Preface to the Second and Third Editions

In the first edition of this book, which appeared in 1998, we endeavored to produce a self-contained exposition of the structure theory of compact groups. The focus was indeed on structure theory and not on representation theory or abstract harmonic analysis, yet the book does contain not only an introduction to, but a wealth of material on, these topics. Indeed, the first four chapters of the book provide a more than adequate introduction to the basics of representation theory of compact groups and the judiciously pedestrian style of these chapters is meant to serve graduate students. Because of our aim for the book to be self-contained it also included material such as an introduction to linear Lie groups, to abstract abelian groups, and to category theory. A prominent feature of the book is the derivation of the structure theory of arbitrary compact groups using an extension of Lie theory, unfettered by dimension restrictions. We record again our apt comments in the preface to the first edition: "It is generally believed that the approximation of arbitrary compact groups by Lie groups settles any issue on the structure of compact groups as soon as it is resolved for Lie groups. ...Our book will go a long way to shed light on this paradigm.... In fact the structure theory presented in this book, notably in Chapter 9, in most cases removes the necessity to use projective limit arguments at all. The structure theorems presented in this book are richer and often more precise than information gleaned directly from approximation arguments."

The driving forces behind the writing of the second edition, which was published in 2006, and this third edition were new material discovered by the authors and others since the first edition appeared, various questions about the structure of compact groups put to the authors by our readers over the ensuing years, and our wish to clarify some aspects of the book which we feel needed improvement. However, in writing the second and third editions we were cognizant of the fact that the book was already sizable, and our commitment to self-containment was not feasible if the book were to remain at less than a thousand pages. The second and third editions also provided us with opportunities to correct many typographical errors which inevitably occur in such a large book, and a small number of mathematical errors which were easily corrected and not of serious consequence.

The list of references has increased as we include recent publications which are most pertinent to the specific content of the book. We still do not claim completeness in these additional references any more than we did in the first edition, otherwise we would have produced an unreasonably bulky list of references.

Both the second and third editions, however, remain unchanged in one important way: The numbering system of the first edition for Definitions, Lemmas, Remarks, Propositions, Theorems, and Main Theorems remains completely intact in the subsequent editions. Therefore all earlier citations and references made to them by numbers remain valid for users of later editions. This cannot hold for references to page numbers, since, due to the augmentation of the text, they have changed. We were able to accomodate all additions organically in the text. Frequently, we added new pieces of information to existing propositions and theorems. In two cases we added a second portion to an existing theorem and numbered it Theorem 8.36bis and Theorem 9.76bis. Sometimes we added a subsection at the end of a chapter, thereby avoiding any recasting of existing numbering.

As already indicated, one of our mathematical philosophies is the emphasis and application of *Lie theory* pervading the book from Chapter 5. By that we mean a consistent use of not necessarily finite dimensional Lie algebras and the associated exponential function wherever it is feasible and advances the structural insight. In this sense we should alert readers to our publications [∗], [†], and [‡], which may be considered as an extension of the Lie theoretical aspects of this book in the direction of a wide-ranging Lie theory for the class of pro-Lie groups: Indeed the book [∗] may be considered a sequel to this book. A pro-Lie group is a limit of a projective system of finite dimensional real Lie groups in which the kernels of the bonding morphisms are not necessarily compact. If we call a topological group G *almost connected* whenever the group G/G_0 of connected components is compact, then the structure and Lie theory of pro-Lie groups covers that of locally compact almost connected groups which includes all connected locally compact groups and all compact groups. However, not all pro-Lie groups are locally compact—as even infinite products $\mathbb{R}^X$ illustrate. It is shown in [∗] that a topological group is a pro-Lie group if and only if it is isomorphic to a closed subgroup of a product of finite dimensional real Lie groups; so this example is already representative.

Judging the significance of recent developments for further research and application is tricky. However, with this reservation in mind, we mention a few among the numerous topics which were added to the content of the second and third editions.

In the second edition, we clarified the ambiguity in the common terminology surrounding the concept of a simple compact connected Lie group. In abstract group theory a simple group is a group without nonsingleton proper normal subgroups. However, in classical Lie group theory a connected Lie group is called "simple" if its Lie algebra is simple, that is, has no nonzero proper ideals. This is equivalent to saying that every nonsingleton closed proper normal subgroup is discrete or, equivalently, that it is locally isomorphic to a compact connected Lie group without nonsingleton proper *closed* normal subgroups. But is a compact group of the latter type *simple* in the sense of abstract group theory? Could it perhaps contain some nonclosed nontrival normal subgroups? The answer is that a compact group having no nontrivial closed normal subgroups has no nontrivial normal subgroups at all and thus is simple in the sense of abstract group theory. A proof is surprisingly nontrivial in so far as it requires Yamabe's Theorem that an arcwise connected subgroup of a Lie group is an analytic subgroup. We recorded these matters around Theorem 9.90. Having clarified the fact that a compact algebraically simple group is either finite or connected, we call a compact group *weakly reductive* if it is isomorphic to a cartesian product of simple compact groups. This allowed us to record also our *Countable Layer Theorem* 9.91 which says that *every compact group G contains a canonical finite or countably infinite descending sequence:*

$$G = N_0 \supset N_1 \supset N_2 \supset \cdots$$

of closed normal subgroups such that, firstly, each quotient group N_j/N_{j+1} *is strictly reductive and, secondly, that* $\bigcap_j N_j$ *is the identity component of the center of the identity component* G_0. It is quite surprising that countability should occur without the hypothesis of metrizability. The Countable Layer Theorem confirms the intuition that compact groups, no matter how large their weight is, are "wide" rather than "deep". We use the Countable Layer Theorem in our proof of Theorem 10.40 that every compact group is dyadic.

It reasonable to ask what is the probability that two randomly chosen elements commute in a compact group. A so-called FC-*group* is a group with finitely many conjugacy classes. In this third edition, from Definition 9.92 through Example 9.100 we discuss the structure of compact FC-groups. It is probably not surprising that from the viewpoint of topological groups these are "almost abelian," and what this means is said in Theorem 9.99. This gives rise to the subsequent discussion of the *commutativity degree* $d(G)$ of a compact group, that is, the probability that a pair (x, y) of randomly picked elements x and y satisfies $xy = yx$. (Thus $d(G)$ is the Haar measure in $G \times G$ of the set $\{(x, y)|xy = yx\}$.) The nature of $d(G)$ is completely clarified in Theorem 9.102. This question lies in the realm of topological combinatorial group theory.

Hilbert's Fifth Problem can be cast in the following form: Is a locally euclidean group a Lie group? This was answered affirmatively in the 1950s by Gleason, Montgomery and Zippin [263]. In 1974, Szenthe [347] formulated a much used transformation group theory extension saying, in the context of a compact group G, that a transitive action of G on a space X causes X to be a real analytic manifold provided X is locally contractible. Our Chapter 10 on actions of compact groups is now augmented by a rather detailed and important discussion of transitive actions of a compact group G on a space X. From Definition 10.60 through Corollary 10.93 we discuss what happens if X is rationally and mod 2 acyclic. Subsequently we consider the situation when X has some open subset contractible to a point in X. In this latter case, X is a compact manifold. All locally contractible spaces fall into this category. The first of these two topics emerged around 1965 in the context of compact monoid theory but has attracted renewed and recent interest by researchers indicating a need for alerting the audience of this book to this aspect of compact transformation group theory. The second consolidates Szenthe's theory. Renewed interest in this issue was kindled by Antonyan's discovery [9] of a serious gap in Szenthe's original proof. Several recent publications provide alternative proofs (see [11], [120], [172]), thereby closing the gap; our presentation in this book is close to [172], but not identical with it.

Chapter 12 on cardinality invariants of compact groups has been expanded and revised in several places so as to include Theorem 12.31 saying that for any compact group G of weight $w(G)$ and every infinite cardinal $\aleph \leq w(G)$ there is a closed subgroup $H \subseteq G$ such that $w(H) = \aleph$.

We mention, finally, that Appendix 5 and Appendix 6 were added. Appendix 5 discusses, from scratch, the compact semigroup $P(G)$ of all probability measures on a compact group G under convolution. The fact that $P(G)$ has a zero element is a classical result of Wendel, which for us secures the existence and uniqueness of Haar measure on a compact group. In previous editions that information was packed into a long drawn-out exercise in Chapter 2. So, in this one particular aspect providing a proof of the existence and uniqueness of normalized Haar measure on a compact group, the present third edition is even more self-contained than the earlier editions were.

Appendix 6 reports very briefly on a technique, which was introduced in the second edition of Pontryagin's famed book [295] on topological groups, namely, the representation of compact groups in terms of projective limits of certain well-ordered inverse systems. This technique yields itself to the application of transfinite induction and therefore has been used in several recent publications as well. In this appendix we mention, in particular, the theorem that the underlying space of every compact group is supercompact. Appendix 6 also contains the theorem that a compact group can be isomorphic to the full homeomorphism group of a completely regular Hausdorff space only if it is profinite.

Selected references for the Preface of the Third Edition

[∗] The Lie Theory of Connected Pro-Lie Groups—A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups, European Mathematical Society Publishing House, 2006, xii + 663pp.

[†] *The Structure of Almost Connected Pro-Lie Groups*, J. of Lie Theory **21** (2011), 347–383.

[‡] *Local Splitting of Locally Compact Groups and Pro-Lie Groups*, J. of Group Theory **14** (2011), 931–935.

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