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Algebraic (2, 2)-transformation groups

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Abstract. In this paper we determine all algebraic transformation groups G, defined over an algebraically closed field k, which operate transitively, but not primitively, on a variety Ω, subject to the following conditions. We require that the (non-effective) action of G on the variety of blocks is sharply 2-transitive, as well as the action on a block Δ of the normalizer G_Δ. Also we require sharp transitivity on pairs (X, Y) of independent points of Ω, i.e. points contained in different blocks.

Although classifications of imprimitive permutation groups first appeared at the beginning of the last century (see [10]) and imprimitive actions play an important role in geometry, the corresponding literature is less well developed than that concerning primitive groups. For finite groups there are some classification results (see for instance [1], [5] and [11]). In [1], by using wreath products, the best-known construction principle for obtaining imprimitive groups, a classification is achieved of finite imprimitive groups acting highly transitively on blocks and satisfying conditions very common in geometry.

The aim of the present paper is to obtain classifications for infinite imprimitive groups belonging to well-studied categories. We start with an imprimitive algebraic group G, over an algebraically closed field k, operating on an algebraic variety Ω of positive dimension in such a way that the induced actions on the set $\bar{\Omega}$ of blocks and on a block Δ are both sharply 2-transitive. Moreover we require that the group acts sharply transitively on pairs of points lying in different blocks. The latter condition, frequently occurring in geometry (see for instance [2]), provides a manageable class of groups. For the classification we do not require that the group actions are bi-regular morphisms but merely that the orbit maps be separable morphisms. It turns out that G is the semidirect product of a 3-dimensional unipotent connected group G_u by a 1-dimensional connected torus T, both acting on the points of an affine plane over k with a full set of parallel lines as the blocks.

There are two subgroups which play a fundamental role for the classification: the kernel $G_{[\bar{\Omega}]}$ of the representation on $\bar{\Omega}$ (the so-called *inertia subgroup*) and its stabilizer $G_{[\bar{\Omega}]_O}$ of a fixed point O, which turns out to be even the pointwise stabilizer of the block containing O. There exists a G-invariant transversal L of G with respect to $G_{[\bar{\Omega}]_O}$ which is essential for the classification. Moreover L is a subgroup precisely if $G_u/\mathfrak{z}(G_u)$ is commutative, and then $G_u/\mathfrak{z}(G_u)$ is even a vector group. Given the structure of L, the classification (see Theorem 15) depends on four (not necessarily independent) integer parameters which distinguish the isomorphism class of G. If $\operatorname{char} \mathsf{k} > 0$, then for suitable values of the integer parameters L can be both a vector group and a non-commutative group.

We refer to [13] for well-known results about non-affine algebraic groups and to [9] for facts about affine algebraic groups.

§1. Throughout the paper G will denote an algebraic group defined over an algebraically closed field k, operating effectively on the points of a variety Ω of positive dimension. We assume that the orbit maps $g \mapsto g(X)$ are separable morphisms $G \to \Omega$ and that G acts transitively with a non-trivial system of imprimitivity $\bar{\Omega}$. We define

the *normalizer* $G_\Delta := \{g \in G : g(\Delta) = \Delta\}$ of $\Delta \in \bar{\Omega}$,

the *centralizer* $G_{[\Delta]} := \{g \in G_\Delta : g(X) = X \text{ for all } X \in \Delta\}$ of $\Delta \in \bar{\Omega}$,

the *inertia* subgroup $G_{[\bar{\Omega}]} := \{g \in G : g(\Delta) = \Delta \text{ for all } \Delta \in \bar{\Omega}\}$.

We assume the following transitivity hypotheses:

(1) $G_\Delta/G_{[\Delta]}$ acts sharply 2-transitively on Δ;

(2) $G/G_{[\bar{\Omega}]}$ acts sharply 2-transitively on $\bar{\Omega}$;

(3) G acts sharply transitively on $\Lambda := \{(X, Y) \in \Omega^2 : \Delta_X \neq \Delta_Y\}$, where $\Delta_Z \in \bar{\Omega}$ denotes the block containing $Z \in \Omega$.

We call a triple $\mathsf{G} = (G, \Omega, \bar{\Omega})$ satisfying these hypotheses a $(2,2)$-*imprimitive algebraic group*. Since the stabilizer of a point is not trivial, conditions (3) and (1) guarantee that G has trivial centre. Hence G must be an affine algebraic group.

Proposition 1. (i) *Every block* $\Delta \in \bar{\Omega}$ *is closed and* $G_\Delta = G_{[\bar{\Omega}]} G_X$ *for any* $X \in \Delta$.

(ii) *The inertia subgroup* $G_{[\bar{\Omega}]}$ *is closed.*

Proof. Every block $\Delta \in \bar{\Omega}$ is a constructible set as the union, for $X \in \Delta$, of two G_X-orbits, $\{X\}$ and $\Delta \backslash \{X\}$, and so Δ is closed by [7, Theorem 1.6]. Then $G_{[\bar{\Omega}]}$ is the intersection of all closed subgroups G_Δ. Finally $G_\Delta = G_{[\bar{\Omega}]} G_X$ follows from the fact that the normal subgroup $G_{[\bar{\Omega}]}$ acts transitively on Δ. □

Remark 2. As orbit maps are separable morphisms $G \to \Omega$, by the universal mapping property we may identify Ω with the homogeneous space G/G_O for a fixed stabilizer $G_O = \{g \in G : g(O) = O\}$, $O \in \Omega$. Likewise, in view of Proposition 1, we may identify $\bar{\Omega}$ with the homogeneous space G/G_Δ.

Proposition 3. *For all* $X \in \Omega$ *the centralizer*

$$G_{[\bar{\Omega}]_X} = \{g \in G_{[\bar{\Omega}]} : g(X) = X\}$$

is contained in $G_{[\Delta_X]}$, *and* $G_{[\bar{\Omega}]} = G_{[\bar{\Omega}]_X} \times G_{[\bar{\Omega}]_Y}$ *for any* $(X, Y) \in \Lambda$.

Proof. $G_{[\bar{\Omega}]_X}$ acts (effectively and) sharply transitively on the block Δ_Y, the centralizer $G_{X,Y}$ being trivial. If blocks contain finitely many points the order of $G_{[\bar{\Omega}]_X}$ is $|\Delta|$. In such a case $G_{[\bar{\Omega}]_X}$ operates non-effectively on $\Delta_X \backslash \{X\}$ with orbits of the same length θ, since $G_{[\bar{\Omega}]_{X,X'}} = G_{[\bar{\Omega}]} \cap G_{[\Delta_X]}$ for any $X' \in \Delta \backslash \{X\}$. But $\gcd(|\Delta| - 1, |\Delta|) = 1$ forces $\theta = 1$.

If blocks contain infinitely many points, $G_{[\bar{\Omega}]_X}$ acts on Δ_Y as the kernel of the Frobenius group $G_{\Delta_Y}/G_{[\Delta_Y]}$. So $G_{[\bar{\Omega}]_X}$ is a 1-dimensional connected unipotent group by [7, Theorem 1.10], and hence must act trivially on Δ_X by [8, Proposition 1]. Therefore in any case $G_{[\bar{\Omega}]_X} < G_{[\Delta_X]}$ and this forces $G_{[\bar{\Omega}]_X}$ to be a normal subgroup of $G_{[\bar{\Omega}]}$. The last claim follows from the sharp transitivity of G on Λ. □

Proposition 4. (a) $\bar{\Omega}$ *contains infinitely many blocks and every block contains infinitely many points.*

(b) G_O *is the semidirect product of the* 1*-dimensional connected unipotent subgroup* $G_{[\bar{\Omega}]_O}$ *by a* 1*-dimensional connected torus* T.

(c) $G/G_{[\bar{\Omega}]}$ *is a* 2*-dimensional Frobenius algebraic group with complement isomorphic to* T.

(d) *For all* $\Delta \in \bar{\Omega}$, $G_\Delta/G_{[\Delta]}$ *is a* 2*-dimensional Frobenius algebraic group whose* 1*-dimensional kernel is isomorphic to* $G_{[\bar{\Omega}]_X}$ *for any* $X \in \Omega \backslash \Delta$.

Proof. The group $G_O/G_{[\bar{\Omega}]_O}$ acts effectively and sharply transitively on $\bar{\Omega} \backslash \{\Delta_O\}$ and maps surjectively onto $G_O/G_{[\Delta_O]}$ by Proposition 3. Thus if $|\bar{\Omega}| < \infty$ then $|\Delta_O| < \infty$ and Ω should be finite. So there are infinitely many blocks and the kernel of the Frobenius algebraic group $G/G_{[\bar{\Omega}]}$ is a 1-dimensional connected unipotent group (from [7, Theorems 1.8, 1.10]) with a 1-dimensional connected torus as the complement $G_{\Delta_O}/G_{[\bar{\Omega}]} = G_{[\bar{\Omega}]} G_O/G_{[\bar{\Omega}]} \simeq G_O/G_{[\bar{\Omega}]_O}$ (by [8, Proposition 1]).

Finally the non-trivial factor group $G_O/G_{[\Delta_O]}$, as a continuous epimorphic image of $G_O/G_{[\bar{\Omega}]_O}$, must be a 1-dimensional connected torus, as well. So G_O must split over the unipotent group $G_{[\bar{\Omega}]_O}$ by a 1-dimensional connected torus T. □

Proposition 5. *G is a solvable connected affine group of dimension* 4 *and G is the semidirect product of its unipotent radical G_u by the torus T. Moreover the centre* $\mathfrak{z}(G_u)$ *of G_u is contained in* $G_{[\bar{\Omega}]}$ *and for any* $X \in \Omega$ *we have* $G_{[\bar{\Omega}]} = \mathfrak{z}(G_u) \times G_{[\bar{\Omega}]_X}$.

Proof. As $G_{[\bar{\Omega}]}$ is a 2-dimensional connected unipotent group by Propositions 4 (d) and 3 and $G/G_{[\bar{\Omega}]}$ is a connected solvable 2-dimensional group by Proposition 4 (c), the unipotent radical G_u has codimension 1 and acts transitively on Ω. We have

$\mathfrak{z}(G_u) < G_{[\bar{\Omega}]}$ since $\mathfrak{z}(G_u)$ centralizes each $G_{[\bar{\Omega}]_X}$. Finally $\mathfrak{z}(G_u)$ is transitive on every block Δ, hence sharply transitive, the group $G_\Delta/G_{[\Delta]}$ being primitive. □

Remark 6. Denote by g_u and g_s the images of $g \in G$ under the projections $G_u \times T \to G_u$ and $G_u \times T \to T$, respectively. The mapping $\pi : G \to G_u/G_{[\bar{\Omega}]_O}$ with $\pi(g) = g_u G_{[\bar{\Omega}]_O}$ turns out to be a separable morphism of algebraic varieties. The fibres of π are precisely the cosets gG_O, so $gG_O \mapsto g_u G_{[\bar{\Omega}]_O}$ yields an isomorphism $G/G_O \to G_u/G_{[\bar{\Omega}]_O}$. Therefore we may take the homogeneous space $G_u/G_{[\bar{\Omega}]_O}$ as Ω and

$$(g, hG_{[\bar{\Omega}]_O}) \mapsto ghg_s^{-1}G_{[\bar{\Omega}]_O} \quad (g \in G, h \in G_u)$$

as the action of G on Ω since $(g_1g_2)_u = (g_1)_u(g_1)_s(g_2)_u(g_1)_s^{-1}$. In particular $\Omega \simeq G_u/G_{[\bar{\Omega}]_O}$ is a 2-dimensional (irreducible affine) variety with

$$\bar{\Omega} = \bigcup_{g \in G_u} \Delta_{g(O)} \simeq \bigcup_{g \in G_u} g\mathfrak{z}(G_u)G_{[\bar{\Omega}]_O}.$$

§2. Let $G = U \rtimes T$ be a semidirect product of an n-dimensional connected unipotent group U by a 1-dimensional connected torus T. According to Serre [14, p. 72], the group U has a representation on the affine space k^n such that the subspaces

$$U_i = \{(x_1, \dots, x_n) \in \mathsf{k}^n : x_{i+1} = \dots = x_n = 0)\}$$

are normal subgroups of G, the product is given by

$$(x_1, \dots, x_n)(y_1, \dots, y_n)$$
$$= (x_1 + y_1 + \psi_1(x_2, \dots, x_n, y_2, \dots, y_n), \dots, x_{n-1} + y_{n-1} + \psi_{n-1}(x_n, y_n), x_n + y_n),$$

for suitable polynomials $\psi_j \in \mathsf{k}[x_j, \dots, x_n, y_{j+1}, \dots, y_n]$, and the automorphism of U induced by an element $\tau \in T$ maps $(x_1, \dots, x_n)$ to

$$(a_\tau^{e_1}x_1 + \varphi_1^{(\tau)}(x_2, \dots, x_n), \dots, a_\tau^{e_{n-1}}x_{n-1} + \varphi_{n-1}^{(\tau)}(x_n), a_\tau^{e_n}x_n)$$

with $a_\tau \in \mathsf{k}^*$ depending bi-regularly on τ, the map $\varphi_j^{(\tau)}$ a morphism $U_n/U_j \to U_j/U_{j-1}$ and e_j a fixed integer.

Lemma 7. *Let $n \geqslant 2$. Then for any $\tau \in T$ the morphism $\varphi_{n-1}^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. Moreover we may take as ψ_{n-1}*

(a) *the zero polynomial, if U_n/U_{n-2} is a vector group,*

(b) $\sum_{i=1}^{p-1} p^{-1}\binom{p}{i}x_n^{ip^r}y_n^{(p-i)p^r}$, *if U_n/U_{n-2} is an Abelian group of exponent p^2,*

(c) $x_n^{p^r}y_n^{p^s}$, *if U_n/U_{n-2} is non-commutative,* *where, in cases* (b) *and* (c), $p = \operatorname{char} \mathsf{k} > 0$, *r and s are non-negative integers such that* $r < s$ *and* $e_{n-1} = e_n \deg(\psi_{n-1})$.

Proof. By for instance [6, Lemma 7.1], we may take as $\psi_{n-1}(x_n, y_n)$ the polynomial

$$\begin{cases} 0 & \text{if } U_n/U_{n-2} \text{ is a vector group,} \\ b\sum_{i=1}^{p-1} p^{-1}\binom{p}{i} x_n^{ip^r} y_n^{(p-i)p^r}, & \text{if } U_n/U_{n-2} \text{ is Abelian but not a vector group,} \\ bx_n^{p^r} y_n^{p^s}, & \text{if } U_n/U_{n-2} \text{ is non-commutative,} \end{cases}$$

for some non-negative integers r, s with $r < s$ and a non-zero scalar b, that may be assumed equal to 1 thanks to the isomorphism

$$(\ldots, x_{n-1}, x_n)U_{n-2} \mapsto (\ldots, bx_{n-1}, x_n)U_{n-2}.$$

Now since τ operates on U_n as an automorphism group, the co-boundary

$$\delta^1(\varphi_{n-1}^{(\tau)})(x_n, y_n) = \varphi_{n-1}^{(\tau)}(y_n) - \varphi_{n-1}^{(\tau)}(x_n + y_n) + \varphi_{n-1}^{(\tau)}(x_n)$$

is zero if U_n/U_{n-2} is a vector group and

$$(a_\tau^{e_{n-1}} - a_\tau^{e_n \deg \psi_{n-1}})\psi_{n-1}(x_n, y_n)$$

otherwise. In the latter case the fact that ψ_{n-1} is not a co-boundary forces each a_τ to be a root of the polynomial $\mathrm{T}^{e_n \deg(\psi_{n-1})} - \mathrm{T}^{e_{n-1}}$ and this forces the condition $e_{n-1} = e_n \deg(\psi_{n-1})$. As a consequence $\delta^1(\varphi_{n-1}^{(\tau)})$ must in any case be zero, which means that $\varphi_{n-1}^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. □

Remark 8. It follows from [3] that the action of a 1-dimensional torus on a 2-dimensional connected unipotent group U may be given by 2×2 diagonal matrices with entries in k. The following lemma, which generalizes both the lemma in [12, p. 109] and [7, Corollary 2.9], shows that this can be done without destroying the group structure of U.

Lemma 9. *Let* $\varphi_2^{(\tau)} = \cdots = \varphi_{n-1}^{(\tau)} = 0$ *and assume that* $\varphi_1^{(\tau)}$ *is a group homomorphism* $U_n/U_{n-1} \to U_1$. *Then there exists a bi-regular section* $\sigma : U_n/U_{n-1} \to U_n$ *such that* $\sigma(x_n U_{n-1}) = (f(x_n), 0, \ldots, 0, x_n)$ *with* $\delta^1(f) = 0$ *and* $\sigma(U_n/U_{n-1})$ *invariant under* T.

Proof. We may suppose that $\varphi_1^{(\tau)} \in \mathsf{k}[x_n]$ with $\varphi_1^{(\tau)}(x_n) = \sum_{i \in I, j \in J} c_{ij} a_\tau^j x_n^i$ for some finite sets I and J of integers with

$$I = \begin{cases} \{1\}, & \text{if } \operatorname{char} \mathsf{k} = 0, \\ \text{a finite set of } p\text{-powers}, & \text{if } \operatorname{char} \mathsf{k} = p > 0. \end{cases}$$

The product $\tau_1\tau_2$ of two elements of T gives

$$\varphi_1^{(\tau_1\tau_2)}(x_n) = a_{\tau_1}^{e_1}\varphi_1^{(\tau_2)}(x_n) + \varphi_1^{(\tau_1)}(a_{\tau_2}^{e_n}x_n),$$

and hence for each $i \in I$ we have

$$\sum_{j\in J} c_{ij}a_{\tau_1}^j a_{\tau_2}^j = \sum_{j\in J} c_{ij}(a_{\tau_1}^{e_1}a_{\tau_2}^j + a_{\tau_1}^j a_{\tau_2}^{ie_n}).$$

By comparing we infer that just c_{i,e_1} and c_{i,ie_n} can occur as non-zero entries. So

$$c_{i,e_1}a_{\tau_1}^{e_1}a_{\tau_2}^{e_1} + c_{i,ie_n}a_{\tau_1}^{ie_n}a_{\tau_2}^{ie_n} = c_{i,e_1}(a_{\tau_1}^{e_1}a_{\tau_2}^{e_1} + a_{\tau_1}^{e_1}a_{\tau_2}^{ie_n}) + c_{i,ie_n}(a_{\tau_1}^{e_1}a_{\tau_2}^{ie_n} + a_{\tau_1}^{ie_n}a_{\tau_2}^{ie_n}),$$

or $c_{i,e_1} + c_{i,ie_n} = 0$. Therefore $\varphi_1^{(\tau)}(x_n) = \sum_{i\in I} c_{i,e_1}(a_\tau^{e_1} - a_\tau^{ie_n})x_n^i$ and

$$\left\{\left(-\sum_{i\in I} c_{i,e_1}x_n^i, 0, \dots, 0, x_n\right) : x_n \in \mathsf{k}\right\}$$

turns out to be T-invariant with $\delta^1 : \sum_{i\in I} c_{i,e_1}\mathrm{T}^i \mapsto 0$. □

Set $M := \{(0, \dots, 0, x_n) : x_n \in \mathsf{k}\}$ and let $\upsilon = (0, \dots, 0, u) \in M$.

Lemma 10. *Let $n \geqslant 3$. Assume that the centralizer $\mathfrak{C}_{U_{n-1}}(\upsilon)$ of υ in U_{n-1} satisfies $\mathfrak{C}_{U_{n-1}}(\upsilon) = U_{n-2} \bmod U_{n-3}$ for all $\upsilon \in M$. Then the automorphism ρ_υ of U_{n-1}/U_{n-3} induced by conjugation by υ is given by*

$$(\dots, x_{n-2}, x_{n-1}, 0)U_{n-3} \mapsto (\dots, x_{n-2} + u^h x_{n-1}^k, x_{n-1}, 0)U_{n-3}$$

with h and k p-powers if $\operatorname{char}\mathsf{k} = p > 0$ *and $h = k = 1$ otherwise.*

Proof. As $U_{n-1}/U_{n-2} \leqslant \mathfrak{z}(U_n/U_{n-2})$, we have

$$\rho_\upsilon : (\dots, x_{n-2}, x_{n-1}, 0)U_{n-3} \mapsto (\dots, x_{n-2} + \sigma(u, x_{n-1}), x_{n-1}, 0)U_{n-3}$$

for some additive polynomial $\sigma \in \mathsf{k}[u, x_{n-1}]$, which turns out to be monomial because $\mathfrak{C}_{U_{n-1}}(\upsilon) = U_{n-2} \bmod U_{n-3}$ forces ρ_υ to act fixed-point freely on U_{n-2}/U_{n-3}. Thus $\sigma(u, x_{n-1}) = cu^h x_{n-1}^k$ for some integers h, k and scalar $c \in \mathsf{k}^*$ that we may assume to be 1, up to the isomorphism

$$(\dots, x_{n-2}, x_{n-1}, x_n)U_{n-3} \mapsto (\dots, x_{n-2}, c^{-1/k}x_{n-1}, x_n)U_{n-3}.$$

Clearly the integers h and k have to satisfy the claimed conditions. □

For the rest of the paper we require the torus T to act sharply transitively on U_n/U_{n-1}. This means

$$e_n = \begin{cases} 1, & \text{if } \operatorname{char}\mathsf{k} = 0, \\ \text{a } p\text{-power}, & \text{if } \operatorname{char}\mathsf{k} = p > 0. \end{cases} \tag{1}$$

§3. Now we go back to the $(2,2)$-imprimitive algebraic group $\mathsf{G} = (G, \Omega, \bar{\Omega})$. This section is devoted to the case where the 2-dimensional factor group $G_u/\mathfrak{z}(G_u)$ is *commutative*.

Proposition 11. *$G_u/\mathfrak{z}(G_u)$ is a vector group.*

Proof. Otherwise we have $\operatorname{char} \mathsf{k} \neq 0$ and $G_{[\bar{\Omega}]}/\mathfrak{z}(G_u)$ coincides with the unique 1-dimensional connected algebraic subgroup of $G_u/\mathfrak{z}(G_u)$. Consequently $G_{[\bar{\Omega}]}$ is the unique 2-dimensional connected algebraic normal subgroup of G_u containing $\mathfrak{z}(G_u)$. Furthermore $G_u/\mathfrak{z}(G_u)$ commutative and $\dim \mathfrak{z}(G_u) = 1$ require that $\mathfrak{z}(G_u)$ is the commutator subgroup of G_u, hence that each commutator morphism $\sigma_g : x \mapsto [g, x]$ is a group homomorphism $G_u \to \mathfrak{z}(G_u)$, whose kernel must have dimension at least 2. So $\ker \sigma_g \geqslant G_{[\bar{\Omega}]}$ for any $g \in G_u$, a contradiction since $\bigcap_{g \in G_u} \ker \sigma_g = \mathfrak{z}(G_u)$. □

Proposition 12. *There exists a T-invariant normal subgroup L of G_u containing the centre $\mathfrak{z}(G_u)$ and $G_u = L \rtimes G_{[\bar{\Omega}]_O}$.*

Proof. By [12, Lemma on p. 109] the T-invariant subgroup $G_{[\bar{\Omega}]}/\mathfrak{z}(G_u)$ of $G_u/\mathfrak{z}(G_u)$ has a T-invariant complement, say $L/\mathfrak{z}(G_u)$ for some T-invariant normal subgroup L of G_u containing $\mathfrak{z}(G_u)$. □

In the notation of Section 2 we may take $U_1 = \mathfrak{z}(G_u)$, $U_2 = G_{[\bar{\Omega}]}$, $U_3 = G_u$. In addition we may choose

$$G_{[\bar{\Omega}]_O} = \{(0, x_2, 0) : x_2 \in \mathsf{k}\},$$

the subgroup $G_{[\bar{\Omega}]_O}$ being T-invariant. Since the normal subgroup L of G is not contained in U_2, we may also put

$$L = \{(x_1, 0, x_3) : x_1, x_3 \in \mathsf{k}\}.$$

Thus the product $(x_1, 0, x_3)(y_1, 0, y_3)$ of two elements of L is given by

$$(x_1 + y_1 + \beta(x_3, y_3), 0, x_3 + y_3)$$

and by Lemma 7 we may take

$$\beta(x_3, y_3) = \begin{cases} 0, & \text{if } L \text{ is a vector group,} \\ \sum_{i=1}^{p-1} p^{-1}\binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}, & \text{if } L \text{ is commutative of exponent } p^2, \\ x_3^{p^r} y_3^{p^s}, & \text{if } L \text{ is non-commutative,} \end{cases} \qquad (2)$$

for some integers r, s with $0 \leqslant r < s$. Moreover, an element $\upsilon = (0, 0, u) \in L$ moves the block Δ_O to a different block $\Delta_{\upsilon(O)}$ (from Remark 6), so υ centralizes no element in $G_{[\bar{\Omega}]_O}$, the intersection $G_{[\bar{\Omega}]_O} \cap G_{[\bar{\Omega}]_{l(O)}}$ being trivial. Then Lemma 10 applies and, up to the isomorphism $(x_1, x_2, x_3) \mapsto (x_1, c^{h_2^{-1}} x_2, x_3)$, we may claim

Proposition 13. *The product* $(x_1, x_2, x_3)(y_1, y_2, y_3)$ *in* G_u *may be defined by*

$$(x_1 + y_1 + y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3), x_2 + y_2, x_3 + y_3),$$

where β *is given by* (2) *and each exponent* h_i *is a p-power if* $\operatorname{char} \mathsf{k} = p > 0$, *and* $h_i = 1$ *otherwise.*

As we observed in Remark 8, there is no loss of generality if we assume that the action of the torus T on the affine plane L is given by 2×2 diagonal matrices. But $G_{[\bar{\Omega}]_O}$ occurs as a further T-invariant subgroup of dimension 1, so the diagonal action of each $\tau \in T$ extends to the whole group G_u via

$$(x_1, x_2, x_3) \mapsto (a_\tau^{e_1} x_1, a_\tau^{e_2} x_2, a_\tau^{e_3} x_3). \tag{3}$$

The value of the exponent e_3 was given by (1), whereas the possible relationship occurring between e_1 and e_3 was stated in Lemma 7. Now by imposing that τ is a group homomorphism we find

$$e_1 = e_2 h_2 + e_3 h_3 \tag{4}$$

with h_i arising from the product of G_u given in Proposition 13.

§4. Assume now the factor group $G_u/\mathfrak{z}(G_u)$ to be *non-commutative*. This requires that $\operatorname{char} \mathsf{k} = p > 0$ and we shall also see that $p > 2$.

In the notation of Section 2 we may take again $U_3 = G_u$, $U_2 = G_{[\bar{\Omega}]}$, $U_1 = \mathfrak{z}(G_u)$ and

$$G_{[\bar{\Omega}]_O} = \{(0, x_2, 0) : x_2 \in \mathsf{k}\}.$$

Also, by Lemma 7, we have

$$\psi_2 : (x_3, y_3) \mapsto x_3^{p^m} y_3^{p^n},$$

for some p-powers p^m and p^n such that $m < n$. Furthermore, considering Remark 6, we see that an element $\upsilon = (0, 0, x_3)$ moves the block Δ_O to a different block $\Delta_{\upsilon(O)}$. So υ does not centralize any element of $G_{[\bar{\Omega}]_O}$ because the intersection $G_{[\bar{\Omega}]_O} \cap G_{[\bar{\Omega}]_{\upsilon(O)}}$ is assumed to be trivial. So Lemma 10 applies and, up to an isomorphism, we may assume that the automorphism induced on $G_{[\bar{\Omega}]}$ by an element $(0, 0, x_3)$ has the form

$$(y_1, y_2, 0) \mapsto (y_1 + y_2^{h_2} x_3^{h_3}, y_2, 0)$$

for suitable p-powers $h_i = p^{l_i}$, $i = 2, 3$. If we represent G_u as a non-central extension of the vector group $G_{[\bar{\Omega}]}$ by $G_u/G_{[\bar{\Omega}]}$ using the cross-section

$$(x_1, x_2, x_3) G_{[\bar{\Omega}]} \mapsto (0, 0, x_3),$$

the product $(x_1, x_2, x_3)(y_1, y_2, y_3)$ of two elements in G_u can also be given by

$$(x_1 + y_1 + y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3), x_2 + y_2 + x_3^{p^m} y_3^{p^n}, x_3 + y_3)$$

with β in $\mathsf{k}[x_3, y_3]$ such that $\beta(0, y_3) = \beta(x_3, 0) = 0$ and G_u is determined by taking

$$\psi_1(x_1, x_2, y_1, y_2) = y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3).$$

Now the associative law forces the polynomial

$$\delta^2(\beta)(z_1, z_2, z_3) = \beta(z_1, z_2) + \beta(z_1 + z_2, z_3) - \beta(z_2, z_3) - \beta(z_1, z_2 + z_3)$$

to be

$$\delta^2(\beta)(z_1, z_2, z_3) = z_1^{p^{l_3}} z_2^{p^{l_2+m}} z_3^{p^{l_2+n}} \tag{5}$$

and we can state

Proposition 14. *A necessary and sufficient condition that G_u can be constructed as an extension of $\mathfrak{z}(G_u)$ by a non-commutative connected unipotent group is that there exists a polynomial $\beta \in \mathsf{k}[x_3, y_3]$ satisfying (5) with $\beta(0, y_3) = \beta(x_3, 0) = 0$. In this case we may take $\psi_1(x_2, x_3, y_2, y_3) = y_2^{p^{l_2}} x_3^{p^{l_3}} + \beta(x_3, y_3)$.*

The crucial question now is under what conditions such a polynomial β exists. Using a universal property of the operator δ^2 we have

$$\sum_{\pi \in \mathsf{S}_n} \operatorname{sign}(\pi)\, \delta^2(\beta)(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}) = 0$$

and this, in view of (5), is equivalent to

$$l_3 - l_2 = m, \quad \text{or} \quad l_3 - l_2 = n. \tag{6}$$

Assume now that $\operatorname{char} \mathsf{k} = 2$ and $l_2 + m > 0$, and denote by β_j the homogeneous component of β of degree j. As in our case the operator δ^2 is additive, (5) says that $\delta^2(\beta) = \delta^2(\beta_k)$, where $k = 2^{l_2}(2^q + 2^m + 2^n)$ with either $q = m$ or $q = n$, according as $l_3 - l_2 = m$ or $l_3 - l_2 = n$. Let

$$\beta_k(y_1, y_2) = \sum_{i=0}^{k} a_i y_1^{k-i} y_2^i.$$

Then (5) becomes

$$\sum_{i=0}^{k} a_i(z_1^{k-i}z_2^i + (z_1+z_2)^{k-i}z_3^i + z_2^{k-i}z_3^i + z_1^{k-i}(z_2+z_3)^i) = z_1^{2^{l_2+q}} z_2^{2^{l_2+m}} z_3^{2^{l_2+n}}.$$

Differentiating this identity with respect to z_1 and evaluating at $(0, y_1, y_2)$ we obtain

$$a_{k-1}y_1^{k-1} + \frac{\partial}{\partial y_1}\beta_k(y_1, y_2) + a_{k-1}(y_1+y_2)^{k-1} = 0, \tag{7}$$

whereas differentiating with respect to z_3 and evaluating at $(y_1, y_2, 0)$ we get

$$a_1(y_1+y_2)^{k-1} + a_1y_2^{k-1} + \frac{\partial}{\partial y_2}\beta_k(y_1, y_2) = 0. \tag{8}$$

As char $\mathsf{k} = 2$, $\frac{\partial}{\partial y_1}\beta_k(y_1, y_2)$ and $\frac{\partial}{\partial y_2}\beta_k(y_1, y_2)$ are polynomials in y_1^2 and y_2^2 respectively, and the identities (7) and (8) force $a_{k-1} = a_1 = 0$. Hence

$$\frac{\partial}{\partial y_1}\beta(y_1, y_2) = \frac{\partial}{\partial y_2}\beta(y_1, y_2) = 0$$

and this yields $a_i = 0$ for all odd i. Thus we may make the substitution $(z_1, z_2, z_3) \mapsto (z_1^2, z_2^2, z_3^2)$, and we obtain $(z_1, z_2, z_3) \mapsto (z_1^{2^{l_2+m}}, z_2^{2^{l_2+m}}, z_3^{2^{l_2+m}})$ by iterating the process. So (5) leads to

$$\delta^2(\gamma)(z_1, z_2, z_3) = z_1^{2^{q-m}} z_2 z_3^{2^{n-m}} \tag{9}$$

with $\gamma(y_1^{2^{l_2+m}}, y_2^{2^{l_2+m}}) = \beta(y_1, y_2)$. Let γ_t be the homogeneous component of degree $t := 1 + 2^{n-m} + 2^{q-m}$ of γ and let $\gamma_t(y_1, y_2) = \sum_{i=0}^{t} b_i y_1^{t-i} y_2^i$. Then (9) says that $\delta^2(\gamma) = \delta^2(\gamma_t)$, and hence

$$\sum_{i=0}^{t} b_i(z_1^{t-i}z_2^i + (z_1+z_2)^{t-i}z_3^i + z_2^{t-i}z_3^i + z_1^{t-i}(z_2+z_3)^i) = z_1^{2^{q-m}} z_2 z_3^{2^{n-m}}.$$

Likewise we obtain

$$b_{t-1}y_1^{t-1} + \frac{\partial}{\partial y_1}\gamma_t(y_1, y_2) + b_{t-1}(y_1+y_2)^{t-1} = (1-\epsilon)y_1 y_2^{2^{n-m}},$$

and

$$b_1(y_1+y_2)^{t-1} + b_1 y_2^{t-1} + \frac{\partial}{\partial y_2}\gamma_t(y_1, y_2) = 0,$$

where $\epsilon = 0$ or $\epsilon = 1$ according as $q = m$ or $q = n$. So by Euler's identity $t\gamma_t(y_1, y_2)$ is the polynomial

$$b_{t-1}y_1(y_1^{t-1} + (y_1 + y_2)^{t-1} + (1 - \epsilon)y_1y_2^{2^{n-m}}) + b_1y_2((y_1 + y_2)^{t-1} + y_2^{t-1})$$

or the polynomial

$$\begin{aligned} &b_{t-1}(y_1^{1+2^{n-m}}y_2^{2^{q-m}} + y_1^{1+2^{q-m}}y_2^{2^{n-m}} + y_1y_2^{2^{n-m}+2^{q-m}} + (1 - \epsilon)y_1^2y_2^{2^{n-m}}) \\ &\quad + b_1(y_1^{2^{n-m}+2^{q-m}}y_2 + y_1^{2^{n-m}}y_2^{1+2^{q-m}} + y_1^{2^{q-m}}y_2^{1+2^{n-m}}). \end{aligned}$$

Let $q = m$. Then we have the polynomial identity

$$(b_{t-1} + b_1)(y_1^{1+2^{n-m}}y_2 + y_1y_2^{1+2^{n-m}}) + b_1y_1^{2^{n-m}}y_2^2 = 0,$$

which implies $b_{t-1} = b_1 = 0$, and, consequently, $\frac{\partial}{\partial y_1}\gamma_t(y_1, y_2) = y_1y_2^{2^{n-m}}$, a contradiction. Let $q = n$. Then

$$\gamma_t(y_1, y_2) = b_{t-1}y_1y_2^{2^{n-m+1}} + b_1y_1^{2^{n-m+1}}y_2$$

and $\delta^2(\gamma_t)(x_1, x_2, x_3) = 0$. This contradicts (9) and $p \neq 2$ follows.

If $p \neq 2$ the polynomials

$$\beta(x_3, y_3) = \begin{cases} \frac{1}{2}x_3^{2p^{l_3}}y_3^{p^{l_2+n}} & \text{if } l_3 - l_2 = m, \\ x_3^{p^{l_3}+p^{l_2+m}}y_3^{p^{l_3}} + \frac{1}{2}x_3^{p^{l_2+m}}y_3^{2p^{l_3}} & \text{if } l_3 - l_2 = n \end{cases} \tag{10}$$

satisfy the conditions required in Proposition 14. Any other polynomial satisfying the conditions of Proposition 14 differs from (10) for a co-cycle $\kappa(x_3, y_3)$ for a central extension of k_+ by k_+; we are going to show that it is a co-boundary.

From Remark 8 we may assume that any element $\tau \in T$ acts on G_u via

$$(x_1, x_2, x_3) \mapsto (a_\tau^{e_1}x_1 + \varphi_1^{(\tau)}(x_3), a_\tau^{e_2}x_2, a_\tau^{e_3}x_3),$$

with the morphism $\varphi_1^{(\tau)}$ depending only on x_3 because $G_{[\bar{\Omega}]_O}$ is T-invariant. By imposing that τ operates as a group homomorphism we obtain first

$$e_2 = e_3(p^m + p^n) \quad \text{and} \quad e_1 = e_2h_2 + e_3h_3 = e_3(p^{l_3} + p^{l_2+m} + p^{l_2+n}), \tag{11}$$

but also

$$a_\tau^{e_1}\beta(x_3, y_3) - \beta(a_\tau^{e_3}x_3, a_\tau^{e_3}y_3) + a_\tau^{e_1}\kappa(x_3, y_3) - \kappa(a_\tau^{e_3}x_3, a_\tau^{e_3}y_3) = \delta^1(\varphi_1^{(\tau)})(x_3, y_3),$$

or

$$a_\tau^{e_1}\kappa(x_3, y_3) - \kappa(a_\tau^{e_3}x_3, a_\tau^{e_3}y_3) = \delta^1(\varphi_1^{(\tau)})(x_3, y_3), \tag{12}$$

because $e_1 = e_3 \deg \beta$ in view of (11). Since e_3 is a p-power and $p > 2$, by (11) the integer e_1 can be neither a p-power nor the sum of two p-powers. Thus [4, Theorem 4.6] guarantees that κ is a co-boundary, i.e. $\kappa = \delta^1(g)$ for some polynomial $g \in \mathsf{k}[\mathrm{T}]$, that may be eliminated using the substitution $x_1 \mapsto x_1 - g(x_3)$. Such a replacement yields $\delta^1(\varphi_1^{(\tau)})(x_3, y_3) = 0$, i.e. $\varphi_1^{(\tau)}$ is additive, and we may assume that the action of T given by diagonal matrices, as Lemma 9 claims.

§5. Now we collect all information obtained in the previous sections and classify G according to the structure of the transversal L. With the aid of Remark 6 we can state our main result:

Main Theorem 15. *Every* $(2,2)$-*imprimitive algebraic group* $\mathsf{G} = (G, \Omega, \bar{\Omega})$ *can be constructed on the affine variety* $\mathsf{k}^3 \times \mathsf{k}^*$ *as follows.*

(1) *Define the unipotent radical* G_u *on the affine space* k^3 *through the product*

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1 + \psi_1(x_3, y_2, y_3), x_2 + y_2 + \psi_2(x_3, y_3), x_3 + y_3),$$

where either

(i) $\psi_2(x_3, y_3) = 0$ *and* $\psi_1(x_3, y_2, y_3) = y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3)$ *with each* h_i *an integer* p-*power* p^{l_i} *in case* $\operatorname{char} \mathsf{k} = p > 0$, $h_i = 1$ *otherwise, and* $\beta(x_3, y_3)$ *one of the polynomials*

$$0, \quad \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}, \quad x_3^{p^r} y_3^{p^s},$$

for suitable non-negative integers r, s *such that* $r < s$,

or

(ii) $\psi_2(x_3, y_3) = x_3^{p^m} y_3^{p^n}$, *with* $p = \operatorname{char} \mathsf{k} > 2$ *and* m, n *non-negative integers such that* $m < n$, *and* $\psi_1(x_3, y_2, y_3)$ *as above with*

$$\beta(x_3, y_3) = \begin{cases} \frac{1}{2} x_3^{2p^{l_3}} y_3^{p^{l_2+n}}, & \text{if } l_3 - l_2 = m, \\ x_3^{p^{l_3} + p^{l_2+m}} y_3^{p^{l_3}} + \frac{1}{2} x_3^{p^{l_2+m}} y_3^{2p^{l_3}}, & \text{if } l_3 - l_2 = n. \end{cases}$$

(2) *Let* $a \in \mathsf{k}^*$ *operate on* k^3 *via*

$$(u_1, u_2, u_3) \mapsto (a^{e_1} u_1, a^{e_2} u_2, a^{e_3} u_3)$$

where

$e_1 = e_2 h_2 + e_3 h_3$, *but also* $e_1 = e_3 \deg \beta$ *if* $\beta \neq 0$,
$e_2 = e_3(\deg \beta - h_3)/(h_2)$ *if* $\beta \neq 0$,
e_3 *is a* p-*power in case* $\operatorname{char} \mathsf{k} = p > 0$, *and* $e_3 = 1$ *otherwise.*

(3) *Identify* Ω *with the affine plane* k^2 *with the parallel lines* $y = k$ *giving the set* $\bar{\Omega}$ *of blocks. Then a transformation* $(u_1, u_2, u_3, a) \in G$ *moves the point* $(x, y) \in \Omega$ *to the point*

$$(u_1 + a^{e_2 h_2 + e_3 h_3} x + \psi_1(u_3, 0, a^{e_3} y), u_3 + a^{e_3} y)). \quad \square$$

The canonical representation of G given by our Main Theorem depends on the polynomial β as well as on the integer parameters e_2, e_3, h_2, h_3, though h_2 and h_3 are already determined by β, e_2 and e_3. Labelling G as $\mathsf{G}_\beta^{(e_2, e_3, h_2, h_3)}$, we ask whether an isomorphism

$$\Phi : \mathsf{G}_\beta^{(e_2, e_3, h_2, h_3)} \to \mathsf{G}_{\beta'}^{(e_2', e_3', h_2', h_3')}$$

between two $(2,2)$-imprimitive algebraic groups with different parameters exists. Of course we may assume the same sets of points and blocks for both groups, so that Φ is a pair (Φ_1, Φ_2) with Φ_1 a group isomorphism $G_\beta^{(e_2, e_3, h_2, h_3)} \to G_{\beta'}^{(e_2', e_3', h_2', h_3')}$ and $\Phi_2 : \mathsf{k}^2 \to \mathsf{k}^2$ a bijective morphism of the affine plane k^2 transforming horizontal lines into horizontal lines such that

$$\Phi_2(g(P)) = \Phi_1(g)(\Phi_2(P)) \quad (g \in G_\beta^{(e_2, e_3, h_2, h_3)}, P \in \mathsf{k}^2).$$

As G_u is transitive on Ω, up to inner automorphisms we may assume that Φ_2 leaves the point $O = (0,0)$ of Ω fixed, hence the line $y = 0$ stable. Then the stabilizer of O, as well as the normalizer and centralizers of Δ_O correspond; in particular

$$\begin{aligned} \Phi_1((0, u_2, 0)) &= (0, b_2 u_2, 0) \quad (b_2 \in \mathsf{k}^*), \\ \Phi_1((u_1, 0, 0)) &= (b_1 u_1, 0, 0) \quad (b_1 \in \mathsf{k}^*), \end{aligned} \tag{13}$$

and, moreover,

$$\begin{aligned} \Phi_1((0, 0, u_3)) &= (f_1(u_3), f_2(u_3), b_3 u_3)) \quad (b_3 \in \mathsf{k}^*), \\ \Phi_2((x, y)) &= (b_1 x + f_1(y), b_3 y), \end{aligned} \tag{14}$$

for suitable polynomials $f_j \in \mathsf{k}[\mathrm{T}]$ such that

$$\begin{aligned} \delta^1(f_2)(x_3, y_3) &= b_2 \psi_2(x_3, y_3) - \psi_2'(b_3 x_3, b_3 y_3) & (x_3, y_3 \in \mathsf{k}), \\ \delta^1(f_1)(x_3, y_3) &= b_1 \psi_1(x_3, 0, y_3) - \psi_1'(b_3 x_3, f_2(y_3), b_3 y_3) & (x_3, y_3 \in \mathsf{k}). \end{aligned} \tag{15}$$

Manifestly tori fixing the point O correspond under Φ_1; in particular we have $\Phi_1(T_\beta^{(e_2, e_3, h_2, h_3)}) = T_{\beta'}^{(e_2', e_3', h_2', h_3')}$ since tori are conjugate under G_u. This means that

$$(u_1, u_2, u_3)^{\Phi_1(\tau)} = (a_\tau^{\varepsilon e_1'} u_1, a_\tau^{\varepsilon e_2'} u_2, a_\tau^{\varepsilon e_3'} u_3),$$

with $\varepsilon = \pm 1$. The identity $\Phi_1((0,0,u_3)^\tau) = (\Phi_1(0,0,u_3))^{\Phi_1(\tau)}$ and the first part of (14) yield $\varepsilon = 1$, $e_3 = e_3'$ and $f_j(a_\tau^{e_3} u_3) = a_\tau^{e_j'} f_j(u_3)$ for $j = 1,2$, whereas the equation $\Phi_1((u_1,u_2,0)^\tau) = (\Phi_1(u_1,u_2,0))^{\Phi_1(\tau)}$ and Equation 13 give $e_1 = e_1'$ and $e_2 = e_2'$. So the polynomials f_j must be monomials and consequently, in case $f_j \neq 0$,

$$e_j = e_3 \deg(f_j) \quad (j = 1,2). \tag{16}$$

Therefore $f_j(\mathrm{T}) = d_j \mathrm{T}^{e_j/e_3}$ with $d_j \in \mathsf{k}$, for $j = 1,2$. Furthermore imposing the condition $\Phi_1(0,0,u_3)\Phi_1(0,v_2,0) = \Phi_1((0,0,u_3)(0,v_2,0))$ we obtain

$$b_2^{h_2'} b_3^{h_3'} u_3^{h_3'} v_2^{h_2'} = b_1 u_3^{h_3} v_2^{h_2},$$

i.e. $(h_2', h_3') = (h_2, h_3)$ and

$$b_1 = b_2^{h_2} b_3^{h_3}. \tag{17}$$

So the first step is achieved:

Proposition 16. *Suppose that* $\mathsf{G}_{\beta'}^{(e_2',e_3',h_2',h_3')}$ *and* $\mathsf{G}_{\beta}^{(e_2,e_3,h_2,h_3)}$ *are isomorphic as algebraic permutation groups. Then*

$$(e_2', e_3', h_2', h_3') = (e_2, e_3, h_2, h_3). \quad \square$$

From [4, Theorem 4.6], the first equation of (15) holds precisely if

$$\delta^1(f_2) = \psi_2 - \psi_2' = 0. \tag{18}$$

Also since we have $e_1 = e_3 \deg(\beta)$ if $\beta \neq 0$ we only need to examine the case where $\operatorname{char} \mathsf{k} = p > 0$, $\beta = 0$ and either $\beta'(x_3, y_3) = \sum_{i=1}^{p-1} p^{-1}\binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}$, or $\beta'(x_3,y_3) = x_3^{p^r} y_3^{p^s}$: by (16) we have $\deg \beta' = \frac{e_1}{e_3} = \deg f_1$ in case $d_1 \neq 0$. Then the second identity of (15) becomes

$$\delta^1(f_1)(x_3, y_3) = -\beta'(b_3 x_3, b_3 y_3) - b_3^{h_3} x_3^{h_3} f_2(y_3)^{h_2} \tag{19}$$

and again [4, Theorem 4.6] excludes the possibility that $f_2 = 0$. Then f_2 is an additive monomial by (18) and (16) forces e_2 to be a p-power. Thus, in view of the Theorem 15, both e_1 and $\deg \beta'$, are sums of two p-powers. So just the following two possibilities can occur: either $\beta'(x_3, y_3) = x_3^{p^r} y_3^{p^s}$, or $\operatorname{char} \mathsf{k} = 2$ and $\beta'(x_3, y_3) = x_3^{2^r} y_3^{2^r}$. Thus Theorem 15 gives either $e_2 h_2 + e_3 h_3 = e_3(p^r + p^s)$, or $e_2 h_2 + e_3 h_3 = e_3 2^{r+1}$, which means that the pair of p-powers (h_2, h_3) is one of the following:

$$\begin{aligned} &(1)\ \ (h_2,h_3) = ((e_3/e_2)p^r, p^s);\\ &(2)\ \ (h_2,h_3) = ((e_3/e_2)p^s, p^r);\\ &(3)\ \ (h_2,h_3) = ((e_3/e_2)2^r, 2^r). \end{aligned} \tag{20}$$

As the right-hand side of (19) must be a co-boundary, (20.1), (20.2) and (20.3) lead respectively to

$$(1)\ d_1 = b_3^{p^r+p^s} = b_3^{p^s} d_2^{(e_3/e_2)p^r}, \qquad \text{hence } d_2 = b_3^{e_2/e_3};$$

$$(2)\ d_1 = 0 \text{ and } b_3^{p^r+p^s} = -b_3^{p^r} d_2^{(e_3/e_2)p^s}, \qquad \text{hence } d_2 = -b_3^{e_2/e_3};$$

$$(3)\ b_3^{2^{r+1}} = b_3^{2^r} d_2^{(e_3/e_2)2^r}, \qquad \text{hence } d_2 = b_3^{e_2/e_3}.$$

Now straightforward calculations show that, for any $b_1, b_2, d_3 \in \mathsf{k}$, the maps

$$\begin{aligned}
&(1)\begin{cases} G_0^{(e_2,e_3,(e_3/e_2)p^r,p^s)} \to G_{x^{p^r}y^{p^s}}^{(e_2,e_3,(e_3/e_2)p^r,p^s)} \\ (u_1,u_2,u_3,a) \mapsto (b_2^{(e_3/e_2)p^r} b_3^{p^s} u_1 + (b_3u_3)^{p^r+p^s}, b_2u_2 + (b_3u_3)^{e_2/e_3}, b_3u_3, a), \end{cases}\\
&(2)\begin{cases} G_0^{(e_2,e_3,(e_3/e_2)p^s,p^r)} \to G_{x^{p^r}y^{p^s}}^{(e_2,e_3,(e_3/e_2)p^s,p^r)}, \\ (u_1,u_2,u_3,a) \mapsto (b_2^{(e_3/e_2)p^s} b_3^{p^r} u_1, b_2u_2 - (b_3u_3)^{e_2/e_3} b_3u_3, a), \end{cases}\\
&(3)\begin{cases} G_0^{(e_2,e_3,(e_3/e_2)2^r,2^r)} \to G_{x^{2^r}y^{2^r}}^{(e_2,e_3,(e_3/e_2)2^r,2^r)} \\ (u_1,u_2,u_3,a) \mapsto (b_2^{(e_3/e_2)2^r} b_3^{2^r} u_1 + d_1u_3^{2^{r+1}}, b_2u_2 + (b_3u_3)^{e_2/e_3}, b_3u_3, a) \end{cases}
\end{aligned} \tag{21}$$

are group isomorphisms corresponding to the values $(20.i)$ of the pair of p-powers (h_2, h_3). Manifestly such isomorphisms supply isomorphisms for the associated permutation groups. Summing up we have

Theorem 17. *The integer parameters e_2, e_3, h_2, h_3 and the polynomial β determine uniquely the isomorphism class of the $(2,2)$-imprimitive algebraic group G, except when the pair (h_2, h_3) takes one of the (integer) values $(20.i)$, and then there are the corresponding isomorphisms $(21.i)$.* □

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