h^q \in \mathbb{R}^{N_h \times N_h}$, $1 \le q \le Q_a$, $\boldsymbol{f}_h^q \in \mathbb{R}^{N_h}$, $1 \le q \le Q_f$, and $\boldsymbol{\ell}_h^q \in \mathbb{R}^{N_h}$, $1 \le q \le Q_\ell$; for example, $(\mathbb{A}_h^q)_{ij} = a^q(\varphi^j, \varphi^i)$, $1 \le i, j \le N_h$, $1 \le q \le Q_a$.

We must now seek affine approximations (4.22) such that $\{\tilde{a}, \tilde{f}, \tilde{\ell}\}$ is sufficiently close to $\{a, f, \ell\}$, and hence $\tilde{u}$ and $\tilde{u}_h$ are sufficiently close to u and u_h, respectively. Towards that end, we first introduce (a restricted form of) the EIM [5]. We are given integer $N_{\text{EIM}} \geq 1$ (in practice, large) and a function $\mathbf{g} : \mathcal{P} \to \mathbb{R}^{N_{\text{EIM}}}$; we further define a train parameter sample of size K_{EIM}, $\Xi_{\text{EIM}} \equiv \{\boldsymbol{\mu}^i \in \mathcal{P}\}_{i=1,\ldots,K_{\text{EIM}}} (\subset \mathcal{P})$, and generate the associated snapshot set $G_{\text{EIM}} \equiv \{\mathbf{g}(\boldsymbol{\mu})\}_{\boldsymbol{\mu} \in \Xi_{\text{EIM}}}$. Lastly, we prescribe norm $\|\cdot\|_{\text{EIM}}$ and associated error tolerance tol_{EIM}.

- In the offline stage, we execute Algorithm 4.1 (presented in detail below) to obtain interpolation indices $\{i_m^* \in \{1, \ldots, N_{\text{EIM}}\}\}_{m=1,\ldots,M}$, associated interpolation vectors $\{\boldsymbol{\xi}^m \in \mathbb{R}^{N_{\text{EIM}}}\}_{m=1,\ldots,M}$, and a nonsingular lower triangular interpolation matrix $B_M \in \mathbb{R}^{M \times M}$.
- In the online stage, given $\boldsymbol{\mu} \in \mathcal{P}$, we approximate $\mathbf{g}(\boldsymbol{\mu})$ as

$$\tilde{\mathbf{g}}(\boldsymbol{\mu}) = \sum_{m=1}^{M} b_m(\boldsymbol{\mu}) \, \boldsymbol{\xi}^m, \tag{4.28}$$

where $b(\boldsymbol{\mu}) \in \mathbb{R}^M$ is the solution of $B_M b(\boldsymbol{\mu}) = \mathbf{g}^*(\boldsymbol{\mu})$ for $\mathbf{g}^*(\boldsymbol{\mu}) \in \mathbb{R}^M$ given by $(\mathbf{g}^*(\boldsymbol{\mu}))_m = (\mathbf{g}(\boldsymbol{\mu}))_{i_m^*}$, $1 \leq m \leq M$. For succinctness in the description of Algorithm 4.1 we define $\mathcal{I}_M : \mathbb{R}^{N_{\text{EIM}}} \to \mathbb{R}^{N_{\text{EIM}}}$ such that (4.28) reads $\tilde{\mathbf{g}}(\boldsymbol{\mu}) = \mathcal{I}_M \mathbf{g}(\boldsymbol{\mu})$ for $M \geq 1$; for $M = 0$ we set $\mathcal{I}_M \equiv 0$.

Algorithm 4.1: Empirical interpolation method (EIM): The EIM algorithm is of the "greedy" variety (also invoked in the identification of V_N, as discussed in Section 4.4.1): on line 4 we choose for the next parameter value the point in Ξ_{EIM} for which the current EIM approximation is worst; within the EIM context, this strategy in conjunction with line 5 also ensures a stable interpolation procedure. We assume that the set G_{EIM} is not embedded in a small finite-dimensional space.

Data: $N_{\text{EIM}}, \Xi_{\text{EIM}}, \mathbf{g} : \mathcal{P} \to \mathbb{R}^{N_{\text{EIM}}}$ (in fact, G_{EIM} suffices), $\|\cdot\|_{\text{EIM}}, \text{tol}_{\text{EIM}}$
Result: M, $\{i_m^* \in \{1, \ldots, N_{\text{EIM}}\}\}_{1 \leq m \leq M}$, $\{\boldsymbol{\xi}^m \in \mathbb{R}^{N_{\text{EIM}}}\}_{1 \leq m \leq M}$, $B_M \in \mathbb{R}^{M \times M}$

1 Set $M = 0$ and err $= \infty$;
2 **while** err $>$ tol_{EIM} **do**
3 Set $M \leftarrow M + 1$;
4 Find $\boldsymbol{\mu}^* = \arg\sup_{\boldsymbol{\mu} \in \Xi_{\text{EIM}}} \|\mathbf{g}(\boldsymbol{\mu}) - \mathcal{I}_{M-1}\mathbf{g}(\boldsymbol{\mu})\|_{\text{EIM}}$;
5 Find $i_M^* = \arg\sup_{i \in \{1,\ldots,N_{\text{EIM}}\}} |(\mathbf{g}(\boldsymbol{\mu}^*) - \mathcal{I}_{M-1}\mathbf{g}(\boldsymbol{\mu}^*))_i|$;
6 Define $\boldsymbol{\xi}^M = (\mathbf{g}(\boldsymbol{\mu}^*) - \mathcal{I}_{M-1}\mathbf{g}(\boldsymbol{\mu}^*)) \,/\, (\mathbf{g}(\boldsymbol{\mu}^*) - \mathcal{I}_{M-1}\mathbf{g}(\boldsymbol{\mu}^*))_{i_M^*}$;
7 Update $(B_M)_{jk} = (\boldsymbol{\xi}^k)_{i_j^*}, 1 \leq j, k \leq M$;
8 Set err $= \|\mathbf{g}(\boldsymbol{\mu}^*) - \mathcal{I}_{M-1}g(\boldsymbol{\mu}^*)\|_{\text{EIM}}$;
9 **end**

By construction, for any $\boldsymbol{\mu} \in \Xi_{\text{EIM}}$, $\|\mathbf{g}(\boldsymbol{\mu}) - \tilde{\mathbf{g}}(\boldsymbol{\mu})\|_{\text{EIM}} \leq \text{tol}_{\text{EIM}}$. However, the EIM approximation may (and must, in practice) be applied to values of $\boldsymbol{\mu} \in \mathcal{P}$ which do not appear in the train parameter sample Ξ_{EIM}.[2] We note that in infinite precision we obtain $M \leq \dim(\text{span}\{G_{\text{EIM}}\})$.

We now apply the EIM to develop approximate affine forms for our PDE. We consider the coefficient function Υ_{00} of (the numerical quadrature version of) (4.6). We set $N_{\text{EIM}} \equiv N_h^{\text{quad},\Omega}$ (or $N_h^{\text{quad},\Gamma_{N,R}}$ for $f_{\Gamma_{N,R}}$, $\ell_{\Gamma_{N,R}}$), identify $(\mathbf{g}(\boldsymbol{\mu}))_i \equiv \Upsilon_{00}(x_i^{\text{quad},\Omega}, \boldsymbol{\mu})$, $1 \leq i \leq N_h^{\text{quad},\Omega}$, and choose $\|\cdot\|_{\text{EIM}} \equiv \|\cdot\|_{\ell^\infty}$. We then apply Algorithm 4.1 to obtain M, the interpolation indices $\{i_m^*\}_{1\leq m\leq M}$, interpolation vectors $\{\boldsymbol{\xi}^m\}_{1\leq m\leq M}$, and interpolation matrix B_M associated to the Υ_{00} contribution to the affine sum for the bilinear form $\tilde{a}$. (Each of the other coefficient functions Υ_{ij}, $0 \leq i,j \leq d$, and $(i,j) \neq (0,0)$, f_Ω, and ℓ_Ω is treated [separately] in the same fashion, as well as $f_{\Gamma_{N,R}}$ and $\ell_{\Gamma_{N,R}}$.) Note that the corresponding $\Theta_a^{\cdot}(\boldsymbol{\mu})$ correspond to the $b_{\cdot}(\boldsymbol{\mu})$ of (4.28) and are thus defined implicitly in terms of interpolation indices, vectors, and matrices; the corresponding $a^{\cdot}(w,v)$ are given by

$$\sum_{k=1}^{N_h^{\text{quad},\Omega}} (\boldsymbol{\xi}^{\cdot})_k \, \varphi^j(x_k^{\text{quad},\Omega}) \, \varphi^i(x_k^{\text{quad},\Omega}) \, \rho_k^{\text{quad},\Omega} \tag{4.29}$$

for quadrature weights and points defined by (4.20).

Assuming that each of the coefficient functions resides on a low-dimensional parametric manifold, we may anticipate that we can obtain a corresponding EIM approximation with relatively few terms, M small. For any given $\mathcal{D} \subset \mathcal{P}$, we define the error induced by the EIM approximation in our bilinear and linear forms as $\epsilon_{\text{EIM}}^{\mathcal{D}}$,

$$\epsilon_{\text{EIM}}^{\mathcal{D}} \equiv \sup_{\boldsymbol{\mu}\in\mathcal{D}} \max\Bigg(\sup_{w\in V, v\in V} \frac{|a(w,v;\boldsymbol{\mu}) - \tilde{a}(w,v;\boldsymbol{\mu})|}{\|w\|_V \|v\|_V}, \sup_{v\in V} \frac{|f(v;\boldsymbol{\mu}) - \tilde{f}(v;\boldsymbol{\mu})|}{\|v\|_V}, \sup_{v\in V} \frac{|\ell(v;\boldsymbol{\mu}) - \tilde{\ell}(v;\boldsymbol{\mu})|}{\|v\|_V} \Bigg). \tag{4.30}$$

For a of the form (4.6) application of the Cauchy–Schwarz inequality yields $\epsilon_{\text{EIM}}^{\Xi_{\text{EIM}}}$ as a function of tol_{EIM}; for example, in the absence of off-diagonal terms in (4.6), $\epsilon_{\text{EIM}}^{\Xi_{\text{EIM}}} = \max(1/c_{L^2}, 1)\,\text{tol}_{\text{EIM}}$. We can further argue that, for Ξ_{EIM} sufficiently rich, $\epsilon_{\text{EIM}}^{\mathcal{P}} \approx \epsilon_{\text{EIM}}^{\Xi_{\text{EIM}}}$; an adaptive procedure has been proposed in [27] to support this argument. Finally, it can readily be demonstrated that

$$\tilde{c}_h^{\text{s}} = c_h^{\text{s}} - \epsilon_{\text{EIM}}^{\mathcal{P}}, \tag{4.31}$$

2 In a similar fashion, in the most general EIM formulation, $\mathbf{g}(\boldsymbol{\mu})$ corresponds to the evaluation of a function $g : \Omega \times \mathcal{P} \to \mathbb{R}$ at a set of (here, quadrature) points in Ω; however, the resulting EIM approximation can then be applied for any $x \in \Omega$.

and also $\tilde{\gamma}_h \le \gamma_h + \epsilon_{\text{EIM}}^{\mathcal{P}}$; it follows that our EIM-perturbed FE problem is well-posed for $\epsilon_{\text{EIM}}^{\mathcal{P}}$ sufficiently small.

There is an alternative EIM approach to the development of the affine forms: the operator EIM (OEIM) [15]. In this case we would apply the EIM method directly (for $N_{\text{EIM}} = N_h$) to $\boldsymbol{f}_h(\boldsymbol{\mu})$, $\boldsymbol{\ell}_h(\boldsymbol{\mu})$ and (for $N_{\text{EIM}} = N_h^2$) to $\mathbf{A}_h \in \mathbb{R}^{N_h^2}$; the latter is the single-index (vector) form of the stiffness matrix $\mathbb{A}_h$, which is then repacked in double-index form once the OEIM is complete. The OEIM has the important advantage of nonintrusiveness: the affine approximation may be deduced solely from the FE stiffness matrix and load vector without any knowledge of the associated formation processes, thus permitting a general interface between FE code and the RB code. It is important to choose a norm $\|\cdot\|_{\text{EIM}}$ for the OEIM procedure to permit ultimate error control of the RB approximation in $\|\cdot\|_V$. For $\boldsymbol{f}_h$ and $\boldsymbol{\ell}_h$ we may choose $\|\boldsymbol{\delta g}\|_{\text{EIM}} = (\boldsymbol{\delta g}^{\text{T}} \mathbb{X}_h^{-1} \boldsymbol{\delta g})^{1/2}$, which can be reasonably rapidly estimated. For $\mathbf{A}_h$, the relevant norm can be evaluated as the square root of the maximum eigenvalue $\lambda_{\max}^{\boldsymbol{\delta A}_h}$ associated to the generalized SPD eigenproblem $\boldsymbol{\delta A}_h^{\text{T}} \mathbb{X}_h^{-1} \boldsymbol{\delta A}_h \chi^{\boldsymbol{\delta A}_h} = \lambda^{\boldsymbol{\delta A}_h} \mathbb{X}_h \chi^{\boldsymbol{\delta A}_h}$, hence somewhat cumbersome. For these norm choices we directly obtain $\epsilon_{\text{EIM}}^{\Xi_{\text{EIM}}} = \text{tol}_{\text{EIM}}$. In summary, the OEIM offers a very easily implemented procedure for the construction of affine approximations.

We briefly revisit Example 1.0, described by equations (4.9)–(4.11). We first note from inspection that we can directly choose $\{\tilde{a}, \tilde{f}, \tilde{\ell}\} = \{a, f, \ell\}$ to obtain an affine representation with $Q_a = 3$, $Q_f = 2$, $Q_\ell = 1$. For our particular example Υ_{00}, Υ_{11}, Υ_{22}, and Υ_{33} are nonzero, and hence the EIM procedure described – which treats each term in the expansion (4.6) separately – would yield $Q_a = 7$; a concatenated EIM – in which we treat all the Υ_{ij}, $0 \le i, j \le d$, within a single EIM – would recover $Q_a = 3$. The OEIM procedure, which treats the entire form, would directly recover $Q_a = 3$, $Q_f = 2$, $Q_\ell = 1$. It is often the case for problems in which the geometry does not depend on the parameter that $\{a, f, \ell\}$ admits an exact affine representation, $\epsilon_{\text{EIM}}^{\mathcal{P}} = 0$, with relatively few terms. However, in the presence of parameter-dependent geometry, and in particular nonaffine geometry transformations, $\{a, f, \ell\}$ will not admit an exact affine representation.

4.2.2 Justification for reduction

The fundamental hypothesis made on the parametric manifold $\mathcal{M} \equiv \{u(\boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \mathcal{P}\}$ introduced in the first section is its "reducibility" in the sense that there supposedly exist(s) some (series of) finite-dimensional space(s) V_N that approximate well $\mathcal{M}$ in the sense that, denoting by $\text{dist}(\mathcal{M}, V_N)$ the deviation of $\mathcal{M}$ from V_N, i. e.,

$$\text{dist}(\mathcal{M}, V_N) = \sup_{u(\boldsymbol{\mu})} \inf_{v_N \in V_N} \|u(\boldsymbol{\mu}) - v_N\|_V,$$

$\text{dist}(\mathcal{M}, V_N)$ is decreasing fast with N increasing. The question we want to raise here is: Why should it be so? And also, what is that (series of) finite-dimensional space(s) V_N?

This hypothesis is formally stated by going one step further in the definition of the deviation, i. e., introducing the quantity

$$d_N(\mathcal{M}, V) = \inf_{V_N,\ \dim V_N = N} \operatorname{dist}(\mathcal{M}, V_N), \tag{4.32}$$

which is known as the Kolmogorov N-width [30, 23] and represents the ability of $\mathcal{M}$ to be approximated by some optimally chosen vectorial space of dimension N.

As has already been remarked above, the "optimal" choice V_N depends on $\mathcal{M}$, and the optimal choice for $\mathcal{M}$ will not be valid for another set of functions.

This notion is the right one, indeed, when, e. g., $d_N(\mathcal{M}, V)$ goes to 0, like ρ^N with $0 < \rho < 1$ or even like cN^{-p} with p large enough (say, $p \geq 6$), then we are in a good shape, expecting that, for any $\boldsymbol{\mu} \in \mathcal{P}$, very few (well-chosen) degrees of freedom (the coefficient in some appropriate basis of V_N) will be sufficient to approximate well any $u(\boldsymbol{\mu})$.

We know that in some cases, e. g., a linear structure of the PDE, by superposition it is possible to check that $\mathcal{M}$ is finite-dimensional. Of course this is neither the generic case nor the case we are interested in. Then typically, regularity of the solutions with respect to the spatial variable may lead to propose, for every $\boldsymbol{\mu}$, a high-order (say, polynomial or spectral) approximation that converges rapidly, even exponentially (when the degree of the polynomial increases) and thus, X_N can be chosen as the set of polynomials of degree $\leq cN^{1/d}$. This will not be the optimal space but the optimal choice is better than the polynomial choice and thus the best polynomial fit provides an upper bound for the Kolmogorov N-width. However, the fact that polynomials approximate well any regular function, regardless of the property of the whole set $\mathcal{M}$, i. e., its structure, shape, coherence, makes it understandable that this "generic" choice cannot be the optimal one for $\mathcal{M}$, and that it may be much, much better.

As noted in [12], if the mapping $\boldsymbol{\mu} \in \mathcal{P} \mapsto u(\boldsymbol{\mu})$ is linear continuous, then the Kolmogorov N-width of $\mathcal{M}$ is upper bounded by a constant times the Kolmogorov N-width of $\mathcal{P}$. The generalization of this statement that is proposed in [12] is that if the previous mapping is holomorphic (meaning that $u(\boldsymbol{\mu})$ has a Fréchet derivative at any parameter $\boldsymbol{\mu}$ belonging to a compact set $K \subset \mathcal{P}$), then, for any $s > 1$ and $t < s - 1$,

$$\sup_{n \geq 1} n^s d_n(K, \mathcal{P}) < \infty \implies \sup_{n \geq 1} n^t d_n(u(K), V) < \infty, \tag{4.33}$$

where $u(K) = \{u(\boldsymbol{\mu}), \boldsymbol{\mu} \in K\}$. The loss of 1 (with respect to the linear case) may be not optimal but this result provides extra reasons for $d_N(\mathcal{M}, V)$ to be small when $u(\boldsymbol{\mu})$ is the solution to some parameter-dependent PDE. Indeed, it is well known that, by differentiating the PDE, $\nabla_{\boldsymbol{\mu}} u(\boldsymbol{\mu})$ is the solution to a similar PDE as $u(\boldsymbol{\mu})$, and thus under a reasonable hypothesis, this set satisfies the holomorphic assumption.

In order to explain faster rates of convergence, the first analysis we know of is [26], where exponential convergence was proven for a simple one-dimensional parameter space elliptic PDE. More recently a general extension was performed in [3], where, by

using, in a constructive way, low-rank tensor approximations for families of elliptic diffusion PDEs parameterized by the diffusion coefficients, the authors have been able to derive exponential convergence rates in a much more general framework.

A typical case where the Kolmogorov N-width is not small (at least in a straightforward manner) is when the problem is convection-dominated. Indeed, the set of solutions to a simple pure linear convection problem, when the velocity is among the parameters, is the initial solution, properly translated: If it is not regular, then the set of all solutions is of very large Kolmogorov N-width. There are ways to circumvent this (see, e. g., [10, 7]) but it is out of the scope of this contribution. Another case where, a priori, the Kolmogorov N-width is not small (but it is simple to fix) is for elliptic problems where the right-hand side contains pointwise singularities, the position of which may vary and is one parameter of the problem. The "trace" of these singularities of the right-hand side can be clearly "seen" on the solution $u(\boldsymbol{\mu})$ itself and this may lead also to a large Kolmogorov N-width. A simple postprocessing of the solution's manifold through a change of variable that maps the singularities' positions to a fixed reference position allows to better compare the solutions and check that indeed, when the singularities are sort of "aligned," the set of all (postprocessed) solutions is of small Kolmogorov N-width.

4.2.3 Extensions

4.2.3.1 Generalization of linear elliptic problems

The extension of our parameterized PDE formulation to vector fields is very simple. In particular, and if we assume that Dirichlet conditions at any point on the boundary are always applied to all components of the vector, we need only redefine $V^d \rightarrow V$. Thus linear elasticity readily falls into the framework considered in this chapter. It is interesting to emphasize here that the manifold of all vector fields (solutions to the problem of interest when the parameter varies) can be considered as a whole, which leads us to approximate a vector solution as a linear combination of vector (reduced) basis functions with scalar coefficients, thereby further decreasing the complexity of the online procedure. Note that we shall not consider here saddle problems: These mixed formulations – such as the incompressible Stokes equations or the equations of linear elasticity for Poisson ratio approaching 1/2 – require special RB treatment [34].

The extension of our parameterized PDE formulation to complex fields – "complexification" for short – is also relatively simple. We must change $\mathbb{R}$ to $\mathbb{C}$, interpret $|\cdot|$ as complex modulus, consider spaces V of complex-valued functions, conjugate the argument of all linear (now anti-linear) forms and the second argument of all inner products and bilinear (now sesquilinear) forms, and in our discrete representations replace transpose $^{\mathrm{T}}$ with Hermitian $^{\mathrm{H}}$. We provide a simple illustration, the Helmholtz problem of acoustics, which we shall denote Example 2.0.

We consider $\Omega \subset \mathbb{R}^{d=3}$. We set $p = 2$ and introduce parameter $\boldsymbol{\mu} \equiv (\mu_1, \mu_2) \in \mathcal{P} \subset \{\nu \in \mathbb{R}^2 \mid \nu_1 \geq 0, \nu_2 \geq 0\}$. We then define

$$a(w, v; \boldsymbol{\mu}) = \int_\Omega (1 + \mathrm{i}\mu_2) \nabla w \cdot \nabla \bar{v} - \mu_1 w \bar{v}, \quad \forall w, v \in V^2, \tag{4.34}$$

as well as suitable $f(\cdot; \boldsymbol{\mu})$ and $\ell(\cdot; \boldsymbol{\mu})$. Here $\mathrm{i}^2 = -1$ and $\bar{v}$ denotes the complex conjugate of v. For $\mu_2 > 0$ (positive dissipation), a is coercive. For $\mu_2 = 0$ (no dissipation), a is inf-sup stable unless μ_1 is an eigenvalue λ of the associated "resonance" problem: Find $(\chi \in V, \lambda_{\text{resonance}} \in \mathbb{R})$ such that $\int_\Omega \nabla\chi \cdot \nabla v = \lambda_{\text{resonance}} \int_\Omega \chi v$, $\forall v \in V$, where here V is our standard real space. In some applications, such as the elastodynamics extension of (4.34), the actual dissipation can be substantial; in other applications, such as acoustics, μ_2 must often be interpreted as a numerical regularization parameter.

4.2.3.2 Evolution problems: parabolic PDEs

We shall consider here only parabolic PDEs. In fact, RB methods can also readily be applied to hyperbolic PDEs, for example the second-order wave equation, however in the absence of adequate dissipation the treatment of rough initial conditions and limited regularity remains an outstanding issue as indicated above. We shall assume for our parabolic PDEs that our bilinear form a is coercive; the noncoercive case is more difficult in particular as regards effective a posteriori error estimation [36].

We introduce the time variable t and temporal domain $(0, T]$, and the space $L^2(\Omega)$ and associated inner product $(\cdot, \cdot)_0$ and induced norm $\|\cdot\|_0$. We further define inner product $\boldsymbol{\mu} \in \mathcal{P} \mapsto m(\cdot, \cdot; \boldsymbol{\mu}) : V \times V \to \mathbb{R}$ which induces norm $m^{1/2}(\cdot, \cdot; \boldsymbol{\mu})$ equivalent to $\|\cdot\|_0$. We now state the weak form of our parabolic PDE: Given $\boldsymbol{\mu} \in \mathcal{P}$, we look for $u(\boldsymbol{\mu}) \in C^0((0, T]; L^2(\Omega)) \cap L^2((0, T]; V)$ such that, for any time t,

$$m\left(\frac{\partial u(t; \boldsymbol{\mu})}{\partial t}, v; \boldsymbol{\mu}\right) + a(u(t; \boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tau(t) f(v; \boldsymbol{\mu}), \quad \forall v \in V, \tag{4.35}$$

where $\tau \in L^2((0, T])$ and $f(\cdot; \boldsymbol{\mu}) \in L^2(\Omega)$. We take for initial condition $u(t = 0; \boldsymbol{\mu}) = u_0 \in L^2(\Omega)$; in actual practice, we may also permit parameter-dependent initial conditions. Note that we shall not explicitly present the treatment of the linear functional output, as (in the absence of an adjoint) the latter differs little between the elliptic and parabolic cases.

We inherit the underlying FE approximation from the elliptic problem of Section 4.2.1.2. We further assume that m admits an (EIM-approximate) affine expansion,

$$\tilde{m}(w, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_m} \Theta_m^q(\boldsymbol{\mu}) m^q(w, v), \tag{4.36}$$

for $\Theta_m^q : \mathcal{P} \to \mathbb{R}$ and parameter-independent $m^q : V \times V \to \mathbb{R}$, $1 \le 1 \le Q_m$. We may then further define our FE mass matrix, $\tilde{\mathbb{M}}_h(\boldsymbol{\mu}) \in \mathbb{R}^{N_h \times N_h}$,

$$\left(\tilde{\mathbb{M}}_h(\boldsymbol{\mu})\right)_{kn} = \tilde{m}(\varphi^n, \varphi^k; \boldsymbol{\mu}), \quad 1 \le k, n \le N_h, \tag{4.37}$$

which we may form as

$$\tilde{\mathbb{M}}_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_m} \Theta_m^q(\boldsymbol{\mu}) \mathbb{M}_h^q, \tag{4.38}$$

for $\mathbb{M}_h^q = m^q(\varphi^n, \varphi^k), 1 \le k, n \le N_h, 1 \le q \le Q_m$. We also introduce a "truth" finite difference discretization in time: we choose $\Delta t = T/J$ and define $t^j = j\,\Delta t$, $0 \le j \le J$.

We can now state the FE approximation: Given $\boldsymbol{\mu} \in \mathcal{P}$, we look for $\tilde{u}_{h,\Delta t}^j(\boldsymbol{\mu})(\approx u(t^j, .; \boldsymbol{\mu})) \in V_h$, $j = 1, \ldots, J$, such that

$$\tilde{m}\left(\frac{\tilde{u}_{h,\Delta t}^j(\boldsymbol{\mu}) - \tilde{u}_{h,\Delta t}^{j-1}(\boldsymbol{\mu})}{\Delta t}, v; \boldsymbol{\mu}\right) + \tilde{a}(\tilde{u}_{h,\Delta t}^j(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tau(t^j)\tilde{f}(v; \boldsymbol{\mu}), \quad \forall v \in V_h; \tag{4.39}$$

we impose the initial condition $\tilde{u}_{h,\Delta t}^{j=0}(\boldsymbol{\mu}) = u_0$. We can then state the discrete equations to be solved at each time t^j:

$$\left(\tilde{\mathbb{A}}_h(\boldsymbol{\mu}) + \frac{1}{\Delta t}\tilde{\mathbb{M}}_h(\boldsymbol{\mu})\right)\tilde{\boldsymbol{u}}_{h,\Delta t}^j(\boldsymbol{\mu}) = \frac{1}{\Delta t}\tilde{\mathbb{M}}_h(\boldsymbol{\mu})\boldsymbol{u}_{h,\Delta t}^{j-1}(\boldsymbol{\mu}) + \tau(t^j)\tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}). \tag{4.40}$$

Although we consider here Euler backward temporal treatment, the methodology readily extends to higher-order temporal discretizations.

4.2.3.3 A nonlinear elliptic problem

Although our presentation of linear elliptic PDEs and also linear parabolic PDEs is rather general, the scope of this article does not permit similar treatment of nonlinear problems. We thus focus on a particular nonlinear elliptic PDE with rather simple structure and underlying theory. In the linear part of this chapter our goal is to provide a complete picture of the state of the art, albeit with a balance between performance and simplicity, and indeed an emphasis on the latter. In this nonlinear thread our goal is less ambitious: We highlight the new difficulty introduced by nonlinearity and the corresponding new ingredient – hyperreduction – developed to address this difficulty. The hyperreduction treatment presented, as well as other hyperreduction approaches [35, 17, 40], is broadly applicable. However, the a posteriori error estimator takes advantage of our particular simple nonlinearity, and our offline computational procedure is rather inefficient.

We consider a particular nonlinear elliptic PDE with monotonic nondecreasing nonlinearity: Given $\boldsymbol{\mu} \in \mathcal{P} \subset \{\nu \in \mathbb{R} \mid \nu \geq 0\} \subset \mathbb{R}^{p=1}$, find $u(\boldsymbol{\mu}) \in V$ such that

$$\int_\Omega \nabla u(\boldsymbol{\mu}) \cdot \nabla v + \int_\Omega \eta(u(\boldsymbol{\mu}))\, v = \boldsymbol{\mu} \int_\Omega v, \quad \forall v \in V. \tag{4.41}$$

We require that $\eta \in C^1(\mathbb{R})$ and furthermore $\eta(z_2) - \eta(z_1) \geq 0$ for $z_2 > z_1$. Two examples are the classical smooth test problem given by $\eta(z) = z^3$ and the more relevant (and nonpolynomial) drag law $\eta(z) = |z|z$. It can be shown that (4.41) admits a unique solution. (Note for this nonlinear problem we prefer explicit rather than abstract representation of the weak form.)

We may then directly introduce the corresponding FE approximation: Given $\boldsymbol{\mu} \in \mathcal{P}$, find $u_h(\boldsymbol{\mu}) \in V_h$ such that

$$\int_\Omega \nabla u_h(\boldsymbol{\mu}) \cdot \nabla v + \int_\Omega \eta(u_h(\boldsymbol{\mu}))\, v = \boldsymbol{\mu} \int_\Omega v, \quad \forall v \in V_h. \tag{4.42}$$

In actual practice, the integrals in (4.42) should be interpreted as quadrature sums, hence

$$\begin{aligned} &\sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega} \nabla u_h(x_j^{\mathrm{quad},\Omega}; \boldsymbol{\mu}) \cdot \nabla v(x_j^{\mathrm{quad},\Omega}) \\ &\qquad + \sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega} \eta(u_h(x_j^{\mathrm{quad},\Omega}; \boldsymbol{\mu}))\, v(x_j^{\mathrm{quad},\Omega}) \\ &\qquad\qquad = \boldsymbol{\mu} \sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega} v(x_j^{\mathrm{quad},\Omega}), \quad \forall v \in V_h. \end{aligned} \tag{4.43}$$

For future reference and for purposes of consistency with previous notation we define for this nonlinear problem $\alpha = \alpha(\boldsymbol{\mu}) = 1$ and $\alpha_h = \alpha_h(\boldsymbol{\mu}) = 1$.

We now prepare an affine version of our nonlinear problem. Towards that end we apply the EIM approach, in particular Algorithm 4.1, to $\mathbf{g}$ given by $(\mathbf{g}(\boldsymbol{\mu}))_i = \eta(u_h(x_i^{\mathrm{quad},\Omega}; \boldsymbol{\mu}))$, $1 \leq i \leq N^{\mathrm{quad},\Omega}$; we specify $N_{\mathrm{EIM}}^{\mathrm{NL}}$, $\Xi_{\mathrm{EIM}}^{\mathrm{NL}}$, $\|\cdot\|_{\mathrm{EIM}}^{\mathrm{NL}} \equiv \|\cdot\|_{\ell^\infty}$, and $\mathrm{tol}_{\mathrm{EIM}}^{\mathrm{NL}}$, and denote by $\mathcal{I}_M : V_h \to V_h$ the resulting interpolation operator as characterized by M, $\{i_m^*\}_{m=1,\ldots,M}$, $\{\boldsymbol{\xi}^i\}_{i=1,\ldots,M}$, and B_M. (We provide an NL superscript to the inputs, but context suffices to indicate the NL for the outputs.) We note that each evaluation of $\mathbf{g}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \Xi_{\mathrm{EIM}}^{\mathrm{NL}}$ is now expensive – solution of the FE approximation to our problem – and not simply evaluation of a coefficient function; more efficient alternatives are proposed in the literature [13]. Then, given $\boldsymbol{\mu} \in \mathcal{P}$, we look for $\tilde{u}_h(\boldsymbol{\mu}) \in V_h$ such that

$$\sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} \nabla \tilde{u}_h(x_j^{\text{quad},\Omega};\boldsymbol{\mu}) \cdot \nabla v(x_j^{\text{quad},\Omega}) + \sum_{m,m'=1}^{M} (B_M^{-1})_{mm'} \eta(\tilde{u}_h(x_{i_{m'}^*}^{\text{quad},\Omega},\boldsymbol{\mu})) \left[\sum_{j=1}^{N^{\text{quad},\Omega}} (\boldsymbol{\xi}^m)_j v(x_j^{\text{quad},\Omega}) \rho_j^{\text{quad},\Omega} \right] = \boldsymbol{\mu} \sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} v(x_j^{\text{quad},\Omega}), \quad \forall v \in V_h. \quad (4.44)$$

Under the assumption of exact quadrature we may write (4.44) as

$$\int_\Omega \nabla \tilde{u}_h(\boldsymbol{\mu}) \cdot \nabla v + \int_\Omega \mathcal{I}_M[\eta((\tilde{u}_h)(\boldsymbol{\mu}))]\, v = \boldsymbol{\mu} \int_\Omega v, \quad \forall v \in V_h, \quad (4.45)$$

where $\mathcal{I}_M$ is the EIM interpolant operator. We emphasize that $\mathcal{I}_M$ is developed through Algorithm 4.1 for $\eta(u_h(x_{i=1,\ldots,N^{\text{quad},\Omega}}^{\text{quad},\Omega};\boldsymbol{\mu}))$, but then applied in (4.45) to $\eta(\tilde{u}_h(x_{i=1,\ldots,N^{\text{quad},\Omega}}^{\text{quad},\Omega};\boldsymbol{\mu}))$. We say that (4.44) is "affine" in the sense that $\eta(\tilde{u}_h(\cdot;\boldsymbol{\mu}))$ no longer appears in the quadrature sum associated with the nonlinear term. The computational importance of this simplification will become clear when we consider RB projection.

4.3 Projection

4.3.1 Elliptic problems

4.3.1.1 Galerkin projection

We consider here the real case, but note that our "complexification" transformation may be directly applied. We are given a hierarchical set of RB spaces $\{V_N\}_{N=1,\ldots,N_{\max}}$. In fact, this section is applicable to any RB spaces, but for purposes of concreteness we sketch here the particular space we shall propose in Section 4.4.1. We first introduce parameter sample $S_{N_{\max}} \equiv \{\boldsymbol{\mu}^j \in \mathcal{P}\}_{j=1,\ldots,N_{\max}}$, the optimal choice of which shall be discussed in Section 4.4.1. The space V_N, for any given N, is then defined as $\text{span}\{\tilde{u}_h(\boldsymbol{\mu}^j), j = 1,\ldots,N\}$. Note that, since $\boldsymbol{\mu}^j, 1 \le j \le N_{\max}$, are independent of N, our RB spaces are nested: $V_1 \subset V_2 \subset \cdots \subset V_{N_{\max}}$.

We now define the RB-Galerkin approximation for some given $N \in \{1,\ldots,N_{\max}\}$: Given $\boldsymbol{\mu} \in \mathcal{P}$, find $\tilde{u}_N(\boldsymbol{\mu}) \in V_N$ such that

$$\tilde{a}(\tilde{u}_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = \tilde{f}(v;\boldsymbol{\mu}), \quad \forall v \in V_N, \quad (4.46)$$

and evaluate the scalar output $\tilde{s}_N(\boldsymbol{\mu}) \in \mathbb{R}$ as $\tilde{s}_N(\boldsymbol{\mu}) = \tilde{\ell}(\tilde{u}_N(\boldsymbol{\mu});\boldsymbol{\mu})$. We recall also that we shall denote by $u_N(\boldsymbol{\mu})$ the RB approximation in the absence of EIM errors; the latter corresponds to an effectively exact affine expansion such that $\epsilon_{\text{EIM}}^{\mathcal{P}} = 0$ and thus

$\{\tilde{a}, \tilde{f}, \tilde{\ell}\} = \{a, f, \ell\}$. We denote the RB stability and continuity constants (hence over V_N) associated to $\tilde{a}$ by $\tilde{\alpha}_N(\boldsymbol{\mu})$, $\tilde{\gamma}_N(\boldsymbol{\mu})$, and $\tilde{\beta}_N(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathcal{P}$; we may then also define the corresponding worst case constants (over $\mathcal{P}$) as $\tilde{\alpha}_N$, $\tilde{\gamma}_N$, and $\tilde{\beta}_N$. We also introduce $\tilde{c}_N^{\mathrm{s}}(\boldsymbol{\mu}) \equiv \max(\tilde{\alpha}_N(\boldsymbol{\mu}), \tilde{\beta}_N(\boldsymbol{\mu}))$ and corresponding worst case (minimum over $\mathcal{P}$) constant $\tilde{c}_N^{\mathrm{s}}$. It is readily demonstrated that $\tilde{\alpha}_N \geq \tilde{\alpha}_h$, however in general $\tilde{\beta}_N \not\geq \tilde{\beta}_h$. In the noncoercive case we must therefore incorporate $\tilde{\beta}_N > 0$ as an additional hypothesis. Alternatively, we may consider a minimum-residual projection [25], which ensures a stable RB approximation, indeed $\tilde{\beta}_N \geq \tilde{\beta}_h$; however, Galerkin projection – the simplest and least expensive – is typically quite effective in practice.

We also introduce a basis for V_N, $\{\zeta_h^n\}_{n=1,\ldots,N}, 1 \leq N \leq N_{\max}$. We shall choose, for purposes of stability, an orthonormal basis: $(\zeta_h^n, \zeta_h^m)_V = \delta_{mn}, 1 \leq m, n \leq N$, for δ_{mn} the Kronecker delta symbol. Given any function $v_N \in V_N$, we denote by $\mathbf{v}_N \in \mathbb{R}^N$ the corresponding vector of basis coefficients. The discrete RB equations then read

$$\tilde{\mathbb{A}}_N(\boldsymbol{\mu})\tilde{\mathbf{u}}_N(\boldsymbol{\mu}) = \tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}) \tag{4.47}$$

and

$$\tilde{s}_N(\boldsymbol{\mu}) = \tilde{\boldsymbol{\ell}}_N^{\mathrm{T}}(\boldsymbol{\mu})\tilde{\mathbf{u}}_N(\boldsymbol{\mu}), \tag{4.48}$$

where $\tilde{\mathbb{A}}_N(\boldsymbol{\mu}) \in \mathbb{R}^{N\times N}$, $\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}) \in \mathbb{R}^N$, and $\tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu}) \in \mathbb{R}^N$ are given by

$$(\tilde{\mathbb{A}}_N(\boldsymbol{\mu}))_{mn} = \tilde{a}(\zeta_h^n, \zeta_h^m; \boldsymbol{\mu}), \quad (\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}))_m = \tilde{f}(\zeta_h^m; \boldsymbol{\mu}), \quad (\tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu}))_m = \tilde{\ell}(\zeta_h^m; \boldsymbol{\mu}), \quad 1 \leq m, n \leq N. \tag{4.49}$$

We further note from (4.22) and (4.49) that

$$\tilde{\mathbb{A}}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\mathbb{A}_N^q, \quad \tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu})\boldsymbol{f}_N^q, \quad \tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_\ell} \Theta_\ell^q(\boldsymbol{\mu})\boldsymbol{\ell}_N^q, \tag{4.50}$$

for parameter-independent $\mathbb{A}_N^q \in \mathbb{R}^{N\times N}, 1 \leq q \leq Q_a$, $\boldsymbol{f}_N^q \in \mathbb{R}^N, 1 \leq q \leq Q_f$, and $\boldsymbol{\ell}_N^q \in \mathbb{R}^N$, $1 \leq q \leq Q_\ell$; for example, $(\mathbb{A}_N^q)_{mn} = a^q(\zeta_h^m, \zeta_h^n), 1 \leq m, n \leq N, 1 \leq q \leq Q_a$.

In actual practice we may express our RB matrices and vectors in a nonintrusive fashion which invokes only standard operators and operations readily available in the FE context. To begin, we introduce, for $1 \leq N \leq N_{\max}$, the RB "basis matrix" $\mathbb{V}_N \in \mathbb{R}^{N_h\times N}$, $(\mathbb{V}_N)_{jn} = (\boldsymbol{\zeta}_h^n)_j, 1 \leq j \leq N_h, 1 \leq n \leq N$. It then follows from (4.49) that

$$\tilde{\mathbb{A}}_N(\boldsymbol{\mu}) = \mathbb{V}_N^{\mathrm{T}}\tilde{\mathbb{A}}_h(\boldsymbol{\mu})\mathbb{V}_N, \quad \tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}) = \mathbb{V}_N^{\mathrm{T}}\tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}), \quad \tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu}) = \mathbb{V}_N^{\mathrm{T}}\boldsymbol{\ell}_h(\boldsymbol{\mu}). \tag{4.51}$$

In the same fashion, from (4.50) (or directly (4.27)), we may write

$$\mathbb{A}_N^q = \mathbb{V}_N^{\mathrm{T}}\mathbb{A}_h^q\mathbb{V}_N, \ 1 \leq q \leq Q_a, \quad \boldsymbol{f}_N^q = \mathbb{V}_N^{\mathrm{T}}\boldsymbol{f}_h^q, \ 1 \leq q \leq Q_f, \quad \boldsymbol{\ell}_N^q = \mathbb{V}_N^{\mathrm{T}}\boldsymbol{\ell}_h^q, \ 1 \leq q \leq Q_\ell. \tag{4.52}$$

We recall that the FE matrices for our nodal basis are sparse, and hence the operation count to form (say) $\mathbb{A}_N^1$, for $\mathbb{A}_h^1$ already formed and represented in sparse format, is $\mathcal{O}(N^2 N_h)$ floating point operations (FLOPs). We discuss operation counts further in the context of the offline-online decomposition.

4.3.1.2 A priori error estimation

It is a simple application of Céa's lemma and Strang's first lemma to demonstrate [18] that, for any $\boldsymbol{\mu} \in \mathcal{P}$,

$$\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V \leq \left(1 + \frac{\tilde{\gamma}_N}{\tilde{c}_N^s}\right) \underbrace{\inf_{w_N \in V_N} \|u_h(\boldsymbol{\mu}) - w_N\|_V}_{\text{best-approximation error}} + \frac{\epsilon_{\text{EIM}}^{\mathcal{P}}}{\tilde{c}_N^s}\left(1 + \frac{\|f(\cdot;\boldsymbol{\mu})\|_{V'}}{c_h^s}\right). \tag{4.53}$$

(We can then readily develop an associated error bound for $s_h - \tilde{s}_N$; we defer discussion of the latter to the a posteriori context.) Note in the error estimate (4.53) that $u_h(\boldsymbol{\mu})$ is the "true" FE approximation on which we build the RB approximation and with respect to which we estimate the RB accuracy, and $\tilde{u}_N(\boldsymbol{\mu})$ is the actual RB approximation including EIM approximation of the bilinear and linear forms; hence the bound (4.53) reflects both RB and EIM contributions to the error. In Section 4.4.1 we shall develop estimates for the decay of the best-approximation error with N relative to the corresponding Kolmogorov N-width associated with our parametric manifold.

Finally, we note that in the coercive case the RB discrete equations are provably well-conditioned under the assumption that $\epsilon_{\text{EIM}}^{\mathcal{P}} < \alpha$; the essential ingredient is orthonormalization of the RB basis functions with respect to the V inner product. In particular, it is readily demonstrated that, in the coercive case, the condition number of $\mathbb{A}_N(\mu)$ (measured in the usual ℓ_2-norm) is bounded by $(\gamma + \epsilon_{\text{EIM}}^{\mathcal{P}})/(\alpha - \epsilon_{\text{EIM}}^{\mathcal{P}})$ for all $\boldsymbol{\mu} \in \mathcal{P}$ and independent of N.

4.3.1.3 A posteriori error estimation

4.3.1.3.1 Dual norm of residual

A posteriori error estimators shall serve to control the error in the RB approximation: in the offline stage to (inexpensively) identify good spaces V_N; in the online stage to verify any particular query $\boldsymbol{\mu} \mapsto \tilde{u}_N(\boldsymbol{\mu}), \tilde{s}_N(\boldsymbol{\mu})$. In principle we would wish to control in both the offline stage and the online stage the total error $\|u(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$, and in certain particular (but important) cases this is indeed possible [39]. More generally, we write $\|u(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V \leq \|u(\boldsymbol{\mu}) - u_h(\boldsymbol{\mu})\|_V + \|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$: In the offline stage we control both $\|u - u_h(\boldsymbol{\mu})\|_V$ (as described in Section 4.2.1.2) and $\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$ over respective (finite-cardinality) train subsets of the parameter domain $\mathcal{P}$, Ξ_{FE} and Ξ_{EIM}, Ξ_{RB}; in the online stage, for *any* $\boldsymbol{\mu} \in \mathcal{P}$, we control – and in particular verify – only $\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$. We justify the online emphasis on only the FE-RB error: the operation count for the online stage shall not depend explicitly on N_h, and thus we may choose the FE approximation space somewhat conservatively; the latter would then accommodate the difference between the offline parameter train set Ξ_{FE} and the full (online) parameter domain $\mathcal{P}$. In the remainder of this section we consider only $\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$ (and, briefly, $|s_h(\boldsymbol{\mu}) - \tilde{s}_N(\boldsymbol{\mu})|$).

To begin, we consider the case in which $\epsilon_{\text{EIM}}^{\mathcal{P}} = 0$ and hence $\tilde{u}_N(\mu) = u_N(\mu)$; for the purposes of this analysis, we explicitly remove the $\tilde{\cdot}$. We introduce the error $e(\boldsymbol{\mu}) \equiv u_h(\boldsymbol{\mu}) - u_N(\boldsymbol{\mu})$ as well as the residual $\boldsymbol{\mu} \mapsto r_h(\cdot;\boldsymbol{\mu}) \in V'$ given by

$$r_h(v;\boldsymbol{\mu}) \equiv f(v;\boldsymbol{\mu}) - a\big(u_N(\boldsymbol{\mu}), v;\boldsymbol{\mu}\big), \quad \forall v \in V_h. \tag{4.54}$$

It is then standard to derive the error–residual relationship,

$$a\big(e(\boldsymbol{\mu}), v;\boldsymbol{\mu}\big) = r_h(v;\boldsymbol{\mu}), \quad \forall v \in V_h. \tag{4.55}$$

For future reference we also introduce the Riesz representation of the residual,

$$R_h(\boldsymbol{\mu}) \equiv \mathcal{R}_h r_h(\cdot,\boldsymbol{\mu}), \tag{4.56}$$

for $\mathcal{R}_h$ given by (4.13); we recall from (4.14) that $\|r_h(\boldsymbol{\mu})\|_{V_h'} = \|R_h(\boldsymbol{\mu})\|_V$.

We now define our a posteriori error estimator, $\boldsymbol{\mu} \in \mathcal{P} \mapsto \Delta_N(\boldsymbol{\mu}) \in \mathbb{R}_{0+}$:

$$\Delta_N(\boldsymbol{\mu}) \equiv \frac{\|R_h(\boldsymbol{\mu})\|_V}{c_h^{\text{s,app}}(\boldsymbol{\mu})}, \tag{4.57}$$

where $c_h^{\text{s,app}}(\boldsymbol{\mu})$ is a (nonnegative) approximation to $c_h^{\text{s}}(\boldsymbol{\mu})$. It is then a simple matter to demonstrate [33] that

$$\frac{c_h^{\text{s}}(\boldsymbol{\mu})}{c_h^{\text{s,app}}(\boldsymbol{\mu})} \le \frac{\Delta_N(\boldsymbol{\mu})}{\|e(\boldsymbol{\mu})\|_V} \le \frac{\gamma_h(\boldsymbol{\mu})}{c_h^{\text{s,app}}(\boldsymbol{\mu})}. \tag{4.58}$$

We observe from the left inequality that, if $0 < c_h^{\text{s,app}}(\boldsymbol{\mu}) \le c_h^{\text{s}}(\boldsymbol{\mu})$, then $\|e(\boldsymbol{\mu})\|_V \le \Delta_N(\boldsymbol{\mu})$: Our error estimator is an error bound. We conclude from the right inequality that the error bound may overestimate the true error but by a factor which is bounded independent of N.

We note that we did not in fact use any special properties of $u_N(\boldsymbol{\mu})$ in our derivation of (4.58), and in particular we did not take advantage of the Galerkin projection. Hence, for the residual defined as (4.54) – with *unperturbed* f and a – our bound (4.58) in fact remains valid also for $\tilde{u}_N(\boldsymbol{\mu})$, and indeed for any function in V_h. However, we shall see that for ($\epsilon_{\text{EIM}}^{\mathcal{P}} \neq 0$ and thus certainly) a nonaffine we cannot compute $\|R_h(\boldsymbol{\mu})\|_V$ efficiently within the offline-online decomposition; more precisely, the operation count for evaluation of $\|R_h(\boldsymbol{\mu})\|_V$ directly as $(\mathbf{r}_h(\boldsymbol{\mu})\mathbb{X}_h^{-1}\mathbf{r}_h(\boldsymbol{\mu}))^{1/2}$ will not be independent of N_h.[3] Furthermore, our "aggregate" error estimator (4.57) does not permit us to deduce, and hence control, the individual contributions of the RB approximation and EIM affine representation to the error $\|u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})\|_V$. We next present an a

3 We note, however, that at least for problems in two space dimensions ($d = 2$), this direct evaluation though not ideal is nevertheless feasible, in particular since the parameter-independent sparse optimally ordered $\mathbb{X}_h$ can be Cholesky-factorized once.

posteriori error estimator for $\epsilon_{\text{EIM}}^{\mathcal{P}} \neq 0$ which addresses these two issues. We reinstate the $\tilde{\cdot}$ notation.

We first define $\tilde{e}(\boldsymbol{\mu}) \equiv u_h(\boldsymbol{\mu}) - \tilde{u}_N(\boldsymbol{\mu})$; we next introduce the perturbed residual $\boldsymbol{\mu} \mapsto \tilde{r}_h(\cdot;\boldsymbol{\mu}) \in V'$ given by

$$\tilde{r}_h(v;\boldsymbol{\mu}) \equiv \tilde{f}(v;\boldsymbol{\mu}) - \tilde{a}(\tilde{u}_N(\boldsymbol{\mu}), v;\boldsymbol{\mu}) \tag{4.59}$$

and the associated Riesz representation $\tilde{R}_h(\boldsymbol{\mu}) = \mathcal{R}_h \tilde{r}_h(\cdot,\boldsymbol{\mu})$. Our revised a posteriori error estimator is then given by

$$\tilde{\Delta}_N(\boldsymbol{\mu}) \equiv \frac{\|\tilde{R}_h(\boldsymbol{\mu})\|_V + \epsilon_{\text{EIM}}^{\mathcal{P}}(1 + \|\tilde{u}_N(\boldsymbol{\mu})\|_V)}{\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu}) - \epsilon_{\text{EIM}}^{\mathcal{P}}}, \tag{4.60}$$

where $\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu})$ is a (nonnegative) approximation to $\tilde{c}_h^{\text{s}}(\boldsymbol{\mu})$. It can be shown that, if $\epsilon_{\text{EIM}}^{\mathcal{P}} < \tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu}) \leq \tilde{c}_h^{\text{s}}(\boldsymbol{\mu})$, then $\|\tilde{e}(\boldsymbol{\mu})\|_V \leq \tilde{\Delta}_N(\boldsymbol{\mu})$. We emphasize that the bound $\tilde{\Delta}_N(\boldsymbol{\mu})$ reflects – and thus can serve to efficiently control – both the RB and the EIM contributions to the error. Finally, we highlight the two constants which must be evaluated in (4.60): $\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu})$, to be discussed shortly, and $\epsilon_{\text{EIM}}^{\mathcal{P}}$, as introduced in (4.30). As regards the latter, we recall that we have direct control only over $\epsilon_{\text{EIM}}^{\Xi_{\text{EIM}}}$, and we must then assume that Ξ_{EIM} is sufficiently rich to represent $\mathcal{P}$.

We also develop a simple a posteriori error estimator for our output:

$$|s_h(\boldsymbol{\mu}) - \tilde{s}_N(\boldsymbol{\mu})| \leq (\|\tilde{\ell}(\cdot,\boldsymbol{\mu})\|_{V_h'} + \epsilon_{\text{EIM}}^{\mathcal{P}})\tilde{\Delta}_N(\boldsymbol{\mu}) + \epsilon_{\text{EIM}}^{\mathcal{P}}\|\tilde{u}_N(\boldsymbol{\mu})\|_V. \tag{4.61}$$

Note for $\epsilon_{\text{EIM}}^{\mathcal{P}} = 0$, symmetric coercive problems, and compliant outputs – $\ell(\cdot;\boldsymbol{\mu}) = f(\cdot;\boldsymbol{\mu})$ – the bound (4.61) is demonstrably pessimistic; the latter is remedied by adjoint techniques [31], [33], which also provide for better approximation of noncompliant outputs.

4.3.1.3.2 Approximate stability constant

We recall that we wish to apply the error bound in the online stage. We shall show in the next section that the dual norm of the residual, which appears in the numerators of (4.57) and (4.60), in fact admits a very efficient offline-online procedure. It remains to develop a formulation for $\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu})$ which also admits an efficient offline-online procedure and furthermore either rigorously, or plausibly, satisfies $0 < c_0 \leq \tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu}) \leq \tilde{c}_h^{\text{s}}(\boldsymbol{\mu})$, $\forall \boldsymbol{\mu} \in \mathcal{P}$. There are a variety of approaches [33]. In some cases, we can explicitly deduce $\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu})$ in terms of the PDE coefficients: In Example 1.0, described by equations (4.9)–(4.11), for $c_{L^2} = 0$ in our inner product (4.1), we may choose $\tilde{c}_h^{\text{s,app}}(\boldsymbol{\mu}) = \min(1, \mu_1) \leq \alpha(\boldsymbol{\mu})$. However, for geometry variation, in particular in the vector case, inspection no longer suffices; and for the noncoercive case the situation is even more difficult. Although there are approaches which can treat the general case rigorously, such as the successive constraint method [22], these techniques are unfortunately quite complicated and often prohibitively expensive in the offline stage.

The simplest approach might be to take $\tilde{c}_h^{s,\mathrm{app}}(\boldsymbol{\mu}) =$ (say) $\tilde{c}_N^s(\boldsymbol{\mu})/2$ [25]. If indeed $\tilde{c}_N^s(\boldsymbol{\mu})$ converges to $\tilde{c}_h^s(\boldsymbol{\mu})$ as N increases, then this simple recipe provides, at least asymptotically, a lower bound. However, the RB spaces V_N are not designed to well approximate the stability constant [39]. We thus present the obvious extension: a colateral RB approximation for the eigenproblem associated with the stability constant. Since even an order-unity error in the stability constant will yield a good error estimator and indeed a bound, even a modest RB approximation should perform very well. This approach also serves several secondary purposes relevant to this handbook: a brief summary of RB treatment of (albeit somewhat nonstandard) eigenproblems [24]; further reinforcement of RB concepts.

To begin, we recall the definition of the supremizer operators $\tilde{T}_h(\boldsymbol{\mu})$ (respectively, $\tilde{T}_N(\boldsymbol{\mu})$): For any $w \in V_h$, $\boldsymbol{\mu} \in \mathcal{P} \mapsto \tilde{T}_h(\boldsymbol{\mu}) \in \mathcal{L}(V_h, V_h)$ such that, for any $w \in V_h$, $(\tilde{T}_h(\boldsymbol{\mu})w, v)_V = \tilde{a}(w, v; \boldsymbol{\mu})$, $\forall v \in V_h$ (respectively, for any $w \in V_N$, $\boldsymbol{\mu} \in \mathcal{P} \mapsto \tilde{T}_N(\boldsymbol{\mu}) \in \mathcal{L}(V_N, V_N)$ such that, for any $w \in V_N$, $(\tilde{T}_N(\boldsymbol{\mu})w, v)_V = \tilde{a}(w, v; \boldsymbol{\mu})$, $\forall v \in V_N$). Here $\mathcal{L}(W, W)$ denotes the space of continuous mappings from W to W. We now introduce the following generalized symmetric eigenproblems: For the coercive case, find $\boldsymbol{\mu} \in \mathcal{P} \mapsto (\Psi_h(\boldsymbol{\mu}), \lambda_h(\boldsymbol{\mu})) \in V_h \times \mathbb{R}_+$ such that

$$\frac{1}{2}\big(\tilde{a}(\Psi_h(\boldsymbol{\mu}), v; \boldsymbol{\mu}) + \tilde{a}(v, \Psi_h(\boldsymbol{\mu}); \boldsymbol{\mu})\big) = \lambda_h(\boldsymbol{\mu})(\Psi_h(\boldsymbol{\mu}), v)_V, \quad \forall v \in V_h; \tag{4.62}$$

and for the noncoercive case, find $\boldsymbol{\mu} \in \mathcal{P} \mapsto (\Phi_h(\boldsymbol{\mu}), \sigma_h^2(\boldsymbol{\mu})) \in V_h \times \mathbb{R}_+$ such that

$$(\tilde{T}_h(\boldsymbol{\mu})\Phi_h(\boldsymbol{\mu}), \tilde{T}_h(\boldsymbol{\mu})v)_V = \sigma_h^2(\boldsymbol{\mu})(\Phi_h(\boldsymbol{\mu}), v)_V, \quad \forall v \in V_h; \tag{4.63}$$

we enumerate the modes in order of increasing magnitude of the eigenvalue. Note that the inner product constant c_{L^2} in (4.1) should be chosen large enough to ensure adequate separation of the lowest eigenvalues.[4]

It is readily demonstrated that

$$\tilde{\alpha}_h(\boldsymbol{\mu}) = \big(\lambda_h(\boldsymbol{\mu})\big)_1 = \frac{\tilde{a}((\Psi_h(\boldsymbol{\mu}))_1, (\Psi_h(\boldsymbol{\mu}))_1; \boldsymbol{\mu})}{((\Psi_h(\boldsymbol{\mu}))_1, (\Psi_h(\boldsymbol{\mu}))_1)_V} \tag{4.64}$$

and

$$\tilde{\beta}_h(\boldsymbol{\mu}) = \big(\sigma_h(\boldsymbol{\mu})\big)_1 = \sqrt{\frac{(\tilde{T}_h(\boldsymbol{\mu})(\Phi_h(\boldsymbol{\mu}))_1, \tilde{T}_h(\boldsymbol{\mu})(\Phi_h(\boldsymbol{\mu}))_1)_V}{((\Phi_h(\boldsymbol{\mu}))_1, (\Phi_h(\boldsymbol{\mu}))_1)_V}}. \tag{4.65}$$

We thus observe that a good approximation for the eigenfunction will yield a good approximation for the respective eigenvalue. We can now proceed to RB approximation.

4 We suggest a value $c_{L^2} = \max_{\boldsymbol{\mu}\in\mathcal{P}} \max(\|\Upsilon_{00}(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}, \max_{i\in\{1,2,3\}} \|\Upsilon_{ii}(\cdot;\boldsymbol{\mu})\|_{L^\infty(\Omega)}/l^2)$, where l is a characteristic minimum length scale associated with Ω.

In particular, we can envision that, just as u_h resides on a low-dimensional parametric manifold, so do $(\Psi_h)_1$ and $(\Phi_h)_1$. We may thus construct corresponding RB spaces and bases, respectively: For the coercive case, V_N^Ψ and $\mathbb{V}_N^\Psi$ for $1 \le N \le N_{\max}^\Psi$; for the inf-sup stable case, V_N^Φ and $\mathbb{V}_N^\Phi$ for $1 \le N \le N_{\max}^\Phi$. The manifolds may not be smooth for eigenproblems; however, we can accommodate mode crossing through proper choice of a sufficiently rich RB space and in particular through incorporation of the first few eigenfunctions. We note that our interest here is in the eigenvalue, not the eigenfunction, and in particular the latter serves only to develop a good approximation for the former.

The Galerkin weak statements of the RB approximations directly follow: For the coercive case, find $\boldsymbol{\mu} \in \mathcal{P} \mapsto (\Psi_N(\boldsymbol{\mu}), \lambda_N(\boldsymbol{\mu})) \in V_N^\Psi \times \mathbb{R}_+$ such that

$$\frac{1}{2}\left(\tilde{a}(\Psi_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) + \tilde{a}(v, \Psi_N(\boldsymbol{\mu}); \boldsymbol{\mu})\right) = \lambda_N(\boldsymbol{\mu})(\Psi_N(\boldsymbol{\mu}), v)_V, \quad \forall v \in V_N^\Psi, \tag{4.66}$$

and set $\tilde{c}_h^{\mathrm{s,app}}(\boldsymbol{\mu}) = (\lambda_N(\boldsymbol{\mu}))_1$; for the noncoercive case, we must consider $\boldsymbol{\mu} \in \mathcal{P} \mapsto (\Phi_N(\boldsymbol{\mu}), \sigma_N^2(\boldsymbol{\mu})) \in V_N^\Phi \times \mathbb{R}_+$ such that

$$(\tilde{T}_h(\boldsymbol{\mu})\Phi_N(\boldsymbol{\mu}), \tilde{T}_h(\boldsymbol{\mu})v)_V = \sigma_N^2(\boldsymbol{\mu})(\Phi_N(\boldsymbol{\mu}), v)_V, \quad \forall v \in V_N^\Phi, \tag{4.67}$$

and set $\tilde{c}_h^{\mathrm{s,app}}(\boldsymbol{\mu}) = (\sigma_N(\boldsymbol{\mu}))_1$. Note in (4.67) we retain $\tilde{T}_h(\boldsymbol{\mu})$, and do not substitute $\tilde{T}_N(\boldsymbol{\mu})$. In practice, we might even deflate $(\lambda_N(\boldsymbol{\mu}))_1$ and $(\sigma^N(\boldsymbol{\mu}))_1$ by some factor, say, 1/2, to ensure that our estimates approach the true respective stability constants from below.

We also provide here the associated discrete equations: $(\boldsymbol{\Psi}_N(\boldsymbol{\mu}) \in \mathbb{R}^N, \lambda_N(\boldsymbol{\mu}) \in \mathbb{R}_+)$ satisfies

$$\underbrace{\left(\mathbb{V}_N^{\Psi\,\mathrm{T}} \frac{1}{2}\left(\tilde{\mathbb{A}}_h(\boldsymbol{\mu}) + \tilde{\mathbb{A}}_h^{\mathrm{T}}(\boldsymbol{\mu})\right) \mathbb{V}_N^\Psi\right)}_{\mathbb{E}_N^\Psi(\boldsymbol{\mu})} \boldsymbol{\Psi}_N(\boldsymbol{\mu}) = \lambda_N(\boldsymbol{\mu})(\mathbb{V}_N^{\Psi\,\mathrm{T}} \mathbb{X}_h \mathbb{V}_N^\Psi)\boldsymbol{\Psi}_N(\boldsymbol{\mu}); \tag{4.68}$$

similarly, $(\boldsymbol{\Phi}_N(\boldsymbol{\mu}) \in \mathbb{R}^N, \sigma_N^2(\boldsymbol{\mu}) \in \mathbb{R}_+)$ satisfies

$$\underbrace{\left(\mathbb{V}_N^{\Phi\,\mathrm{T}} \left(\tilde{\mathbb{A}}_h^{\mathrm{T}}(\boldsymbol{\mu}) \mathbb{X}_h^{-1} \tilde{\mathbb{A}}_h(\boldsymbol{\mu})\right) \mathbb{V}_N^\Phi\right)}_{\mathbb{E}_N^\Phi(\boldsymbol{\mu})} \boldsymbol{\Phi}_N(\boldsymbol{\mu}) = \sigma_N^2(\boldsymbol{\mu})(\mathbb{V}_N^{\Phi\,\mathrm{T}} \mathbb{X}_h \mathbb{V}_N^\Phi)\boldsymbol{\Phi}_N(\boldsymbol{\mu}). \tag{4.69}$$

The RB matrices associated with these eigenproblems admit an affine decomposition: For (4.68), $\mathbb{E}_N^\Psi(\boldsymbol{\mu})$ can be expressed as

$$\mathbb{E}_N^\Psi(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) \left(\mathbb{V}_N^{\Psi\,\mathrm{T}} \frac{1}{2}\left(\mathbb{A}_h^q + \mathbb{A}_h^{q\,\mathrm{T}}\right)\mathbb{V}_N^\Psi\right), \tag{4.70}$$

which is the usual single-sum affine expansion; for (4.69), $\mathbb{E}_N^\Phi(\boldsymbol{\mu})$ can be expressed as

$$\mathbb{E}_N^\Phi(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \sum_{q'=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) \Theta_a^{q'}(\boldsymbol{\mu}) (\mathbb{V}_N^{\Phi\,\mathrm{T}} (\mathbb{A}_h^{q\,\mathrm{T}} \mathbb{X}_h^{-1} \mathbb{A}_h^{q'}) \mathbb{V}_N^\Phi), \tag{4.71}$$

which is now a double-sum affine expansion.

4.3.1.4 Offline-online computational procedures

We describe here the offline-online computational procedure. In the offline stage we prepare a parameter-independent data set. In the online (or deployed) stage we perform RB queries: $\boldsymbol{\mu} \mapsto (\tilde{u}_N(\boldsymbol{\mu}), \tilde{s}_N(\boldsymbol{\mu}))$. In general, the offline stage (respectively, a single online RB query) is very expensive (respectively, very inexpensive) relative to a single FE query $\boldsymbol{\mu} \mapsto \tilde{u}_h(\boldsymbol{\mu})$. As described in Section 4.1, the RB method, and in particular the offline expense, can be justified in the real-time context or the many-query context.

In Section 4.4.1 we shall describe the procedure by which we identify our RB spaces. The latter is performed as part of the offline procedure. In the current section we presume that the RB spaces are *given* in the form of the RB basis matrix $V_{N_{\max}}$. We consider here the offline-online procedure for subsequent (i) formation and solution of the RB discrete equations to obtain $\tilde{\boldsymbol{u}}_N(\boldsymbol{\mu})$ and $\tilde{s}_N(\boldsymbol{\mu})$, and (ii) evaluation of the dual norm of the residual, $\|\tilde{R}_h(\boldsymbol{\mu})\|_V$, as required by our a posteriori error indicator (4.60). More generally, the former is an example of a single-sum affine expansion, whereas the latter is an example of double-sum affine expansion. Other examples of single-sum and double-sum expansions include evaluation of $(\lambda_N(\boldsymbol{\mu}))_1$ and $(\sigma_N(\boldsymbol{\mu}))_1$, respectively, as required for the stability-constant approximation; the latter thus follow offline-online strategies very similar to those described below for the RB linear system.

4.3.1.4.1 RB linear system: formation and solution

In the offline stage, we form $\mathbb{A}^q_{N_{\max}}, 1 \le q \le Q_a$, $\boldsymbol{f}^q_{N_{\max}}, 1 \le q \le Q_f$, and $\boldsymbol{\ell}^q_{N_{\max}}, 1 \le q \le Q_\ell$ of (4.52); the operation count, taking into account FE sparsity, is $\mathcal{O}(Q_a N^2_{\max} N_h) + \mathcal{O}(Q_f N_{\max} N_h) + \mathcal{O}(Q_\ell N_{\max} N_h)$. It is important to note that we form these matrices and vectors for the largest space, $V_{N_{\max}}$; as the RB spaces are hierarchical, we can then readily obtain the matrices (respectively, vectors) for any other RB space, V_N, by simply extracting the $N \times N$ first entries (respectively, N first entries). In the online stage, for any given N and $\boldsymbol{\mu} \in \mathcal{P}$, we first form $\tilde{\mathbb{A}}_N(\boldsymbol{\mu})$, $\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu})$, and $\tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu})$ from (4.50) in $\mathcal{O}(Q_a N^2) + \mathcal{O}(Q_f N) + \mathcal{O}(Q_\ell N)$ FLOPs; we then solve (4.47) for $\boldsymbol{u}_N(\boldsymbol{\mu})$ in $\mathcal{O}(N^3)$ FLOPs – in general, $A_N(\boldsymbol{\mu})$ shall be a dense matrix; finally, we evaluate $s_N(\boldsymbol{\mu})$ from (4.48) in $\mathcal{O}(N)$ FLOPs. Note that if we wish to visualize the full field, then an additional $\mathcal{O}(NN_h)$ FLOPs are required to evaluate the FE basis coefficients of the RB approximation as $\mathbb{V}_N \boldsymbol{u}_N(\boldsymbol{\mu})$; the online operation count is independent of N_h except for this (elective) full field reconstruction. Note however that, if the visualization of each RB function is stored and prepared offline as a frame on a GPU, only the linear combination of these frames is required and we are back to an $\mathcal{O}(N)$ complexity.

4.3.1.4.2 Residual dual norm evaluation

To begin, we form the FE representation of $\tilde{r}_h(\cdot;\boldsymbol{\mu})$, $\tilde{r}_h(\varphi^j;\boldsymbol{\mu}), 1 \le j \le N_h$:

$$\tilde{\mathbf{r}}_h(\boldsymbol{\mu}) = \tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}) - \tilde{\mathbb{A}}_h(\boldsymbol{\mu})\mathbb{V}_N \boldsymbol{u}_N(\boldsymbol{\mu}). \tag{4.72}$$

It is then readily demonstrated from $\tilde{R}_h(\boldsymbol{\mu}) = \mathcal{R}_h \tilde{r}_h(\cdot;\boldsymbol{\mu})$ and (4.14) that the dual norm of the residual, hence $\|\tilde{R}_h(\boldsymbol{\mu})\|_V$, is given by

$$\|\tilde{R}_h(\boldsymbol{\mu})\|_V = \left(\tilde{\boldsymbol{f}}_h(\boldsymbol{\mu}) - \tilde{\mathbb{A}}_h(\boldsymbol{\mu})\mathbb{V}_N \boldsymbol{u}_N(\boldsymbol{\mu})\right)^{\mathrm{T}} \mathbb{X}_h^{-1} \left(\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}) - \tilde{\mathbb{A}}_h(\boldsymbol{\mu})\mathbb{V}_N \boldsymbol{u}_N(\boldsymbol{\mu})\right). \tag{4.73}$$

We now introduce $\boldsymbol{\Theta}_N \in \mathbb{R}^{Q_f + Q_a N}$ as

$$\begin{aligned}\boldsymbol{\Theta}_N(\boldsymbol{\mu}) \equiv \big(&\Theta_f^1(\boldsymbol{\mu}), \ldots, \Theta_f^{Q_f}(\boldsymbol{\mu}), \Theta_a^1(\boldsymbol{\mu})(\boldsymbol{u}_N(\boldsymbol{\mu}))_1, \ldots, \Theta_a^1(\boldsymbol{\mu})(\boldsymbol{u}_N(\boldsymbol{\mu}))_N, \\ &\ldots, \Theta_a^{Q_a}(\boldsymbol{\mu})(\boldsymbol{u}_N(\boldsymbol{\mu}))_1, \ldots, \Theta_a^{Q_a}(\boldsymbol{\mu})(\boldsymbol{u}_N(\boldsymbol{\mu}))_N\big)^{\mathrm{T}},\end{aligned} \tag{4.74}$$

and also $\boldsymbol{L}_N \in \mathbb{R}^{N_h \times (Q_f + Q_a N)}$ as

$$\boldsymbol{L}_N \equiv \big(\underbrace{\boldsymbol{f}_h^1}_{N_h \times 1} \mid \ldots \mid \boldsymbol{f}_h^{Q_f} \mid \underbrace{-\tilde{\mathbb{A}}_h^1 \mathbb{V}_N}_{N_h \times N} \mid \ldots \mid -\tilde{\mathbb{A}}_h^{Q_a} \mathbb{V}_N \big); \tag{4.75}$$

note $\boldsymbol{L}_N$ is expressed in block column form with | as block delimiters. We now combine (4.73), the affine representations of the FE operators, (4.27), (4.74), and (4.75) to obtain

$$\|\tilde{R}_h(\boldsymbol{\mu})\|_V = \big(\boldsymbol{\Theta}_N^{\mathrm{T}}(\boldsymbol{\mu}) \underbrace{\boldsymbol{L}_N^{\mathrm{T}} \mathbb{X}_h^{-1} \boldsymbol{L}_N}_{W_N} \boldsymbol{\Theta}_N(\boldsymbol{\mu})\big)^{1/2}; \tag{4.76}$$

note that $W_N \in \mathbb{R}^{(Q_f + Q_a N) \times (Q_f + Q_a N)}$. We can now describe the offline-online decomposition.

In the offline stage we form $W_{N_{\max}}$. In the (say) direct-solution context, we would perform a sparse Cholesky of (optimally ordered) $\mathbb{X}_h$ once. We would then perform $Q_f + Q_a N_{\max}$ forward/back substitutions to find $\boldsymbol{L}'_{N_{\max}} \equiv \mathbb{X}_h^{-1} \boldsymbol{L}_{N_{\max}}$; we would then complete the formation of $W_{N_{\max}}$ by matrix multiplication, $\boldsymbol{L}_{N_{\max}}^{\mathrm{T}} \boldsymbol{L}'_{N_{\max}}$, at cost $\mathcal{O}((Q_f + Q_a N)^2 N_h)$ FLOPs. Note that $W_{N_{\max}}$ is *parameter-independent*. In the online stage, given $\boldsymbol{\mu} \in \mathcal{P}$ and our associated RB approximation $\boldsymbol{u}_N(\boldsymbol{\mu})$, we first extract submatrix W_N from $W_{N_{\max}}$ and evaluate $\boldsymbol{\Theta}_N(\boldsymbol{\mu})$. We then perform the sum (4.76), $(\boldsymbol{\Theta}_N^{\mathrm{T}}(\boldsymbol{\mu}) W_N \boldsymbol{\Theta}_N(\boldsymbol{\mu}))^{1/2}$. The operation count is $\mathcal{O}((Q_f + Q_a N)^2)$ FLOPs, which we note is independent of N_h.[5] We observe that

5 We can now readily define minimum-residual projection: Find $\tilde{\boldsymbol{u}}_N^*(\boldsymbol{\mu})$ and hence $\boldsymbol{\Theta}^*(\boldsymbol{\mu})$ which minimizes $\|\tilde{R}_h(\boldsymbol{\mu})\|_V$. It follows from our offline-online discussion that minimum-residual projection will be more expensive than Galerkin projection but only as regards formation of the RB linear system and in particular for larger Q_f, Q_a.

the operation count scales quadratically with both Q_f, Q_a and also N, which emphasizes the important role of a posteriori error estimation to control both the RB but also the EIM costs.

Finally, we note one shortcoming of the offline-online approach. Let us denote machine precision by ϵ_{prec}. The construction (4.76) computes a small number, $\|\tilde{R}_h(\boldsymbol{\mu})\|_V$, as the square root of the cancellation of (many) large summands. To illustrate the difficulty, consider $\epsilon_{\text{prec}} = 1\times10^{-16}$, assume $\|\tilde{R}_h(\boldsymbol{\mu})\|_V = 1\times10^{-10}$, and furthermore say (for simplicity) that $\boldsymbol{\Theta}_N^{\text{T}}(\boldsymbol{\mu})W_N\boldsymbol{\Theta}_N(\boldsymbol{\mu})$ is the sum of just two terms, respectively $1 + \frac{1}{2}\times10^{-20}$ and $-1 + \frac{1}{2}\times10^{-20}$. Clearly upon truncation in finite precision we obtain for the sum (respectively, the square root of the sum) not 10^{-20} (respectively, 10^{-10}) but rather 0. This finite-precision effect can in principle compromise numerical convergence tests for sufficiently high accuracy, and remedies are proposed in the literature [11, 4]. However, for engineering calculations, the limitation is not significant: It will very rarely, if ever, be the case that the data for an engineering problem are known to sufficient digits to warrant a numerical error as small as 1×10^{-8}.

4.3.2 Extensions

4.3.2.1 Evolution problems: parabolic PDEs

4.3.2.1.1 Galerkin projection

As for elliptic PDEs, we are given a hierarchical set of RB spaces $\{V_N\}_{N=1,\ldots,N_{\max}}$. We are further provided, for $1 \le N \le N_{\max}$, with an associated basis $\{\zeta_h^n\}_{n=1,\ldots,N}$, and corresponding basis matrix $\mathbb{V}_N \in \mathbb{R}^{N_h\times N}$, $(\mathbb{V}_N)_{kn} = (\boldsymbol{\zeta}_h^n)_k$, $1 \le k \le N_h$, $1 \le n \le N$. The method by which we shall develop $V_{N_{\max}}$ for the parabolic case [20], related to but also different from the method with which we identify $V_{N_{\max}}$ for the elliptic case, shall be summarized in Section 4.4.3.1. We note here only that the construction shall ensure that u_0 (our initial condition) resides in V_N.

We can now state the Galerkin projection [19, 20]: Given $N \in \{1,\ldots,N_{\max}\}$ and $\boldsymbol{\mu} \in \mathcal{P}$, we look for $\tilde{u}_{N,\Delta t}^j(\boldsymbol{\mu})(\approx u_h(t^j;\boldsymbol{\mu})) \in V_N$, $j = 1,\ldots,J$, such that

$$\tilde{m}\left(\frac{\tilde{u}_{N,\Delta t}^j(\boldsymbol{\mu}) - \tilde{u}_{N,\Delta t}^{j-1}(\boldsymbol{\mu})}{\Delta t}, v;\boldsymbol{\mu}\right) + \tilde{a}(\tilde{u}_{N,\Delta t}^j(\boldsymbol{\mu}), v;\boldsymbol{\mu}) = \tau(t^j)\tilde{f}(v;\boldsymbol{\mu}), \quad \forall v \in V_N; \tag{4.77}$$

we impose the initial condition $\tilde{u}_{N,\Delta t}^{j=0}(\boldsymbol{\mu}) = u_0$ ($\in V_N$, by construction). We note that in (4.77) there is no reduction in the temporal dimension: the RB projection (4.77) retains the "true" finite difference discretization of the FE projection (4.39); the RB acceleration is effected solely through the dimension reduction in the spatial dimension.

To develop the discrete equations we require the RB mass matrix, $\tilde{\mathbb{M}}_N(\boldsymbol{\mu}) \in \mathbb{R}^{N\times N}$,

$$(\tilde{\mathbb{M}}_N(\boldsymbol{\mu}))_{kn} = \tilde{m}(\zeta_h^n, \zeta_h^k;\boldsymbol{\mu}), \quad 1 \le k, n \le N, \tag{4.78}$$

which we may form as

$$\tilde{\mathbb{M}}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_m} \Theta_m^q(\boldsymbol{\mu}) \mathbb{M}_N^q, \tag{4.79}$$

for $\mathbb{M}_N^q = m^q(\zeta_h^n, \zeta_h^k), 1 \le k, n \le N, 1 \le q \le Q_m$. We may directly formulate our RB mass matrix and constituents in terms of the corresponding FE quantities and our basis matrix:

$$\tilde{\mathbb{M}}_N(\boldsymbol{\mu}) = \mathbb{V}_N^{\mathrm{T}} \tilde{\mathbb{M}}_h(\boldsymbol{\mu}) \mathbb{V}_N, \quad \mathbb{M}_N^q = \mathbb{V}_N^{\mathrm{T}} \mathbb{M}_h^q \mathbb{V}_N, \quad 1 \le q \le Q_m. \tag{4.80}$$

We can then state the discrete equations to be solved at each time t^j:

$$\underbrace{\left(\tilde{\mathbb{A}}_N(\boldsymbol{\mu}) + \frac{1}{\Delta t}\tilde{\mathbb{M}}_N(\boldsymbol{\mu})\right)}_{\tilde{\mathbb{H}}_N(\boldsymbol{\mu})} \tilde{\boldsymbol{u}}_{N,\Delta t}^j(\boldsymbol{\mu}) = \frac{1}{\Delta t}\tilde{\mathbb{M}}_N(\boldsymbol{\mu})\, \boldsymbol{u}_{N,\Delta t}^{j-1}(\boldsymbol{\mu}) + \tau(t^j)\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu}). \tag{4.81}$$

Note that, in general for RB methods, we prefer implicit temporal discretizations, since inversion of the small RB discrete operators is relatively inexpensive (and furthermore the RB mass matrix is not close to diagonal in particular given our V-orthonormalization of the basis) and implicit methods allow larger time steps.

4.3.2.1.2 Error estimation

We directly consider the more practically important case of a posteriori error estimation. We shall provide here an estimator for the case in which $\epsilon_{\mathrm{EIM}}^{\mathcal{P}} = 0$ and hence we shall suppress the $\tilde{\cdot}$ for the purposes of this analysis; extension to $\epsilon_{\mathrm{EIM}}^{\mathcal{P}} \neq 0$ is not difficult. We first introduce the error $e^j(\boldsymbol{\mu}) \equiv u_{h,\Delta t}^j - u_{N,\Delta t}^j(\boldsymbol{\mu})$, $1 \le j \le J$. We next define, for $1 \le j \le J$, the residual $\boldsymbol{\mu} \mapsto r_{h,\Delta t}^j(\cdot;\boldsymbol{\mu}) \in V'$,

$$r_{h,\Delta t}^j(v;\boldsymbol{\mu}) \equiv \tau(t^j)\tilde{f}(v;\boldsymbol{\mu}) - m\left(\frac{u_{N,\Delta t}^j(\boldsymbol{\mu}) - u_{N,\Delta t}^{j-1}(\boldsymbol{\mu})}{\Delta t}, v;\boldsymbol{\mu}\right) - a(u_{N,\Delta t}^j(\boldsymbol{\mu}), v;\boldsymbol{\mu}), \quad \forall v \in V_h, \tag{4.82}$$

and associated Riesz representation, $R_{h,\Delta t}^j(\boldsymbol{\mu}) \equiv \mathcal{R}_h r_{h,\Delta t}^j(\cdot;\boldsymbol{\mu})$. We then define, for $1 \le j \le J$, our error estimator $\Delta_{N,\Delta t}^j(\boldsymbol{\mu})$:

$$\Delta_{N,\Delta t}^j(\boldsymbol{\mu}) \equiv \left(\frac{\Delta t}{c_h^{\mathrm{s,app}}(\boldsymbol{\mu})} \sum_{j'=1}^{j} \|R_{h,\Delta t}^{j'}\|_V^2\right)^{1/2}, \tag{4.83}$$

where $c_h^{\mathrm{s,app}}(\boldsymbol{\mu})$ is an approximation to $\alpha_h(\boldsymbol{\mu})$, for example as developed in Section 4.3.1.3. It can then be shown [19] that, if $c_h^{\mathrm{s,app}}(\boldsymbol{\mu}) \le \alpha_h(\boldsymbol{\mu})$, then

$$\left(m(e^j(\boldsymbol{\mu}), e^j(\boldsymbol{\mu});\boldsymbol{\mu}) + c_h^{\mathrm{s,app}}(\boldsymbol{\mu}) \sum_{j'=1}^{j} \|e^{j'}(\boldsymbol{\mu})\|_V^2\right)^{1/2} \le \Delta_{N,\Delta t}^j(\boldsymbol{\mu}), \quad j = 1, \ldots, J. \tag{4.84}$$

We note that, consistent with the continuous problem formulation, we control both the (discrete-time) $C^0((0,T];L^2(\Omega))$ error and the $L^2((0,T];V)$-error.

4.3.2.1.3 Offline-online computational procedures

As for the elliptic case, we presume here that the RB basis matrix, $V_{N_{\max}}$, is provided; we discuss the associated procedure, part of the offline stage, in Section 4.4.3.1.

We first consider the formation and solution of the RB linear system. In the offline stage, we form $\mathbb{A}^q_{N_{\max}}, 1 \le q \le Q_a$, $\mathbb{M}^q_{N_{\max}}, 1 \le q \le Q_m$, $\boldsymbol{f}^q_{N_{\max}}, 1 \le q \le Q_f$, and $\boldsymbol{\ell}^q_{N_{\max}}, 1 \le q \le Q_\ell$ of (4.52) and (4.80); the operation count, taking into account FE sparsity, is $\mathcal{O}((Q_a+Q_m)N^2_{\max}N_h)+\mathcal{O}(Q_f N_{\max}N_h)+\mathcal{O}(Q_\ell N_{\max}N_h)$. In the online stage, for any given N and $\boldsymbol{\mu} \in \mathcal{P}$, we first form $\tilde{\mathbb{A}}_N(\boldsymbol{\mu})$, $\tilde{\mathbb{M}}_N(\boldsymbol{\mu})$, $\tilde{\mathbb{H}}_N(\boldsymbol{\mu})$, $\tilde{\boldsymbol{f}}_N(\boldsymbol{\mu})$, and $\tilde{\boldsymbol{\ell}}_N(\boldsymbol{\mu})$ from (4.50) and (4.79) in $\mathcal{O}((Q_a+Q_m)N^2)+\mathcal{O}(Q_f N)+\mathcal{O}(Q_\ell N)$ FLOPs; we next perform the LU decomposition of $\tilde{\mathbb{H}}_N(\boldsymbol{\mu})$, once, at cost $\mathcal{O}(N^3)$; we then solve (4.81) as J forward/back substitutions at cost $\mathcal{O}(JN^2)$; finally, we evaluate $s_N(\boldsymbol{\mu})$ in $\mathcal{O}(N)$ FLOPs. Note that we take advantage here of the linear *time-invariant* nature of our operator: For J not too small, we expect the online cost of the J parabolic updates to scale as $\mathcal{O}(JN^2)$ FLOPs, hence roughly independent of $Q_\cdot$ and only quadratically with N.

We next turn to the evaluation of the error bound, and in particular the contribution of the dual norm of the residual. We now introduce $\boldsymbol{\Theta}^j_{N,\Delta t} \in \mathbb{R}^{Q_f+Q_m N+Q_a N}, 1 \le j \le J$, as

$$
\begin{aligned}
\boldsymbol{\Theta}^j_{N,\Delta t}(\boldsymbol{\mu}) \equiv \Bigg(& \Theta^1_f(\boldsymbol{\mu}), \ldots, \Theta^{Q_f}_f(\boldsymbol{\mu}), \\
& \Theta^1_m(\boldsymbol{\mu}) \frac{(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_1 - (\boldsymbol{u}^{j-1}_{N,\Delta t}(\boldsymbol{\mu}))_1}{\Delta t}, \ldots, \Theta^1_m(\boldsymbol{\mu}) \frac{(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_N - (\boldsymbol{u}^{j-1}_{N,\Delta t}(\boldsymbol{\mu}))_N}{\Delta t}, \ldots, \\
& \Theta^{Q_m}_m(\boldsymbol{\mu}) \frac{(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_1 - (\boldsymbol{u}^{j-1}_{N,\Delta t}(\boldsymbol{\mu}))_1}{\Delta t}, \ldots, \Theta^{Q_m}_m(\boldsymbol{\mu}) \frac{(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_N - (\boldsymbol{u}^{j-1}_{N,\Delta t}(\boldsymbol{\mu}))_N}{\Delta t}, \\
& \Theta^1_a(\boldsymbol{\mu})(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_1, \ldots, \Theta^1_a(\boldsymbol{\mu})(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_N, \ldots, \\
& \Theta^{Q_a}_a(\boldsymbol{\mu})(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_1, \ldots, \Theta^{Q_a}_a(\boldsymbol{\mu})(\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu}))_N \Bigg)^{\mathrm{T}},
\end{aligned}
\tag{4.85}
$$

and also $\boldsymbol{L}_{N,\Delta t} \in \mathbb{R}^{N_h \times (Q_f+(Q_m+Q_a)N)}$ as

$$
\boldsymbol{L}_{N,\Delta t} \equiv (\boldsymbol{f}^1_h | \ldots | \boldsymbol{f}^{Q_f}_h | -\mathbb{M}^1_h \mathbb{V}_N | \ldots | -\mathbb{M}^{Q_m}_h \mathbb{V}_N | -\tilde{\mathbb{A}}^1_h \mathbb{V}_N | \ldots | -\tilde{\mathbb{A}}^{Q_a}_h \mathbb{V}_N); \tag{4.86}
$$

note $\boldsymbol{L}_{N,\Delta t}$ is expressed in block column form with | as block delimiters. We now combine (4.73), the affine representations of the FE operators, (4.27), (4.85), and (4.86) to obtain

$$
\|\tilde{R}^j_{h,\Delta t}(\boldsymbol{\mu})\|_V = \big((\boldsymbol{\Theta}^j_{N,\Delta t})^{\mathrm{T}}(\boldsymbol{\mu}) \underbrace{\boldsymbol{L}^{\mathrm{T}}_{N,\Delta t} \mathbb{X}^{-1}_h \boldsymbol{L}_{N,\Delta t}}_{W_{N,\Delta t}} \boldsymbol{\Theta}^j_{N,\Delta t}(\boldsymbol{\mu})\big)^{1/2}; \tag{4.87}
$$

note that $W_{N,\Delta t} \in \mathbb{R}^{(Q_f+(Q_m+Q_a)N)\times(Q_f+(Q_m+Q_a)N)}$.

In the offline stage we form $W_{N_{\max},\Delta t}$. In the (say) direct-solution context, we would perform a sparse Cholesky of (optimally ordered) $\mathbb{X}_h$ once. We would then perform $Q_f + (Q_m+Q_a)N_{\max}$ forward/back substitutions to find $\boldsymbol{L}'_{N_{\max},\Delta t} \equiv \mathbb{X}_h^{-1}\boldsymbol{L}_{N_{\max},\Delta t}$; we would then complete the formation of $W_{N_{\max},\Delta t}$ by multiplication, $\boldsymbol{L}^{\mathrm{T}}_{N_{\max},\Delta t}\boldsymbol{L}'_{N_{\max},\Delta t}$, at cost $\mathcal{O}((Q_f + (Q_m + Q_a)N)^2 N_h)$ FLOPs. Note that $W_{N_{\max},\Delta t}$ is *parameter-independent*. In the online stage, given $\boldsymbol{\mu} \in \mathcal{P}$ and our associated RB approximation $\boldsymbol{u}^j_{N,\Delta t}(\boldsymbol{\mu})$, $1 \le j \le J$, we first extract submatrix $W_{N,\Delta t}$ from $W_{N_{\max},\Delta t}$ and evaluate $\boldsymbol{\Theta}^j_{N,\Delta t}(\boldsymbol{\mu})$, $1 \le j \le J$. We then perform the sum (4.87), $((\boldsymbol{\Theta}^j_{N,\Delta t}(\boldsymbol{\mu}))^{\mathrm{T}}\, W_{N,\Delta t}\, \boldsymbol{\Theta}^j_{N,\Delta t}(\boldsymbol{\mu}))^{1/2}$, $1 \le j \le J$; the operation count (for given j) is $\mathcal{O}((Q_f + (Q_m + Q_a)N)^2)$ FLOPs, which we note again is independent of N_h.

4.3.2.2 A nonlinear elliptic problem

4.3.2.2.1 Galerkin projection

As for linear elliptic problems, we are given a hierarchical set of RB spaces $\{V_N \subset V_h\}_{N=1,\ldots,N_{\max}}$ and associated basis $\{\zeta^i_h\}_{i=1,\ldots,N_{\max}}$. We may then define our RB-Galerkin approximation for (4.41): Given $N \in \{1,\ldots,N_{\max}\}$ and $\boldsymbol{\mu} \in \mathcal{P}$, find $\tilde{u}_N(\boldsymbol{\mu}) \in V_N$ such that

$$\begin{aligned}
&\sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega}\, \nabla\tilde{u}_N(x_j^{\mathrm{quad},\Omega};\boldsymbol{\mu}) \cdot \nabla v(x_j^{\mathrm{quad},\Omega}) \\
&+ \sum_{m,m'=1}^{M} (B_M^{-1})_{mm'}\, \eta(\tilde{u}_N(x^{\mathrm{quad},\Omega}_{i^*_{m'}},\boldsymbol{\mu}))\left[\sum_{j=1}^{N^{\mathrm{quad},\Omega}} (\boldsymbol{\xi}^m)_j\, v(x_j^{\mathrm{quad},\Omega})\, \rho_j^{\mathrm{quad},\Omega}\right] \\
&= \boldsymbol{\mu} \sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega} v(x_j^{\mathrm{quad},\Omega}), \quad \forall v \in V_N, \qquad (4.88)
\end{aligned}$$

where the EIM interpolation system is defined in Section 4.2.3.3 (recall (4.44)). We may further introduce the Newton update equation associated with (4.88): For given current iterate $\tilde{u}^k_N(\boldsymbol{\mu}) \in V_N$, find $\delta\tilde{u}_N(\boldsymbol{\mu}) \in V_N$ such that

$$\begin{aligned}
&\sum_{j=1}^{N^{\mathrm{quad},\Omega}} \rho_j^{\mathrm{quad},\Omega}\, \nabla\delta\tilde{u}_N(x_j^{\mathrm{quad},\Omega};\boldsymbol{\mu}) \cdot \nabla v(x_j^{\mathrm{quad},\Omega}) \\
&+ \sum_{m,m'=1}^{M} (B_M^{-1})_{mm'}\, \dot{\eta}(\tilde{u}^k_N(x^{\mathrm{quad},\Omega}_{i^*_{m'}},\boldsymbol{\mu}))\, \delta\tilde{u}^k_N(x^{\mathrm{quad},\Omega}_{i^*_{m'}};\boldsymbol{\mu})\left[\sum_{j=1}^{N^{\mathrm{quad},\Omega}} (\boldsymbol{\xi}^m)_j\, v(x_j^{\mathrm{quad},\Omega})\, \rho_j^{\mathrm{quad},\Omega}\right] \\
&= \tilde{r}^k_h(v;\boldsymbol{\mu}), \quad \forall v \in V_N. \qquad (4.89)
\end{aligned}$$

Note $\dot{\eta}$ is the derivative of η (we reserve prime for dummy indices): For example, for $\eta(z) = z^3$, $\dot{\eta}(z) = 3z^2$, and for $\eta(z) = |z|z$, $\dot{\eta}(z) = 2|z|$; in general, $\dot{\eta} > 0$ from our

assumption of monotonic nondecreasing η. The right-hand side of (4.89), residual $\boldsymbol{\mu} \in \mathcal{P} \mapsto \tilde{r}_h^k(\cdot;\boldsymbol{\mu}) \in V_h'$, is defined as

$$\begin{aligned}\tilde{r}_h^k(v;\boldsymbol{\mu}) \equiv \boldsymbol{\mu} & \sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} v(x_j^{\text{quad},\Omega}) \\ & - \sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} \nabla \tilde{u}_N^k(x_j^{\text{quad},\Omega};\boldsymbol{\mu}) \cdot \nabla v(x_j^{\text{quad},\Omega}) \\ & - \sum_{m,m'=1}^{M} (B_M^{-1})_{mm'} \eta(\tilde{u}_N^k(x_{i_{m'}^*}^{\text{quad},\Omega},\boldsymbol{\mu})) \left[\sum_{j=1}^{N^{\text{quad},\Omega}} (\boldsymbol{\xi}^m)_j v(x_j^{\text{quad},\Omega}) \rho_j^{\text{quad},\Omega} \right], \quad \forall v \in V_N. \end{aligned} \tag{4.90}$$

The left-hand side of (4.89) is the Gâteaux derivative of our RB nonlinear operator.

We now proceed to the discrete equations. We first introduce parameter-independent matrices and vectors $\mathbb{B}_N \in \mathbb{R}^{N\times N}$, $\mathbb{C}_N \in \mathbb{R}^{M\times N}$, and $\boldsymbol{f}_N^1 \in \mathbb{R}^N$ given by

$$(\mathbb{B}_N)_{nn'} \equiv \sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} \nabla \zeta_h^{n'}(x_j^{\text{quad},\Omega}) \cdot \nabla \zeta_h^n(x_j^{\text{quad},\Omega}), \quad 1 \le n, n' \le N, \tag{4.91}$$

$$(\mathbb{C}_N)_{m'n} \equiv \sum_{m=1}^{M} (B_M^{-1})_{mm'} \sum_{j=1}^{N^{\text{quad},\Omega}} (\boldsymbol{\xi}^m)_j \zeta_h^n(x_j^{\text{quad},\Omega}) \rho_j^{\text{quad},\Omega}, \quad 1 \le m' \le M, 1 \le n \le N, \tag{4.92}$$

$$(\boldsymbol{f}_N^1)_n \equiv \sum_{j=1}^{N^{\text{quad},\Omega}} \rho_j^{\text{quad},\Omega} \zeta_h^n(x_j^{\text{quad},\Omega}), \tag{4.93}$$

respectively. Note that $\mathbb{B}_N$, $\mathbb{C}_N$, and $\boldsymbol{f}_N^1$ are all parameter-independent.

Our Newton update may then be expressed as

$$\begin{aligned}\sum_{n'=1}^{N} (\mathbb{J}_N^k(\boldsymbol{\mu}))_{nn'} (\delta \boldsymbol{u}_N^k(\boldsymbol{\mu}))_{n'} = \boldsymbol{\mu}(\boldsymbol{f}_N^1)_n & - \sum_{n'=1}^{N} (\mathbb{B}_N)_{nn'} (\boldsymbol{u}_N^k(\boldsymbol{\mu}))_{n'} \\ & - \sum_{m'=1}^{M} \eta_{N\,m'}^k(\boldsymbol{\mu}) (\mathbb{C}_N)_{m'n}, \quad 1 \le n \le N, \end{aligned} \tag{4.94}$$

where the Jacobian matrix $\mathbb{J}_N^k(\boldsymbol{\mu}) \in \mathbb{R}^{N\times N}$ is given by

$$(\mathbb{J}_N^k(\boldsymbol{\mu}))_{nn'} \equiv (\mathbb{B}_N)_{nn'} + \sum_{m'=1}^{M} \dot{\eta}_{N\,m'}^k(\boldsymbol{\mu}) (\mathbb{C}_N)_{m'n} \zeta_h^{n'}(x_{i_{m'}^*}^{\text{quad},\Omega}), \quad 1 \le n, n' \le N. \tag{4.95}$$

Here $\eta_{N\,m'}^k$, $\dot{\eta}_{N\,m'}^k$, $1 \le m' \le M$, are given by

$$\eta_{N\,m'}^k(\boldsymbol{\mu}) = \eta\left(\sum_{n=1}^{N} (\tilde{\boldsymbol{u}}_N^k(\boldsymbol{\mu}))_n \zeta_h^n(x_{i_{m'}^*}^{\text{quad},\Omega}) \right), \quad 1 \le m' \le M, \tag{4.96}$$

$$\dot{\eta}^k_{N\,m'}(\boldsymbol{\mu}) = \dot{\eta}\left(\sum_{n=1}^{N}(\tilde{\boldsymbol{u}}^k_N(\boldsymbol{\mu}))_n \zeta^n_h(x^{\text{quad},\Omega}_{i^*_{m'}})\right), \quad 1 \le m' \le M. \tag{4.97}$$

We note that for $\eta(z) = z^3$ it can be shown that $\mathbb{J}^k_N(\boldsymbol{\mu})$ of (4.95) is symmetric positive-definite for $\epsilon^{\text{NL}}_{\text{EIM}}$ sufficiently small.

4.3.2.2.2 Error estimation

We directly consider the more practically important case of a posteriori error estimation. We shall continue to assume (effectively) exact quadrature. By way of preliminaries, we define the Laplacian bilinear form $b : V \times V \to \mathbb{R}$,

$$b(w, v) = \int_\Omega \nabla w \cdot \nabla v, \quad \forall w, v \in V^2. \tag{4.98}$$

We then define (for consistency with earlier parts of the chapter)

$$\tilde{c}^{\text{s}}_h(\boldsymbol{\mu}) \equiv \inf_{v \in V_h} \frac{|b(v, v)|}{\|v\|_V}; \tag{4.99}$$

in the current context, $\tilde{c}^{\text{s}}_h(\boldsymbol{\mu})$ is a coercivity constant which is in fact parameter-independent.

We now define our error as $\tilde{e}^k(\boldsymbol{\mu}) \equiv u_h(\boldsymbol{\mu}) - \tilde{u}^k_N(\boldsymbol{\mu})$. We further introduce the Riesz representation of our residual, $\tilde{R}^k_h = \mathcal{R}_h \tilde{r}^k_h(\cdot; \boldsymbol{\mu})$ for $\tilde{r}^k_h(\cdot; \boldsymbol{\mu})$ defined in (4.90). We may then define our a posteriori error estimator,

$$\tilde{\Delta}^k_N(\boldsymbol{\mu}) = \frac{\|\tilde{R}^k_h(\boldsymbol{\mu})\|_V + c^{-1/2}_{L^2} |\Omega|^{1/2} \text{tol}^{\text{NL}}_{\text{EIM}}}{\tilde{c}^{\text{s,app}}_h(\boldsymbol{\mu})}. \tag{4.100}$$

Here c_{L^2} is the weight of the $L^2(\Omega)$-contribution to our norm $\|\cdot\|_V$, $|\Omega|$ is the measure of our domain in $\mathbb{R}^d$, and $\tilde{c}^{\text{s,app}}_h(\boldsymbol{\mu})$ is an approximation to the coercivity constant $\tilde{c}^{\text{s}}_h(\boldsymbol{\mu})$, (4.99), as developed in Section 4.3.1.3. We can then show that, for any $\boldsymbol{\mu} \in \Xi^{\text{NL}}_{\text{EIM}}$, and $\tilde{c}^{\text{s,app}}_h(\boldsymbol{\mu}) \le \tilde{c}^{\text{s}}_h(\boldsymbol{\mu})$,

$$\|\tilde{e}^k(\boldsymbol{\mu})\|_V \le \tilde{\Delta}^k_N(\boldsymbol{\mu}); \tag{4.101}$$

we can plausibly extend our result to any $\boldsymbol{\mu} \in \mathcal{P}$, perhaps with a safety factor, depending on the richness of $\Xi^{\text{NL}}_{\text{EIM}}$. The a posteriori error bound (4.100), (4.101), identifies three contributions to the error: the (incomplete) convergence of the Newton iteration, as reflected in superscript k; the RB approximation error, as reflected in the dual norm of the residual; and the perturbation of our problem to affine form, as reflected in the EIM tolerance parameter. We emphasize that, within this nonlinear iterative context, the error bound can serve not only to assess and control RB and EIM errors but also as an iterative termination criterion.

We briefly derive this result in particular to emphasize the simplification afforded by our monotonic nondecreasing nonlinearity and conversely the challenges associated with more difficult nonlinearities. We first note that (under our assumption of exact quadrature) our residual may be expressed as

$$\tilde{r}_h^k(v;\boldsymbol{\mu}) \equiv \boldsymbol{\mu} \int_\Omega v - b(\tilde{u}_N^k(\boldsymbol{\mu}), v) - \int_\Omega \mathcal{I}_M[\eta(\tilde{u}_N^k(\cdot;\boldsymbol{\mu}))]v, \quad \forall v \in V_h. \tag{4.102}$$

We also note that $u_h(\boldsymbol{\mu})$ satisfies

$$b(u_h(\boldsymbol{\mu}), v) + \int_\Omega \eta(u_h(\boldsymbol{\mu}))v = \boldsymbol{\mu} \int_\Omega v, \quad \forall v \in V_h. \tag{4.103}$$

It thus follows that

$$\begin{aligned} b(u_h(\boldsymbol{\mu}) - \tilde{u}_N^k(\boldsymbol{\mu}), v) + \int_\Omega [\eta(u_h(\boldsymbol{\mu})) - \eta(\tilde{u}_N^k(\boldsymbol{\mu}))]v \\ = \tilde{r}_h^k(v;\boldsymbol{\mu}) + \int_\Omega [\mathcal{I}_M[\eta(\tilde{u}_N^k(\cdot;\boldsymbol{\mu}))] - \eta(\tilde{u}_N^k(\boldsymbol{\mu}))]v, \quad \forall v \in V_h. \end{aligned} \tag{4.104}$$

We now choose $v = u_h(\boldsymbol{\mu}) - \tilde{u}_N^k(\boldsymbol{\mu})$ and note from our assumption of monotonic nondecreasing η that

$$\int_\Omega [\eta(u_h(\boldsymbol{\mu})) - \eta(\tilde{u}_N^k(\boldsymbol{\mu}))][u_h(\boldsymbol{\mu}) - \tilde{u}_N^k(\boldsymbol{\mu})] > 0. \tag{4.105}$$

We then consider the interpolation error term and invoke the Cauchy–Schwarz inequality and our definition of the norm $\|\cdot\|_V$; the result directly follows. We emphasize that without the property (4.105) we would need to estimate the inf-sup constant associated with the Jacobian and then consider a contraction argument per the Brezzi–Rappaz–Raviart theory [37] – hence much more difficult to realize in practice if we wish to quantitatively evaluate the necessary constants.

4.3.2.2.3 Offline-online computational procedures

As for the linear case, we presume here that the RB space and basis are provided.

We first consider the formation and solution of the RB nonlinear system. In the offline stage, we perform the EIM for the nonlinear term and subsequently form $\mathbb{B}_{N_{\max}}$, $\mathbb{C}_{N_{\max}}$, and $\boldsymbol{f}^1_{N_{\max}}$. Then at each Newton iteration, we first form η_N^k and $\dot{\eta}_N^k$ at cost $\mathcal{O}(MN)$ FLOPs. We next form the Jacobian at cost $\mathcal{O}(MN^2)$ and the residual at cost $\mathcal{O}(MN)$. Finally, we invert the Jacobian at cost $O(N^3)$ FLOPs.

We next consider the a posteriori error estimator and in particular the dual norm of the residual. In fact, once we evaluate $(\eta_{N\,m}^k(\boldsymbol{\mu}))_{m=1,\ldots,M}$, the residual of our nonlinear

elliptic problem is computationally analogous to the residual of a linear elliptic problem, with the $(\eta^k_{N\,m}(\boldsymbol{\mu}))_{m=1,\ldots,M}$ playing a very similar role to $(\mathbf{u}_N(\boldsymbol{\mu}))_{n=1,\ldots,N}$. The online operation count for the dual norm of the residual is then $\mathcal{O}((N+M)^2)$, hence quite inexpensive; of course, if the weak form included other parameters, hence $p > 1$, the operation count would increase commensurately.

We close this section by emphasizing the importance of hyperreduction. We first consider nonpolynomial nonlinearity, for example $\eta(z) = |z|z$: In the absence of hyperreduction, the online cost would depend on N_h, both for formation of the Jacobian and for evaluation of the dual norm of the residual. We next consider polynomial nonlinearities, hence $\eta(z) = z^s$ for s an odd integer (in order to honor our monotonic nondecreasing assumption): In the absence of hyperreduction, the online cost for formation of the residual is $\mathcal{O}(N^{s+1})$ and the online cost for evaluation of the dual norm of the residual is $\mathcal{O}(N^{2s})$, and thus potentially prohibitive even for $s = 3$. We illustrate the difficulty for a polynomial nonlinearity $\eta(z) = z^3$. In this case the nonlinear contribution to the residual is given by

$$\underbrace{\sum_{n=1}^{N}\sum_{n'=1}^{N}\sum_{n''=1}^{N}\left(\tilde{\boldsymbol{u}}_N(\boldsymbol{\mu})\right)_n\left(\tilde{\boldsymbol{u}}_N(\boldsymbol{\mu})\right)_{n'}\left(\tilde{\boldsymbol{u}}_N(\boldsymbol{\mu})\right)_{n''}\underbrace{\int_\Omega \zeta_h^n\zeta_h^{n'}\zeta_h^{n''}\zeta_h^{n'''}}_{\text{form offline}}}_{\text{evaluate online}}, \tag{4.106}$$

where $\zeta_h^{n'''}$ plays the role of test function.

4.4 Approximation spaces

4.4.1 Elliptic problems: weak greedy method

Perhaps the most simple – and the most popular – approach to the identification of reduced-order approximation spaces is the proper orthogonal decomposition (POD).[6] It consists in building a matrix with entries $(\tilde{u}_h(\boldsymbol{\mu}), \tilde{u}_h(\boldsymbol{\mu}'))_V$, $\boldsymbol{\mu}$, and $\boldsymbol{\mu}'$ belonging to some subset Ξ_{POD} of $\mathcal{P}$ with cardinal K_{POD} (large enough since Ξ_{POD} is supposed to scan well $\mathcal{P}$), and considering the eigenvectors associated to the largest values of its largest eigenvalues. The interest of this approach is two-fold:

– it is very simple to implement and does not rely on further mathematical ingredients as is required in other methods (such as error estimators for the weak greedy approach that will be explained next);

6 The POD can also be interpreted in terms of singular value decomposition, principal component analysis, or even Karhunen–Loève transform.

– it provides some hope that an RB exists (when the eigenvalues' decay is fast enough), but also, on the contrary, a proof that the Kolmogorov N-width is (in some bad situations) large since $d_N(\mathcal{M}, V)$ is always larger than $\sum_{j>N} \sigma_j^2$.

However, in the parametric RB context, the POD has several drawbacks:
– formation of the covariance matrix requires many (K_{POD}) queries to the FE approximation, $\boldsymbol{\mu} \mapsto \tilde{u}_h(\boldsymbol{\mu})$; identification of the POD modes requires solution of a dense $K_{\text{POD}} \times K_{\text{POD}}$ eigenproblem;
– the norm in which the approximation is optimal is $L^2(\mathcal{P}; V)$, which is not ideal for pointwise (in parameter) queries, $\boldsymbol{\mu} \mapsto \tilde{u}_N(\boldsymbol{\mu})$.

Alternatives, widely used in the RB context, are the greedy and (because more amenable for implementation) the weak greedy approach: The optimality norm $L^\infty(\mathcal{P}; V)$ is more appropriate, consistent with the Kolmogorov N-width definition; only $N_{\max}$ FE queries are required to identify the RB spaces $\{V_N\}_{N=1,\dots,N_{\max}}$; the resulting approximation space nevertheless reflects information from $K_{\text{trial}} \gg N_{\max}$ points on the parametric manifold. The POD is discussed in depth in several other chapters in this handbook; we focus here on the greedy-type methods.

We first present the algorithm: Let Ξ_{trial} be a given subset of $\mathcal{P}$ with cardinal K_{trial} (large enough since Ξ_{trial} is supposed to scan well $\mathcal{P}$).

Algorithm 4.2: Greedy method: We assume that the set $\{\tilde{u}_h(\boldsymbol{\mu})\}_{\boldsymbol{\mu}\in\Xi_{\text{trial}}}$ is not embedded in a small finite-dimensional space.

Data: $\Xi_{\text{trial}}, \tilde{u}_h : \mathcal{P} \to V_h, \|\cdot\|_{\text{Greedy}}, \text{tol}_{\text{Greedy}}$
Result: M, $\{\boldsymbol{\mu}_m \in \Xi_{\text{trial}}\}_{1\le m\le M}$, $\{\boldsymbol{\zeta}^i \in V_h\}_{1\le i\le M}$, $V_M \subset V, \dim V_M = M$
1 Set $M = 0$, $V_0 = \emptyset$, and err $= \infty$;
2 **while** err $>$ tol$_{\text{Greedy}}$ **do**
3 Set $M \leftarrow M + 1$;
4 Find $\boldsymbol{\mu}_M = \arg\sup_{\boldsymbol{\mu}\in\Xi_{\text{trial}}} \|\tilde{u}_h(\boldsymbol{\mu}) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu})\|_{\text{Greedy}}$ (where Π_{M-1} denotes a projection approach – like an orthogonal projection in V or a Galerkin projection – onto V_{M-1} according to $\|\cdot\|_{\text{Greedy}}$);
5 Define $\boldsymbol{\zeta}^M = (\tilde{u}_h(\boldsymbol{\mu}_M) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu}_M)) / \|(\tilde{u}_h(\boldsymbol{\mu}_M) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu}_M))\|_{\text{Greedy}}$;
6 Update $V_M = \text{Span}\{\boldsymbol{\zeta}^i, i = 1, \dots, M\}$;
7 Set err $= \|\tilde{u}_h(\boldsymbol{\mu}_M) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu}_M)\|_{\text{Greedy}}$;
8 **end**

The pure greedy method is associated to the choice $\Xi_{\text{trial}} = \mathcal{P}$ and $\|\cdot\|_{\text{Greedy}} = \|\cdot\|_V$, the projection operator being the V-orthogonal projection. It is a theoretical algorithm as the computation of $\boldsymbol{\mu}^*$ on line 4 above is quite impossible to determine as it requires in particular the knowledge of every $\tilde{u}_h(\boldsymbol{\mu})$, for any $\boldsymbol{\mu} \in \Xi_{\text{trial}}$. Alternatively a weak greedy approach can be implemented where the exact V-norm is replaced by a surrogate,

an example being given by the a posteriori error estimate for Galerkin-RB approximations, as the ones that have been presented in Section 3 (see [38] for the first actual use of this weak greedy approach). From (4.58), we can thus interpret the weak version with respect to the pure greedy method by replacing line 4 above by

$$\text{Select a } \boldsymbol{\mu}_m \text{ such that } \|\tilde{u}_h(\boldsymbol{\mu}_m) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu}_m)\|_V \geq \gamma \sup_{\boldsymbol{\mu}\in\mathcal{P}} \|\tilde{u}_h(\boldsymbol{\mu}) - \Pi_{M-1}\tilde{u}_h(\boldsymbol{\mu})\|_V \quad (4.107)$$

with some given γ, such that $0 < \gamma < 1$ (in this context, the denomination "weak greedy" was first introduced in [8]). Another element that enters in the weak greedy is the choice $\Xi_{\text{trial}} \subset_{\neq} \mathcal{P}$ that should be large enough and well chosen so that (4.107) remains true.

4.4.2 Optimality of the weak greedy method

In this subsection we state recent results about the performance of the weak greedy algorithm compared to the Kolmogorov N-width optimal choice given by $d_N(\mathcal{M}, V)$. First of all, we define $\sigma_N(\mathcal{M}, V)$: a comparable quantity with respect to $d_N(\mathcal{M}, V)$ which is associated to the series of spaces $\{V_n\}_{n\geq 0}$ defined in the greedy algorithm

$$\sigma_n(\mathcal{M}, V) = \text{dist}(\mathcal{M}, V_n) = \sup_{\boldsymbol{\mu}} \inf_{v_n\in V_n} \|\tilde{u}_h(\boldsymbol{\mu}) - v_n\|_V \quad (4.108)$$

that characterizes the approximation space resulting from the weak greedy algorithm.

Of course, if $(\sigma_n(\mathcal{M}, V))_{n\geq 0}$ decays at a rate comparable to $(d_n(\mathcal{M}, V))_{n\geq 0}$, this means that the greedy selection provides essentially the best possible accuracy attainable by n-dimensional subspaces. The first comparison between $(\sigma_n(\mathcal{M}, V))_{n\geq 0}$ and $(d_n(\mathcal{M}, V))_{n\geq 0}$ was given in [9]: $\sigma_n(\mathcal{M}, V) \leq Cn2^n d_n(\mathcal{M}, V)$ which, of course, is only useful if $d_n(\mathcal{M}, V)$ decays to zero faster than $n^{-1}2^{-n}$. A more conservative estimate results from the series of papers from [8] and [14]. We present here their proof in the Hilbertian context.

For any choice Ξ_{trial} with cardinal $K_{\text{trial}} > 0$ large enough, let us consider the set $X_{K_{\text{trial}}}$ of all vectors $\tilde{u}_h(\boldsymbol{\mu})$, $\boldsymbol{\mu} \in \Xi_{\text{trial}}$. It is included in $\mathcal{M}$ and thus we derive obviously

$$\forall m > 0, \quad d_m(X_{K_{\text{trial}}}, V) \leq d_m(\mathcal{M}, V). \quad (4.109)$$

This means that there exist m vectors $b_1, b_2, \ldots, b_m$, generating a finite-dimensional space $\mathcal{H}_m$ such that

$$\max_{\boldsymbol{\mu}\in\Xi_{\text{trial}}} \min_{\mathbf{c}_m\in\mathcal{H}_m} \|\tilde{u}_h(\boldsymbol{\mu}) - \mathbf{c}_m\|_V \leq d_m(\mathcal{M}, V). \quad (4.110)$$

By taking the projection of these vectors $b_1, b_2, \ldots, b_m$ in $\text{Span}(X_{K_{\text{trial}}})$, we can determine m orthonormal vectors in $\text{Span}(X_{K_{\text{trial}}})$, which we denote by $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m$ that generate

a finite-dimensional space $\widetilde{\mathcal{H}}_m \subset \mathrm{Span}(X_{K_{\mathrm{trial}}})$ of dimension m[7] such that

$$\max_{\boldsymbol{\mu}\in\Xi_{\mathrm{trial}}} \min_{\widetilde{\mathbf{c}_m}\in\widetilde{\mathcal{H}}_m} \|\tilde{u}_h(\boldsymbol{\mu}) - \widetilde{\mathbf{c}_m}\|_V \leq d_m(\mathcal{M}, V). \tag{4.111}$$

Let us now introduce the scalar products $a_{n,j} = (\tilde{u}_h(\boldsymbol{\mu}_n), \boldsymbol{\zeta}^j)_V$, $1 \leq j, n \leq K_{\mathrm{trial}}$ where $\boldsymbol{\zeta}^{\cdot}$ are defined on line 5 of Algorithm 4.2). We easily get $a_{n,j} = 0$ for $j > n$ and $\tilde{u}_h(\boldsymbol{\mu}_n) = \sum_{j=0}^{n} a_{n,j}\zeta_j$. The matrix A with entries $a_{n,j}$ is lower triangular and incorporates all the information about the weak greedy algorithm on $\mathcal{M}$. In addition, whatever the definition of the space V, this matrix leads us to an analysis in $\ell^2(\mathbb{N})$ (even $\ell^2(\{1, 2, \ldots, K_{\mathrm{trial}}\})$) more simple than the analysis in V. From the very definition of the greedy selection, we get the two following properties:

$\mathbf{P_1}$: The diagonal elements of A satisfy $\gamma\sigma_n(\mathcal{M}, V) \leq |a_{n,n}| \leq \sigma_n(\mathcal{M}, V)$.

$\mathbf{P_2}$: For every $m \geq n$, one has $\sum_{j=n}^{m} a_{m,j}^2 \leq \sigma_n(\mathcal{M}, V)^2$.

For any K and m, $1 \leq m < K$, and any $M \geq 0$, let us consider the $K \times K$ matrix $G = (g_{i,j})$ extracted from A by considering the rows and columns of A with indices from $\{M + 1, \ldots, M + K\}$. Each row $\boldsymbol{\gamma}_i$ of G is the coordinate of the projection of $\tilde{u}_h(\boldsymbol{\mu}_{M+i})$ in the vector space spanned by $\zeta_{M+1}, \zeta_{M+2}, \ldots, \zeta_{M+K}$. Similarly as what we have done on the projection of $\tilde{u}_h(\boldsymbol{\mu}_{M+i})$ in the vector space spanned by $\zeta_{M+1}, \zeta_{M+2}, \ldots, \zeta_{M+K}$, we project each vector $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m$ in the vector space spanned by $\zeta_{M+1}, \zeta_{M+2}, \ldots, \zeta_{M+K}$ and further normalize them. The associated vectors are denoted as $\widehat{b}_i$, $1 \leq i \leq m$ and span a finite-dimensional space $\widehat{\mathcal{H}}_m$ that satisfies, from (4.111),

$$\max_{i} \min_{\widetilde{\mathbf{c}_m}\in\widehat{\mathcal{H}}_m} \|\boldsymbol{\gamma}_i - \widetilde{\mathbf{c}_m}\|_{\ell^2} \leq d_m(\mathcal{M}, V). \tag{4.112}$$

Now we note that the ℓ^2-norm of each $\boldsymbol{\gamma}_i$ is, from property $\mathbf{P_2}$ above,

$$\|\boldsymbol{\gamma}_i\|_{\ell^2} \leq \sigma_M(\mathcal{M}, V).$$

The projection of each of these vectors on the vectorial space spanned by $\widehat{b}_i$, $1 \leq i \leq m$, belongs to a ball (in dimension m) of radius $\leq \sigma_M(\mathcal{M}, V)$: $\mathcal{B}(0, \sigma_M(\mathcal{M}, V))$, due to the m-width property (4.112), each of the vectors $\boldsymbol{\gamma}_i$ thus belong to a "cylinder" with basis $\mathcal{B}(0, \sigma_M(\mathcal{M}, V))$ and height $\leq d_m(\mathcal{M}, V)$ (in dimension $K-m$). The volume of the convex set spanned by these vectors $\boldsymbol{\gamma}_i$ (that is equal to the determinant of G) is thus upper bounded by $\mathcal{V}_m = \mathrm{Vol}(\mathcal{B}(0, \sigma_M(\mathcal{M}, V))) \times d_m(\mathcal{M}, V)^{K-m}$. Recalling that the matrix G is lower diagonal, we thus get from $\mathbf{P_1}$

$$\det G = \Pi_{n=M+1}^{K+M} |a_{n,n}| \leq \mathcal{V}_m \leq C\sigma_M(\mathcal{M}, V))^m \times d_m(\mathcal{M}, V)^{K-m}, \tag{4.113}$$

---

7 Note that if the projection of $\mathcal{H}_m$ in $\mathrm{Span}(X_{K_{\mathrm{trial}}})$ is of dimension $< m$, we choose any vector $\tilde{b}$ in $\mathrm{Span}(X_{K_{\mathrm{trial}}})$ to complement the family $\{\tilde{b}_i\}_{1\leq i\leq m}$.

where C is here a universal constant bounding the volume of the unit-ball, whatever the dimension (we can take for instance $C = 6$).

We thus get (a slightly improved version of) Theorem 3.2 in [14].

Theorem. *For the weak greedy algorithm with constant γ in a Hilbert space V and for any compact set $\mathcal{M}$, we have the following inequalities between $\sigma_n = \sigma_n(\mathcal{M}, V)$ and $d_n = d_n(\mathcal{M}, V)$ for any $M \geq 0$, $1 \leq m < K$:*

$$\Pi_{i=1}^{K} \sigma_{M+i} \leq 6\gamma^{-K} \sigma_M^m d_m^{K-m}. \tag{4.114}$$

Then, as of consequence of various choices of the values M, K, and m, we get the following corollary.

Corollary. *For the weak greedy algorithm with constant γ in V, we have the following:*

- *For any $n \geq 1$, we have*

$$\sigma_n(\mathcal{M}, V) \leq 6^{\frac{1}{n}} \gamma^{-1} \min_{1 \leq m \leq n} d_m^{\frac{n-m}{n}}(\mathcal{M}, V).$$

  In particular $\sigma_{2n}(\mathcal{M}, V) \leq \sqrt{2}\gamma^{-1}\sqrt{d_n(\mathcal{M}, V)}$, $n \geq 1$.
- *If $d_n(\mathcal{M}, V) \leq C_0 n^{-\alpha}$, then $\sigma_n(\mathcal{M}, V) \leq C_1 n^{-\alpha}$ with $C_1 = 2^{5\alpha+1}\gamma^{-2}C_0$.*
- *If $d_n(\mathcal{M}, V) \leq C_0 e^{-c_0 n^{\alpha}}$, then $\sigma_n(\mathcal{M}, V) \leq \sqrt{2C_0}\gamma^{-1} e^{-c_1 n^{\alpha}}$, where $c_1 = 2^{-1-2\alpha} c_0$.*

For the first item, we take $M = 0$, $K = n$, and any $1 \leq m < n$ in the previous theorem, and use the monotonicity of σ and the fact that $\sigma_0 \leq 1$. For the proof of the two other items we refer to [14].

As a final obvious remark we want to point out that $d_m(\mathcal{M}, V)$ depends on $\mathcal{M}$ and thus on $\mathcal{P}$. This suggests a localization greedy procedure so as to lower the effect of the large size of $\mathcal{P}$; we refer to [16, 2, 29] for various approaches to exploit this argument.

4.4.3 Extensions

4.4.3.1 Evolution problems: parabolic PDEs

The greedy method for elliptic problems has been extended to the parabolic case in the POD greedy method first proposed in [20]. On line 4 of Algorithm 4.2 we still identify $\boldsymbol{\mu}_m$ as the "worst-approximated" parameter value, but now the error bound is typically evaluated at the final time, T. The update on line 5 is then given as some number of POD modes associated with the projection error.

4.4.3.2 A nonlinear elliptic problem

For our particular nonlinear problem and associated error estimator the weak greedy Algorithm 4.2 requires little modification. More generally, apart from difficulties asso-

ciated with error estimation, the weak greedy does extend relatively easily to the nonlinear case. However, as already noted, the EIM preparation of the nonlinear problem can be expensive, and for this reason more advanced techniques – which combine the EIM and RB greedy algorithms – are an active area of research (e. g., [13, 6]).

Bibliography

[1] B. O. Almroth, P. Stern, and F. A. Brogan, Automatic choice of global shape functions in structural analysis, *AIAA Journal*, **16** (5) (1978), 525–528.

[2] D. Amsallem, M. J. Zahr, and C. Farhat, Nonlinear model order reduction based on local reduced-order bases, *International Journal for Numerical Methods in Engineering*, **92** (10) (2012), 891–916.

[3] M. Bachmayr and A. Cohen, Kolmogorov widths and low-rank approximations of parametric elliptic PDEs, *Mathematics of Computation*, **86** (304) (2017), 701–724.

[4] O. Balabanov and A. Nouy, Randomized linear algebra for model reduction. Part I: Galerkin methods and error estimation. *Advances in Computational Mathematics*, **45** (2019), 2969–3019.

[5] M. Barrault, Y. Maday, N. Nguyen, and A. T. Patera, An empirical interpolation method: Application to efficient reduced-basis discretization of partial differential equations, *Comptes Rendus de L'Académie Des Sciences. Série 1, Mathématique*, **339** (2004), 667–672.

[6] A. Benaceur, V. Ehrlacher, A. Ern, and S. Meunier, A progressive reduced basis/empirical interpolation method for nonlinear parabolic problems, *SIAM Journal on Scientific Computing*, **40** (5) (2018), A2930–A2955.

[7] F. Bernard, A. Iollo, and S. Riffaud, Reduced-order model for the BGK equation based on POD and optimal transport, *Journal of Computational Physics*, **373** (2018), 545–570.

[8] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk, Convergence rates for greedy algorithms in reduced basis methods, *SIAM Journal on Mathematical Analysis*, **43** (3) (2011), 1457–1472.

[9] A. Buffa, Y. Maday, A. T. Patera, C. Prud'homme, and G. Turinici, *A priori* convergence of the greedy algorithm for the parametrized reduced basis method, *M2AN*, **46** (3) (2012), 595–603.

[10] N. Cagniart, Y. Maday, and B. Stamm, Model order reduction for problems with large convection effects, in *Contributions to Partial Differential Equations and Applications*, pp. 131–150, Springer, 2019.

[11] F. Casenave, A. Ern, and T. Lelièvre, Accurate and online-efficient evaluation of the *a posteriori* error bound in the reduced basis method, *M2AN*, **48** (1) (2014), 207–229.

[12] A. Cohen and R. DeVore, Kolmogorov widths under holomorphic mappings, *IMA Journal of Numerical Analysis*, **36** (1) (2016), 1–12.

[13] C. Daversin and C. Prud'homme, Simultaneous empirical interpolation and reduced basis method for non-linear problems, *Comptes Rendus de L'Académie Des Sciences. Série 1, Mathématique*, **353** (2015), 1105–1109.

[14] R. DeVore, G. Petrova, and P. Wojtaszczyk, Greedy algorithms for reduced bases in Banach spaces, *Constructive Approximation*, **37** (3) (2013), 455–466.

[15] M. Drohmann, B. Haasdonk, and M. Ohlberger, Reduced basis approximation for nonlinear parametrized evolution equations based on empirical operator interpolation, *SIAM Journal on Scientific Computing*, **34** (2) (2012), A937–A969.

[16] J. L. Eftang, A. T. Patera, and E. M. Rønquist, An "hp" certified reduced basis method for parametrized elliptic partial differential equations, *SIAM Journal on Scientific Computing*, **32** (6) (2010), 3170–3200.

[17] C. Farhat, T. Chapman, and P. Avery, Structure-preserving, stability, and accuracy properties of the energy-conserving sampling and weighting method for the hyper reduction of nonlinear finite element dynamic models, *International Journal for Numerical Methods in Engineering*, **102** (5) (2015), 1077–1110.

[18] M. A. Grepl, Y. Maday, C. N. Nguyen, and A. T. Patera, Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations, *M2AN*, **41** (3) (2007), 575–605.

[19] M. A. Grepl and A. T. Patera, *A posteriori* error bounds for reduced-basis approximations of parametrized parabolic partial differential equations, *M2AN*, **39** (1) (2005), 157–181.

[20] B. Haasdonk and M. Ohlberger, Reduced basis method for finite volume approximation of parametrized linear evolution equations, *M2AN*, **42** (2) (2008), 277–302.

[21] J. S. Hesthaven, G. Rozza, and B. Stamm, *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*, Springer, 2016.

[22] D. B. P. Huynh, G. Rozza, S. Sen, and A. T. Patera, A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability constants, *Comptes Rendus de L'Académie Des Sciences. Série 1, Mathématique*, **345** (2007), 473–478.

[23] G. G. Lorentz, Mv. Golitschek, and Y. Makovoz, *Constructive Approximation (Advanced Problems)*, Grundlehren der Mathematischen Wissenschaften, vol. 304, Springer-Verlag, 1966.

[24] L. Machiels, Y. Maday, I. B. Oliveira, A. T. Patera, and D. Rovas, Output bounds for reduced-basis approximations of symmetric positive definite eigenvalue problems, *Comptes Rendus de L'Académie Des Sciences. Série 1, Mathématique*, **331** (2) (2000), 153–158.

[25] Y. Maday, A. T. Patera, and D. Rovas, A blackbox reduced-basis output bound method for noncoercive linear problems, in D. Cioranescu and J. L. Lions (eds.), *Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar*, vol. XIV, pp. 533–569, Elsevier, Amsterdam, 2002.

[26] Y. Maday, A. T. Patera, and G. Turinici, *A priori* convergence theory for reduced-basis approximations of single-parameter elliptic partial differential equations, *Journal of Scientific Computing*, **17** (1-4) (2002), 437–446.

[27] Y. Maday and B. Stamm, Locally adaptive greedy approximations for anisotropic parameter reduced basis spaces, *SIAM Journal on Scientific Computing*, **35** (6) (2013), A2417–A2441.

[28] A. K. Noor and J. M. Peters, Reduced basis technique for nonlinear analysis of structures, *AIAA Journal*, **18** (4) (1980), 455–462.

[29] S. Pagani, A. Manzoni, and A. Quarteroni, Numerical approximation of parametrized problems in cardiac electrophysiology by a local reduced basis method. MATHICSE Technical Report Nr. 25.2017, 2017.

[30] A. Pinkus, *N-widths in Approximation Theory*, Springer Science and Business Media, 1985.

[31] C. Prud'homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici, Reliable real-time solution of parametrized partial differential equations: reduced-basis output bound methods, *Journal of Fluids Engineering*, **124** (1) (2002), 70–80.

[32] A. Quarteroni, A. Manzoni, and F. Negri, *Reduced Basis Methods for Partial Differential Equations*, Springer, 2016.

[33] G. Rozza, D. B. P. Huynh, and A. T. Patera, Reduced basis approximation and *a posteriori* error estimation for affinely parametrized elliptic coercive partial differential equations, *Archives of Computational Methods in Engineering*, **15** (3) (2008), 229–275.

[34] G. Rozza and K. Veroy, On the stability of the reduced basis method for Stokes equations in parametrized domains, *Computer Methods in Applied Mechanics and Engineering*, **196** (7) (2007), 1244–1260.

[35] D. Ryckelynck, *A priori* hyperreduction method: an adaptive approach, *Journal of Computational Physics*, **202** (1) (2005), 346–366.
[36] K. Urban and A. T. Patera, An improved error bound for reduced basis approximation of linear parabolic problems, *Mathematics of Computation*, **83** (2014), 1599–1615.
[37] K. Veroy and A. T. Patera, Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis *a posteriori* error bounds, *International Journal for Numerical Methods in Engineering*, **47** (2005), 773–788.
[38] K. Veroy, C. Prud'homme, D. V. Rovas, and A. T. Patera, *A posteriori* error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations, *AIAA Paper No. 2003-3847*, 2003.
[39] M. Yano, A reduced basis method for coercive equations with an exact solution certificate and spatio-parameter adaptivity: energy-norm and output error bounds, *SIAM Journal on Scientific Computing*, **40** (1) (2018), A388–A420.
[40] M. Yano and A. T. Patera, An LP empirical quadrature procedure for reduced basis treatment of parametrized nonlinear PDEs, *Computer Methods in Applied Mechanics and Engineering*, **344** (2019), 1104–1123.