Preface to the Fourth Edition

In this preface we intend to give the reader a glimpse of those portions and aspects of this book which have been added to its existing body in this present, fourth edition. However, the reader wishing to get an impression of the spirit permeating the book as a whole should also peruse the prefaces of the first, second and third edition. Indeed it might be a good idea to read them in this order. The first edition of this book appeared in 1998, the second in 2006 and the third in 2013. All three were well received by reviewers and frequently quoted by researchers in a variety of areas. Many developments happened in the 20 years since the appearance of the first edition. The present one includes new material amounting to doubling of the size of one of the previous chapters and to the addition of one new appendix. Also there are numerous augmentations clear across the text.

Nothing of the body of the text that accumulated through the years was ever eliminated. In the preface to the second and third edition we have already emphasized an important characteristic of the book and reiterate it now: the original internal numbering system has been retained through all editions. Accordingly all citations of items identified by the internal numbers in any of the previous editions remain intact throughout.

In the past, the Tannaka Duality Theorem for compact groups was not included. Edwin Hewitt and Kenneth Ross [148] referred in 1969 to this result as presented by Tannaka and Krein by writing "Although these theorems were published in 1938 and 1949, respectively, mathematicians have used them very little, and they have not contributed to harmonic analysis on compact non-Abelian groups as the Pontryagin-van Kampen theorem has done for LCA groups." Fifty years later we can say that they have not contributed to the knowledge on the structure of compact groups—the subject of this book, and that this was primarily the reason why we did not include Tannaka Duality in the earlier editions. So why do we include it now?

In the new Appendix 7 we focus on the class $\mathcal{V}$ of vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and on the class $\mathcal{W}$ of weakly complete topological $\mathbb{K}$-vector spaces. Here a topological $\mathbb{K}$-vector space is called *weakly complete* if and only if, as a topological $\mathbb{K}$-vector space, it is isomorphic to $\mathbb{K}^J$, for some set J. These two categories possess two features that deserve to be mentioned at once: Firstly, the dual $V^* = \mathrm{Hom}_{\mathcal{V}}(V, \mathbb{K})$ of a $\mathcal{V}$-object V is a $\mathcal{W}$-object, and the dual $W' = \mathrm{Hom}_{\mathcal{W}}(W, \mathbb{K})$ of a $\mathcal{W}$-object is a $\mathcal{V}$-object and, moreover, there are natural isomorphisms $V \cong (V^*)'$ and $W \cong (W')^*$ establishing a rather elementary duality between $\mathcal{V}$ and $\mathcal{W}$. For $\mathbb{K} = \mathbb{R}$, we have also isomorphisms $\mathrm{Hom}_{\mathcal{V}}(V, \mathbb{R}) \cong \mathrm{Hom}(V, \mathbb{R}/\mathbb{Z}) = \widehat{V}$ and $\mathrm{Hom}_{\mathcal{W}}(W, \mathbb{R}) \cong \mathrm{Hom}(W, \mathbb{R}/\mathbb{Z}) = \widehat{W}$, so that the duality between $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{R}$ is a part of the Pontryagin Duality as we shall discuss in Appendix 7. Indeed their

Pontryagin Duality had been the subject of Chapter 7 already in all preceding editions.

The second noteworthy feature of the dual categories $\mathcal{V}$ and $\mathcal{W}$ is the compatibility of their respective tensor products with duality: $(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^*$ (and dually). These tensor products make $\mathcal{V}$ and $\mathcal{W}$ what in Appendix 3, following a general practice, we call *symmetric monoidal categories*. Therefore both of them have algebras and Hopf algebras.

In particular, we introduce in a natural fashion the class of weakly complete topological algebras A and show that for each of them the set of invertible elements A^{-1} is a topological group, indeed a pro-Lie group. So there is a functor $A \mapsto A^{-1}$ from the category of all weakly complete topological algebras to the category of all topological groups, each with their morphisms. The Adjoint Functor Existence Theorem applies and secures the existence of a left adjoint $\mathbb{K}[-]$ to this functor, which takes a topological group G to the weakly compact topological algebra $\mathbb{K}[G]$. This leads us to study in the new Part 3 of Chapter 3 the *group algebras of compact groups*. If G is a compact group, then the isomorphic copy of G in $\mathbb{R}[G]^{-1}$ can be characterized in terms of the Hopf algebra structure of $\mathbb{R}[G]$. Indeed we shall be able to characterize precisely those real weakly complete symmetric cocommutative Hopf algebras which occur (up to isomorphism) as group Hopf algebras $\mathbb{R}[G]$ for compact groups G. We shall call them *compactlike*. That is, our approach yields the folllowing equivalence theorem: *There is a precise categorical equivalence between the category of compact groups and the category of weakly complete compactlike real symmetric Hopf algebras.*

The relevance of this context for the traditional theory of compact groups is this: The duality between $\mathcal{V}$ and $\mathcal{W}$ implements in a straightforward fashion a duality between weakly complete cocomplete real symmetric Hopf algebras and (abstract) commutative real symmetric Hopf algebras. The (abstract) real symmetric Hopf algebras appearing as dual objects of the weakly complete compactlike real symmetric Hopf algebras (namely, the $\mathbb{R}[G]$ with compact G) are called *reduced Hopf algebras* (following G. Hochschild in [155]). Thus the equivalence theorem above yields the following duality theorem: *The category of compact groups is dual to the category of reduced real Hopf algebras.* This is the Tannaka-Hochschild Duality Theorem. It is now filled with additional significance due to the fact which we establish in Part 3, namely, that the dual $\mathbb{R}[G]'$ of the weakly complete real symmetric group Hopf algebra of a compact group is naturally isomorphic to the real symmetric Hopf algebra $R(G, \mathbb{R})$ of representative functions of the compact group G.

Aside from the innovation regarding the weakly complete group algebras of compact groups, we implemented numerous smaller local improvements of material present in earlier editions. An example is Theorem 6.55 in which it is now clearly formulated that for a compact Lie group G every element of the commutator algebra $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of G is itself a commutator. Another significant improvement of an earlier result is Theorem A1.32 concerning the general theory of divisibility in abelian groups. This material benefitted from the developments of the recent monograph [144] by Herfort, Hofmann and Russo. Among the cardinal numbers naturally attached to mainly compact groups we expanded notably the presentation of *density*. The presentaton of these complementary results begins with the definition of the logarithm of arbitrary cardinal numbers in Definition 12.16a and finally culminates in our systematic comparison of density and weight for arbitrary compact groups in 12.31a, from which we conclude that the density of a closed subgroup of a compact group never exceeds the density of the latter. By a theorem of Itzkowitz [216], a closed subgroup of a separable compact group is separable which now emerges as a special case of the general situation. The Appendix 5 on Measures on Compact Groups is complemented by a subsection on infinite compact groups G dealing with the existence of subgroups of G failing to be measurable with respect to Haar measure of G. This issue sounds simple but leads into complications of set theory and logic.

Selected references for the Preface of the Fourth Edition

[∗] The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras, J. of Lie Theory 28 (2019), 413–455 (by Rafael Dahmen and Karl H. Hofmann).

[†] On Weakly Complete Group Algebras of Compact Groups, J. of Lie Theory **30** (2020), 407–424 (by Karl H. Hofmann and Linus Kramer).

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