Preface to the First Edition

The territory of compact groups seems boundless. How vast it is we realized in the course of teaching the subject off and on for thirty years, after pursuing joint research in the area for eighteen years, and by writing this book. We cover a lot of material in it, but we remain in awe of the enormity of those topics on compact groups we felt we must leave out.

Therefore, we must indicate the drift of the contents and explain our strategy. The theme of this book is the *Structure Theory* of compact groups. One cannot speak about compact groups without dealing with representation theory, and indeed various topics which belong to abstract harmonic analysis. While this book is neither on the representation theory nor on the harmonic analysis of compact groups, it does contain in its early chapters the elements of representation theory of compact groups (and some in great generality). But a large volume of material in the research and textbook literature referring to technical aspects of representation theory and harmonic analysis of compact groups remains outside the purview of this text.

One cannot speak about compact groups without, at some time, examining Lie group and Lie algebra theory seriously. This book contains a completely self-contained introduction to linear Lie groups and a substantial body of material on compact Lie groups. Our approach is distinctive in so far as we define a linear Lie group as a particular subgroup of the multiplicative group of a Banach algebra. Compact Lie groups are recognized at an early stage as being linear Lie groups. This approach avoids the use of machinery on manifolds.

There are quite a number of excellent and accessible sources dealing with such matters as the classification of complex simple Lie algebras and, equivalently, compact simple Lie algebras; we do not have to reserve space for them here.

Two of the results on compact groups best known among the educated mathematical public are that, firstly, a compact group is a limit of compact Lie groups, and, secondly, compact Lie groups are compact groups of matrices. Of course we will prove both of these facts and use them extensively. But we hasten to point out a common misconception even among mathematicians who are reasonably well informed on the subject. It is generally believed that the approximation of arbitrary compact groups by Lie groups settles any issue on the structure of compact groups as soon as it is resolved for Lie groups. There is veracity to this legend, as most legends are founded in reality—somewhere, but this myth is far from reflecting the whole truth. Our book will go a long way to shed light on this paradigm on the structure theory of compact groups. In fact the structure theory presented in this book, notably in Chapter 9, in most cases removes the necessity to use projective limit arguments at all. The structure theorems presented in this book are richer and often more precise than information gleaned directly from approximation arguments. In this spirit we present the structure theory with a goal to be free, in the end, of all dimensional restrictions, in particular, of the manifold aspect of Lie groups.

Finally, one cannot speak about compact groups without talking about totally disconnected compact groups; according to the principle of approximating compact groups by compact Lie groups, totally disconnected groups are approximated by finite groups and are therefore also known as profinite groups. We will, by necessity, discuss this subject—even early on, but the general theory of profinite groups has a strong arithmetic flavor; this is not a main thrust of this book. In contrast with that direction we emphasize the strong interplay between the algebraic and the topological structures underlying a compact group. In a simplified fashion one might say that this is, in the first place, a book on the structure of *connected* compact groups and, in the second, on the various ways that general compact groups are composed of connected compact groups and totally disconnected ones.

The table of contents, fairly detailed as it is, serves the reader as a first vantage point for an overview of the topics covered in the book. The first four chapters are devoted to the basics and to the fundamental facts of linear representation of compact groups. After this we step back and take a fresh approach to another core piece, Chapters 5 and 6, in which we deal with the requisite Lie theory. From there on out, it is all general structure theory of compact groups without dimensional restrictions. It will emerge as one of the lead motives, that so much of the structure theory of compact groups is understood, once *commutative* compact groups are elucidated; the inner and technical reasons will emerge as we progress into the subject. But it is for this reason that we begin emphasizing the abelian group aspect from the first chapter onwards. Compact abelian groups have a territory of their own, called *duality theory*; some of it can be dealt with in the first chapter on a very elementary level—and we do that; some requires more information on characters and we shall have sufficient information in the second chapter to get, at this early stage, a proof of the Pontryagin–van Kampen Duality Theorem for compact abelian groups. Yet the finer aspects of duality and a fuller exploitation is deferred to Chapters 7 and 8. There are very good reasons why, in the context of abelian topological groups, we do not restrict ourselves to compact groups but cover at least locally compact ones, and indeed develop a certain amount of duality even beyond these. The theory of compact abelian groups will lead us deeply into aspects of topology and even set theory and logic. Armed with adequate knowledge of compact abelian groups we finally deal in earnest with the structure of compact groups in Chapter 9. In Chapter 10 we broach the topic of compact transformation groups; part of this material is so basic that it could have been presented in the first chapter. However for the applications of compact transformations group theory that we need, more sophisticated results are required. We can present these only after we completed certain parts of the structure theory such as compact Lie group theory. Therefore we opt for keeping material on transformation groups in one place. The later chapters then discuss a variety of special topics pointing up additional ramifications of the structure of compact groups of large infinite dimension; much of this material reflects some of the authors' own research interests.

Most works in the general area of group theory, topological algebra, and functional analysis will consider compact groups a "classical topic," having its roots in the second and third decade of this century. (For some discussion of the history of Lie groups and topological groups see e.g. [41] (notably p. 287–305), [70], [138], [292].) Nevertheless, interested readers will observe novel aspects in the way we approach the subject and place our emphasis. In the first four chapters it was our aim to progress into the theory with as few prerequisites as possible and with the largest pay-offs possible at the same time. Thus our approach to the basic representation theory, while still reflecting the classical approach by Peter and Weyl has our particular stamp on it. Later on, one of our main motives, emerging from Chapter 5 on, is a very general theory of the exponential function far beyond the more traditional domain of finite dimensional Lie groups; and even there our direct approach places much heavier emphasis on the exponential function as the essential feature of Lie group theory than authors commonly do. But we are certainly not bucking a trend in doing so. In many respects, in the end, one might consider our approach as a *general Lie theory for compact groups*, irrespective of dimension.

Strategies for using the book. We were employing certain strategies ourselves: the first and foremost being to make this a source book which is as self-contained as possible. This caused us to present rather fully some source material one needs in separate appendices of which there are four. Dealing with an advanced topic like this we do not find it always possible to abstain from citing other sources. We make an effort to state the prerequisites at the beginning of each chapter and warn the reader about those rare points where we have to invoke outside source material. In lieu of lengthy introductions to either the book or to the individual chapters we have provided, for an orientation of the reader, a postscript at the end of each chapter with commentaries on the material that was covered.

With regard to our efforts to make this a self-contained source on the structure of compact groups, we think it to be quite justified to call the book a *primer for the student.* The initial chapters should be accessible for the beginning graduate student having had basic analysis, algebra, elementary functional analysis up through the elements of Banach spaces and Banach algebras, and having acquired a small body of background information about point set topology; later chapters will require more background knowledge, including some algebraic topology.

For reasons we have indicated, one might argue with us whether it is right to call the book a *handbook for the expert.* Yet correspondence with mathematicians working in various fields of specialisation who asked us about information regarding the structure of compact groups has convinced us that there is a large body of structural information at hand which is not accessible, or, at least, not *easily and readily* accessible in the textbook and handbook literature, and we hope that some of the material presented here justifies, to some extent, the designation "handbook" as well. We have made an attempt to compile a fairly detailed table of contents and a large alphabetical index. The list of references is substantial but is by no means exhaustive.

The book contains material for several separate courses or seminars. In Chapters 1 through 4 one has the body of an introductory course on compact groups of two semesters. Chapter 5 contains more than enough material for a one-semester introductory course on the general theory of (linear) Lie groups. Chapter 6 has up to two semesters worth of additional material on the structure of compact Lie groups. Appendix 1 makes up for a one semester course in basic abelian group theory, starting from the most basic facts reaching up to rather sophisticated logical aspects of modern algebraic theory of groups. Appendix 3 is an introductory course into category theory for the working mathematician (to borrow a title from MacLane) replete with examples from numerous mathematical endeavors not immediately related to compact group theory. Such courses have been taught by us through the years in one form or another and our lecture notes became the bases for some of the material in the book. Other portions of the book are more likely to lend themselves to seminars or specialized advanced courses rather than to basic courses.

Background and Acknowledgements. Much material that was developed in courses given by the authors in their respective environments made its way into this monograph. The first author has taught courses on Lie group theory at the University of Tübingen, Germany, at Tulane University in New Orleans, and the Technische Hochschule Darmstadt now Technische Universität Darmstadt) from the early sixties through recent years. He has given courses on compact groups at Tulane University repeatedly, beginning 1966, at the Université de Paris VI in the fall of 1973, at the Université Catholique de Louvain-la-Neuve in March 1988, at Technische Hochschule Darmstadt from 1989 on. The Lecture Notes from these courses formed the nucleus of the first four chapters of this book. At the Technische Hochschule Darmstadt, in the context of these courses, he taught courses on abelian groups and on category theory; the lecture notes from these provided the basis material for parts of Chapter 8 and of the Appendices 1 and 3. The second author has had a history of courses of general topology and of topological groups which strongly influenced certain pedagogical aspects of our presentation. From the year 1979 on, both authors cooperated on a research project involving compact groups. Morris brought to this project his expertise on topological groups, varieties of topological groups, and his interest in free topological groups, and Hofmann contributed his knowledge of compact groups and Lie groups. Basically, the project concerned the study of free compact groups and is reflected in many parts of the book from Chapter 8 on, notably in Chapters 11 and 12.

We owe a debt of gratitude to people and institutions. First and foremost to our families. They have shown much appreciated tolerance for the stress caused to them in the process of our research and our writing of this book. Without their encouragement the project would hardly have materialized.

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and Helge Glöckner at the University of Erlangen read large portions of the book and directed our attention to errors we have had in earlier drafts. Their suggestions were most valuable. Dieter Remus of the University of Hannover read several chapters of the book carefully and shared with us his list of typos and his knowledge of the literature. Our students Brigitte Breckner, Robert Graeff, Jürgen Heil, Peter Lietz, Peter Maier, Erhard Weidenauer at Technische Hochschule Darmstadt did their best in error detection on parts of the text as it was developed and distributed. Dong Hoon Lee of Case Western Reserve University, Visiting Professor at Technische Hochschule Darmstadt during the summer of 1997, contributed much through his wise input, notably the matters concerning the automorphism group of compact groups. Richard Bödi of the University of Tübingen carefully read Chapter 5 and helped us greatly with typographical and mathematical errors in earlier versions of the text. Vladimir Pestov of Victoria University of Wellington, New Zealand gave us valuable comments. Several others—too many to name—read some portions of the text and pointed out errors and omissions. None of these readers are, of course, responsible for errors, typographical or otherwise, that remained in the text. Benno Artmann of the Technische Universität Darmstadt helped us by insisting that a book on the structure of compact groups should contain enough information to explain which surfaces and which spheres are compact groups; the answers to these questions are easily formulated but their proofs are far less elementary than one might think at first glance.

We most cordially thank Wolfgang Ruppert of the University of Vienna for providing the pictures in Chapters 1 and 11, and in Appendices 1 and 3. He has been extremely helpful and cooperative. The System Manager of the Computer Network of Technische Universität Darmstadt, Dr. Holger Grothe, has helped us with great patience and endurance in more ways than we could mention here. The authors typeset the book in plain TEX and were allowed to use a program for the alphabetical index written by Ulrike Klein of Technische Universität Darmstadt.

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Very prominently we express our gratitude to our publisher, Verlag Walter de Gruyter in Berlin. Manfred Karbe and Irene Zimmermann, our editors, have worked with deep mathematical insight, diligence and patience on our source material, and polished the final format giving the text the shape in which the reader finds it now. They have tirelessly communicated with us; with the volume of this book under our eyes we thoroughly appreciate their input.

K. H. Hofmann and S. A. Morris Darmstadt and Adelaide, 1998