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Generalizing Camina groups and their character tables

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Abstract. We generalize the definition of Camina groups. We show that our generalized Camina groups are exactly the groups isoclinic to Camina groups, and so many properties of Camina groups are shared by these generalized Camina groups. In particular, we show that if G is a nilpotent, generalized Camina group then its nilpotence class is at most 3. We use the information we collect about generalized Camina groups with nilpotence class 3 to characterize the character tables of these groups.

1 Introduction

Throughout this note, all groups are finite. There are several equivalent formulations of the definition of Camina groups. For our purposes, we will say that G is a *Camina group* if the conjugacy class of every element $g \in G \setminus G'$ is gG'. In this paper, we generalize the definition as follows. The group G is a *generalized Camina group* if the conjugacy class of every element $g \in G \setminus \mathrm{Z}(G)G'$ is gG'. It is very easy to show that if G is a generalized Camina group, then either G has nilpotence class 2 or $G/\mathrm{Z}(G)$ is a Camina group. We will show that generalized Camina groups can be characterized in terms of isoclinisms. (We will remind the reader of the definition of isoclinic groups in Section 2.)

Theorem 1. *Let G be a group. Then G is a generalized Camina group if and only if G is isoclinic to a Camina group.*

Thus, many results for Camina groups will translate immediately to generalized Camina groups. In [9] and [10], Macdonald and Mann proved that if G is a Camina 2-group, then its nilpotence class is 2. In the important paper [1], Dark and Scoppola proved that if G is a Camina p-group with p odd, then the nilpotence class of G is at most 3.

It follows that if G is a nilpotent, generalized Camina group, then G is isoclinic to a nilpotent Camina group H. It is known that H must be a p-group for some prime p. It is not difficult to see that any Camina group isoclinic to G will be a p-group for the same prime p, and we will say that p is the prime associated with G. (It is not difficult to show that p is well defined.) Using the isoclinism, we obtain bounds on the nilpotence class of nilpotent, generalized Camina groups.

Theorem 2. *Let G be a nilpotent, generalized Camina group with associated prime p.*

(1) *If $p = 2$, then G has nilpotence class* 2.

(2) *If p is odd, then G has nilpotence class at most* 3.

Suppose that G is nilpotent of nilpotence class 2. We will show that G is a generalized Camina group if and only if every non-linear irreducible character of G vanishes on $G \setminus \mathrm{Z}(G)$. We studied the character tables of such groups in [6]. In that paper, we called a group G a VZ-group if every non-linear irreducible character of G vanishes on $G \setminus \mathrm{Z}(G)$. While writing that paper, we realized that VZ-groups were a generalization of nilpotent Camina groups of nilpotence class 2. Our goal in this paper is to find an appropriate generalization of Camina groups. We will show that G is a VZ-group if and only if G is a generalized Camina group of nilpotence class 2. Hence, our idea of a generalized Camina group seems to be the generalization which extends the definition of VZ-groups beyond nilpotence class 2.

We now focus on nilpotent generalized Camina groups of nilpotence class 3. Using the isoclinism with Camina groups, we collect the following facts about these groups.

Theorem 3. *Let G be a nilpotent, generalized Camina group of nilpotence class* 3, *and let $Z = \mathrm{Z}(G)$ and $G_3 = [G', G]$. Then the following are true.*

(1) *$G/G'Z$, $G'Z/Z$, and G_3 are elementary abelian p-groups for some prime p.*

(2) *$|G : G'Z| = p^{2n}$ and $|G'Z : Z| = |G' : G_3| = p^n$ for some even integer n.*

(3) *Every non-linear character in $\mathrm{Irr}(G/G_3)$ is fully ramified with respect to $G/G'Z$ and every character in $\mathrm{Irr}(G) \setminus \mathrm{Irr}(G/G_3)$ is fully ramified with respect to G/Z.*

(4) $\mathrm{cd}(G) = \{1, p^n, p^{3n/2}\}$.

(5) *Every element $g \in G'Z \setminus Z$ has the coset gG_3 as its conjugacy class in G.*

(6) *The sizes of conjugacy classes of G are* 1, $|G_3|$, *and* $|G'|$.

Now we turn to the question that motivated our study. In [6], we classified the character tables of VZ-groups, generalizing a result of Nenciu in [13] which classified the character tables of p-groups whose derived subgroups have prime order p. This led us to look for the proper generalization of Camina groups. We believe that the above results indicate that generalized Camina groups are the proper generalization. With our results regarding nilpotent, generalized Camina groups, we are able to classify the character tables of nilpotent, generalized Camina groups of nilpotence class 3. Surprisingly, we have the same characterization as in the nilpotence class 2 case.

Theorem 4. *Let G and H be generalized Camina groups of nilpotence class* 3. *Suppose that there exist isomorphisms*

$$a : \mathrm{Irr}(G/G') \to \mathrm{Irr}(H/H') \quad \textit{and} \quad b : \mathrm{Irr}(\mathrm{Z}(G)) \to \mathrm{Irr}(\mathrm{Z}(H))$$

so that $a(\alpha)_{\mathrm{Z}(H)} = b(\alpha_{\mathrm{Z}(G)})$ for all $\alpha \in \mathrm{Irr}(G/G')$. Then G and H have identical character tables.

For nilpotent Camina groups with nilpotence class 3, this simplifies to the following.

Theorem 5. *Let G and H be Camina groups of nilpotence class* 3. *Then G and H have identical character tables if and only if $|G : G'| = |H : H'|$ and $|[G', G]| = |[H', H]|$.*

2 Isoclinism and generalized Camina groups

We begin with the following general lemma.

Lemma 2.1. *Let g be an element of a group G. Then the following are equivalent*:

(1) *the conjugacy class of g is gG'*;

(2) $|\mathrm{C}_G(g)| = |G : G'|$;

(3) *for every $z \in G'$, there is an element $y \in G$ so that $[g, y] = z$*;

(4) $\chi(g) = 0$ *for all non-linear $\chi \in \mathrm{Irr}(G)$.*

Proof. Since G/G' is abelian, we have $g^x G' = (gG')^{xG'} = gG'$ for all $x \in G$, and so $\mathrm{cl}(g) \subseteq gG'$. Also, $|\mathrm{cl}(g)| = |G : \mathrm{C}_G(g)|$, and hence $|\mathrm{C}_G(g)| = |G|/|\mathrm{cl}(g)|$. It follows that $|\mathrm{C}_G(g)| = |G : G'|$ if and only if $\mathrm{cl}(g) = gG'$. Fix $z \in G'$. Notice that $g^y = gz$ if and only if $[g, y] = z$, and so g is conjugate to every element in gG' if and only if for every $z \in G'$ there is an element $y \in G$ so that $[g, y] = z$. Finally, we know by the second orthogonality relation (see [5, Theorem 2.18]) that

$$|\mathrm{C}_G(g)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)|^2 = |G : G'| + \sum_{\chi \in \mathrm{nlIrr}(G)} |\chi(g)|^2,$$

where $\mathrm{nlIrr}(G)$ is the set of non-linear irreducible characters of G. The last equality in the previous equation comes from the fact that $|\chi(g)|^2 = 1$ for every linear character χ. Since $|\chi(g)|^2$ is a non-negative number, we conclude that $|\mathrm{C}_G(g)| = |G : G'|$ if and only if $\chi(g) = 0$ for all $\chi \in \mathrm{nlIrr}(G)$. □

We call G a VZ-group (vanishing off the center group) if every non-linear irreducible character of G vanishes on $G \setminus \mathrm{Z}(G)$. In the note [6], we proved that if G is a VZ-group, then G is nilpotent of class 2 and $|\mathrm{cd}(G)| = \{1, f\}$ where $f^2 = |G : \mathrm{Z}(G)|$. In fact, every non-linear irreducible character of G is fully ramified with respect to $G/\mathrm{Z}(G)$. Since $|\mathrm{cd}(G)| = 2$ and G is nilpotent, G must have an abelian normal

p-complement for some prime p. In other words, $G = P \times Q$ where P is a p-group and Q is an abelian p'-group. It is not difficult to see that P will also be a VZ-group.

We obtain the following characterization of VZ-groups. This result is Lemma 2.1 applied to VZ-groups. Similar results can be found in [2] and [12].

Lemma 2.2. *Let G be a group. The following are equivalent*:

(1) $\mathrm{cl}(G) = gG'$ *for all* $g \in G \setminus \mathrm{Z}(G)$;

(2) $|\mathrm{C}_G(g)| = |G : G'|$ *for all* $g \in G \setminus \mathrm{Z}(G)$;

(3) *G is a VZ-group*;

(4) *for every $g \in G \setminus \mathrm{Z}(G)$ and $z \in G'$, there exists $y \in G$ so that $[g, y] = z$.*

The last condition appears as a hypothesis of [4, Theorem 7.5]. Since VZ-groups have nilpotence class 2, the hypothesis that $G' \leqslant \mathrm{Z}(G)$ is not needed for that theorem.

Following Hall, we call two groups G and H *isoclinic* if there exist isomorphisms $\alpha : G/\mathrm{Z}(G) \to H/\mathrm{Z}(H)$ and $\beta : G' \to H'$ so that

$$[\alpha(g_1\mathrm{Z}(G)), \alpha(g_2\mathrm{Z}(G))] = \beta([g_1\mathrm{Z}(G), g_2\mathrm{Z}(G)]) \quad \text{for all } g_1, g_2 \in G.$$

In his seminal paper [3], Hall proved many facts regarding isoclinic groups. Two of the facts that we will need are the following: (1) every group G is isoclinic to a group H where $\mathrm{Z}(H) \leqslant H'$ and (2) β identifies the terms of the lower central series of G with the lower central series of H. Therefore, if G and H are isoclinic, then G is nilpotent if and only if H is nilpotent, and if they are nilpotent, then they have the same nilpotence class. In [2, Theorem A], it is shown that VZ-groups can be characterized in terms of their isoclinism class.

Lemma 2.3 ([2, Theorem A]). *Let G be a finite group. Then G is a VZ-group if and only if G is isoclinic to a Camina group of nilpotence class* 2.

We now consider the group-theoretic structure of VZ-groups. Combining the previous lemma with [8], we obtain the following information about $G/\mathrm{Z}(G)$ and G'.

Lemma 2.4. *Let G be a VZ-group. Then $G/\mathrm{Z}(G)$ and G' are elementary abelian p-groups for some prime p. Suppose that $|G : \mathrm{Z}(G)| = p^m$ and $|G'| = p^n$. Then $m \geqslant 2n$.*

Proof. By Lemma 2.3, we know that G is isoclinic to a group H where H is a Camina group of nilpotence class 2. By [8, Theorem 2.2], we know that $H/\mathrm{Z}(H)$ and H' are elementary abelian p-groups for some prime p. Applying [8, Theorem 3.2] to H, we have $|H : \mathrm{Z}(H)| \geqslant |H'|^2$. Since $G/\mathrm{Z}(G) \cong H/\mathrm{Z}(H)$ and $G' \cong H'$, the result follows. □

Throughout the remainder of this paper, we will use GCG to denote generalized Camina groups. Obviously, every Camina group is a generalized Camina group. Furthermore, if G is a Camina group and A an abelian group, then $G \times A$ will be a generalized Camina group. It is not difficult to find other examples of generalized Camina groups that are not Camina groups. For example, let $C = \langle c \rangle$ be a cyclic group of order p^n for some prime p and some integer $n \geqslant 3$. We define an automorphism σ on C by $c^\sigma = c^{p^{n-1}+1}$. We take G to be the semi-direct product of $\langle \sigma \rangle$ acting on C. It is not difficult to see that G is a GCG, and that no proper quotient of G is a Camina group. If G is an arbitrary GCG that is not nilpotent of class 2, then G will have a Camina group as a quotient, as shown in the next lemma.

Lemma 2.5. *Suppose that G is a GCG such that $G' \nleqslant Z$ where $Z = Z(G)$. Then G/Z is a Camina group.*

Proof. We know that $(G/Z, G'Z/Z)$ is a GCP. Since $(G/Z)' = G'Z/Z$, it follows that G/Z_1 is a Camina group. $\square$

Lemma 2.5 shows that if G is a GCG and is nilpotent of class c, then G/Z is a Camina group and is nilpotent of class $c - 1$. Thus results regarding nilpotent Camina groups hold for G/Z, and many results for Camina groups could be directly applied to yield results for G at the cost of one nilpotence class. However, with some work, we can obtain results for G similar to the results for Camina groups.

Lemma 2.6. *Suppose G has nilpotence class 2. Then G is a GCG if and only if G is a VZ-group.*

Proof. We have $G' \leqslant Z$, and so $G'Z = Z$. Thus (G, Z) is a GCP if and only if $(G, G'Z)$ is a GCP. $\square$

The next fact is an application of Lemma 2.1 in terms of GCGs.

Lemma 2.7. *Let G be a group. Then G is a GCG if and only if for every element $g \in G \backslash G'\mathrm{Z}(G)$ and every element $z \in G'$ there is an element $y \in G$ so that $[g, y] = z$.*

With this fact, we can characterize GCGs in terms of isoclinism classes. This result is motivated by Lemma 2.3, and in fact, that lemma is a corollary to this result since VZ-groups are just the GCGs of nilpotence class 2.

Lemma 2.8. *Let G be a group. Then G is a GCG if and only if G is isoclinic to a Camina group.*

Proof. The group G is isoclinic to a group H where $\mathrm{Z}(H) \leqslant H'$. Notice that

$$G'\mathrm{Z}(G)/\mathrm{Z}(G) = (G/\mathrm{Z}(G))',$$

and the isoclinism will map this to $(H/\mathrm{Z}(H))' = H'/\mathrm{Z}(H)$. Also, it is easy to see for all $g_1, g_2 \in G$ that $[g_1\mathrm{Z}(G), g_2\mathrm{Z}(G)] = [g_1, g_2]$. Now, it is not difficult to see that for each $g \in G \backslash G'\mathrm{Z}(G)$ and $z \in G'$ there is an element $y \in G$ so that $[g, y] = z$ if and only if for each $h \in H \backslash H'$ and $x \in H'$ there is an element $w \in H$ so that $[h, w] = x$. Hence G is a GCG if and only if H is a Camina group. □

With the characterization in terms of isoclinic groups, we can prove Theorem 2.

Proof of Theorem 2. Take H to be a Camina group isoclinic to G. We know that H is nilpotent with the same nilpotence class as G. Notice that H will be a p-group. Macdonald proved that if $p = 2$ then H has nilpotence class 2. (This is also proved in [10].) It is also proved that if p is odd then H has nilpotence class at most 3. The isoclinism gives the same results for G. □

Proof of Theorem 3. Let H be the Camina group that is isoclinic to G. By [8, Theorem 2.2], we know that H/H' is an elementary abelian p-group for some prime p. From [10], we know that H' is an elementary abelian p-group. Applying [8, Theorem 5.2], we have $|H : H'| = p^{2n}$ and $|H' : H_3| = p^n$ for some even integer n. Since $G/G'Z(G) \cong H/H'$ and G' is isomorphic to H' under a map that identifies the lower central series of G with the lower central series of H, we obtain conclusions (1) and (2).

It is mentioned in [3] that isoclinic groups have the same character degrees and conjugacy class sizes. This fact for character degrees is proved in [11, Theorem D]. In [8], it is proved that H will have conjugacy class sizes of 1, $|H_3|$, and $|H'|$. Since $|G_3| = |H_3|$ and $|G'| = |H'|$, it follows that the sizes of the conjugacy classes for G are 1, $|G_3|$, and $|G'|$. Observe that $G'Z/G_3$ is central in G/G_3, so if $g \in G'Z \backslash Z$, then the conjugacy class for g must be contained in gG_3. Since g is not in the center of G, its conjugacy class must have size $|G_3|$, and so its conjugacy class must be gG_3.

In [7], it was shown that the non-linear characters in $\mathrm{Irr}(H/H_3)$ are fully ramified with respect to H/H', and the characters in $\mathrm{Irr}(H|H_3)$ are fully ramified with respect to H/H_3. Therefore, $\mathrm{cd}(H) = \{1, p^n, p^{3n/2}\}$, and so we have $\mathrm{cd}(G) = \{1, p^n, p^{3n/2}\}$. We have seen that $G'Z/G_3$ is contained in the center of G/G_3. We know that $|G : G'Z| = p^{2n}$, and G/G_3 is not abelian. Since the square of every degree in $\mathrm{cd}(G/G_3)$ divides the index of the center of G/G_3 and $\mathrm{cd}(G/G_3) \subseteq \mathrm{cd}(G)$, it follows that $\mathrm{cd}(G/G_3) = \{1, p^n\}$, $G'Z/G_3$ is the center of G/G_3, and hence every non-linear character in $\mathrm{Irr}(G/G_3)$ is fully ramified with respect to $G/G'Z$. In [11, Theorem D], it is shown that the number of characters of a given degree in $\mathrm{Irr}(G)$ is equal to $|G|/|H|$ times the number of characters of that degree in $\mathrm{Irr}(H)$. One can show that $|G|/|H| = |G'Z : G'|$. Since all of the irreducible characters of H with degree p^n lie in $\mathrm{Irr}(H/H_3)$, we deduce that the number of such character is $|H' : H_3| - 1 = p^n - 1$. Thus, $\mathrm{Irr}(G)$ will have $|G'Z : G'|(p^n - 1)$ characters of degree p^n.

We now compute the number of characters of degree p^n in $\mathrm{Irr}(G/G_3)$. We note that the isoclinism implies that both $G'Z/Z$ and G'/G_3 are isomorphic to H'/H_3. It follows that $G'Z/Z$ is isomorphic to G'/G_3, and hence $G_3 = Z \cap G'$. In particular, we have $ZG'/G_3 = G'/G_3 \times Z/G_3$. Since the non-linear characters in $\mathrm{Irr}(G/G_3)$ are fully ramified with respect to $G/G'Z$, it follows that they are in one-to-one correspondence with the characters $\lambda \times \alpha$ where $\lambda \in \mathrm{Irr}(G'/G_3)$ with $\lambda \neq 1$ and $\alpha \in \mathrm{Irr}(Z/G_3)$.

It follows that the number of characters of degree p^n is

$$(|G'/G_3| - 1)|Z : G_3| = (p^n - 1)|G'Z : G'|.$$

Thus $\mathrm{Irr}(G/G_3)$ contains all characters in $\mathrm{Irr}(G)$ that have degree p^n. Since the characters in $\mathrm{Irr}(G|G_3)$ are not linear, they must all have degree $p^{3n/2}$. Since $|G : Z| = p^{3n}$, this implies that they are all fully ramified with respect to G/Z. $\square$

3 Identical character tables

In this section, we prove Theorem 4. We say that two groups G and H have identical character tables if there exist bijections $f : \mathrm{Irr}(G) \to \mathrm{Irr}(H)$ and $g : \mathrm{Cl}(G) \to \mathrm{Cl}(H)$ such that $f(\chi)(g(K)) = \chi(K)$ for all $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, where $\chi(K)$ is the value obtained by evaluating χ on an element of K. We showed in [6] that if we take $a : \mathrm{Irr}(G/G') \to \mathrm{Irr}(H/H')$ to be the isomorphism obtained by restricting f to $\mathrm{Irr}(G/G')$ and we define $b : \mathrm{Irr}(\mathrm{Z}(G)) \to \mathrm{Irr}(\mathrm{Z}(H))$ so that $f(\chi)_{\mathrm{Z}(H)} = \chi(1)b(\lambda)$ when $\chi_{\mathrm{Z}(G)} = \chi(1)\lambda$, then b is an isomorphism and $a(\alpha)_{\mathrm{Z}(H)} = b(\alpha_{\mathrm{Z}(G)})$ for all $\alpha \in \mathrm{Irr}(G/G')$. We also showed that if G and H are VZ-groups that satisfy this condition, then G and H have identical character tables.

We now show that if G and H are nilpotent, generalized Camina groups of nilpotence class 3 for which there are isomorphisms

$$a : \mathrm{Irr}(G/G') \to \mathrm{Irr}(H/H') \quad \text{and} \quad b : \mathrm{Irr}(\mathrm{Z}(G)) \to \mathrm{Irr}(\mathrm{Z}(H))$$

such that $a(\alpha)_{\mathrm{Z}(H)} = b(\alpha_{\mathrm{Z}(G)})$ for all $\alpha \in \mathrm{Irr}(G/G')$, then they have identical character tables. We need to make use of several results regarding generalized Camina groups of nilpotence class 3.

Proof of Theorem 4. We begin by noting that $\alpha \in \mathrm{Irr}(G/G'\mathrm{Z}(G))$ if and only if $\alpha_{\mathrm{Z}(G)} = 1_{\mathrm{Z}(G)}$, and similarly, $\beta \in \mathrm{Irr}(H/H'\mathrm{Z}(H))$ if and only if $\beta_{\mathrm{Z}(H)} = 1_{\mathrm{Z}(H)}$. It follows that restricting a gives a bijection from $\mathrm{Irr}(G/G'\mathrm{Z}(G))$ to $\mathrm{Irr}(H/H'\mathrm{Z}(H))$. This implies that $|G : G'\mathrm{Z}(G)| = |H : H'\mathrm{Z}(H)|$. Since $|G' : G_3|$ and $|H' : H_3|$ are both the square root of this number, we conclude that $|G' : G_3| = |H' : H_3|$. Also, G'/G_3 and H'/H_3 are both elementary abelian p-groups, and hence $G'/G_3 \cong H'/H_3$. We know that $\mathrm{Z}(G) \cap G' = G_3$, so $G'\mathrm{Z}(G)/G_3 = G'/G_3 \times \mathrm{Z}(G)/G_3$. This implies that $\beta \in \mathrm{Irr}(\mathrm{Z}(G)/G_3)$ if and only if there exists $\alpha \in \mathrm{Irr}(G/G')$ such that $\alpha_{\mathrm{Z}(G)} = \beta$. Hence the interaction between a and b will imply that $b(\mathrm{Irr}(\mathrm{Z}(G)/G_3)) = \mathrm{Irr}(\mathrm{Z}(H)/H_3)$. Since $\mathrm{Irr}(\mathrm{Z}(G))$ is partitioned into $\mathrm{Irr}(\mathrm{Z}(G)/G_3)$ and $\mathrm{Irr}(\mathrm{Z}(G) \mid G_3)$ and $\mathrm{Irr}(\mathrm{Z}(H))$ is partitioned into $\mathrm{Irr}(\mathrm{Z}(H)/H_3)$ and $\mathrm{Irr}(\mathrm{Z}(H) \mid H_3)$, it follows that

$$b(\mathrm{Irr}(\mathrm{Z}(G) \mid G_3)) = \mathrm{Irr}(\mathrm{Z}(H) \mid H_3).$$

We partition $\mathrm{Irr}(G)$ into three sets: $\mathrm{Irr}(G/G')$, $\mathrm{nlIrr}(G/G_3)$, and $\mathrm{Irr}(G|G_3)$. Obviously, the characters in $\mathrm{Irr}(G/G')$ are linear. All of the characters in $\mathrm{nlIrr}(G/G_3)$ are fully ramified with respect to $G/G'Z(G)$. This yields a bijection between $\mathrm{nlIrr}(G/G_3)$ and the set

$$\mathrm{Irr}(G'Z(G)/G_3 \mid G'/G_3) = \{\gamma \times \delta \mid \gamma \in \mathrm{Irr}(G'/G_3), \delta \in \mathrm{Irr}(Z(G)/G_3), \gamma \neq 1\}.$$

Finally, the characters in $\mathrm{Irr}(G|G_3)$ are all fully ramified with respect to $G/Z(G)$. This gives a bijection between $\mathrm{Irr}(G|G_3)$ and $\mathrm{Irr}(Z(G) \mid G_3)$.

We also partition the conjugacy classes of G into three sets. If $x \in G \backslash G'Z(G)$, then the conjugacy class of x is xG'. If $y \in G'Z(G) \backslash Z(G)$, then the conjugacy class of y is yG_3. Finally, if $z \in Z(G)$, then the conjugacy class of z is $\{z\}$. Notice that if $y \in G'Z(G)$, then $y = wz$ where $w \in G'$ and $z \in Z(G)$. If $w_1 z_1 G_3 = w_2 z_2 G_3$ where $w_1, w_2 \in G'$ and $z_1, z_2 \in Z(G)$, then $w_2^{-1} w_1 G_3 = z_1^{-1} z_2 G_3$. This implies that $w_2^{-1} w_1, z_1^{-1} z_2 \in G' \cap Z(G) = G_3$. We conclude that $w_1 G_3 = w_2 G_3$ and $z_1 G_3 = z_2 G_3$.

Before we can define f and g, we need to define some other functions. First, using [6], we can find bijections $a^* : G/G' \to H/H'$ and $b^* : Z(G) \to Z(H)$ so that $a(\alpha)(a^*(xG')) = \alpha(xG')$ and $b(\beta)(b^*(z)) = \beta(z)$ for all $\alpha \in \mathrm{Irr}(G/G')$, $\beta \in \mathrm{Irr}(Z(G))$, $x \in G$, and $z \in Z(G)$. In addition, we define bijections $c : \mathrm{Irr}(G'/G_3) \to \mathrm{Irr}(H'/H_3)$ and $c^* : G'/G_3 \to H'/H_3$ so that $c(\gamma)(c^*(yG_3)) = \gamma(yG_3)$ for all $\gamma \in \mathrm{Irr}(G'/G_3)$ and $y \in G'$.

We now define f and g as follows. If $\chi \in \mathrm{Irr}(G/G')$, then $f(\chi) = a(\chi)$. Suppose that $\chi \in \mathrm{nlIrr}(G/G_3)$. We know that $\chi_{G'Z(G)} = \chi(1)\gamma \times \beta$ where $\gamma \in \mathrm{Irr}(G'/G_3)$, $\beta \in \mathrm{Irr}(Z(G)/G_3)$, and $\gamma \neq 1$. Now the character $c(\gamma) \times b(\beta)$ is fully ramified with respect to $H/H'Z(H)$. We define $f(\chi)$ to be the unique irreducible constituent of $(c(\gamma) \times b(\beta))^H$. Finally, if $\chi \in \mathrm{Irr}(G|G_3)$, then we have $\chi_{Z(G)} = \chi(1)\beta$ for some $\beta \in \mathrm{Irr}(Z(G) \mid G')$. Now $b(\beta)$ is fully ramified with respect to $H/Z(H)$, and we take $f(\chi)$ to be the unique irreducible constituent of $b(\beta)^H$.

Similarly, suppose $K \in \mathrm{Cl}(G)$. If $K = xG'$ with $x \in G \backslash G'Z(G)$, set $g(K) = a^*(xG')$. If $K = zwG_3$ with $w \in G' \backslash G_3$ and $z \in Z(G)$, then we set $g(K) = b^*(z)c^*(wG_3)$. (If $z_1 w_1 G_3 = z_2 w_2 G_3$, then we have $w_1 G_3 = w_2 G_3$ and $z_1 G_3 = z_2 G_3$. It follows that $b^*(z_1)c^*(w_1 G_3) = b^*(z_2)c^*(w_2 G_3)$. Therefore $g(zwG_3)$ is well defined.) Finally, if $K = \{z\}$, then we set $g(K) = \{b^*(z)\}$.

We now evaluate $f(\chi)(g(K))$ for $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$. We have a number of cases to consider. Suppose first that $\chi \in \mathrm{Irr}(G/G')$. If $K = xG'$ with $x \in G \backslash G'$, then $f(\chi)(g(K)) = a(\chi)(a^*(xG')) = \chi(xG') = \chi(K)$. Suppose that $K = zwG_3$ with $w \in G' \backslash G_3$ and $z \in Z(G)$. Then

$$f(\chi)(g(K)) = a(\chi)(b^*(z)c^*(wG_3)) = b(\chi_{Z(G)})(b^*(z)) = \chi_{Z(G)}(z) = \chi(zwG_3),$$

since $wG_3 \leqslant \ker(\chi)$. When $K = \{z\}$ with $z \in Z(G)$, then

$$f(\chi)(g(K)) = a(\chi)_{Z(H)}(b^*(z)) = b(\chi_{Z(G)})(b^*(z)) = \chi_{Z(G)}(z) = \chi(K).$$

Next suppose that $\chi \in \mathrm{nlIrr}(G/G_3)$. Suppose that $K = xG'$ with $x \in G \backslash G'Z(G)$. We observe that $f(\chi)(g(K)) = 0 = \chi(K)$. Recall that $\chi_{G'Z(G)} = \chi(1)\gamma \times \beta$ where

$\gamma \in \mathrm{Irr}(G'/G_3)$, $\beta \in \mathrm{Irr}(\mathrm{Z}(G)/G_3)$, and $\gamma \neq 1$. We know that

$$f(\chi)_{H'\mathrm{Z}(G)} = \chi(1)c(\gamma) \times b(\beta).$$

Suppose that $K = zwG_3$ for $w \in G' \setminus G_3$ and $z \in \mathrm{Z}(G)$. Hence

$$\begin{aligned} f(\chi)(g(K)) &= \chi(1)(c(\gamma) \times b(\beta))(b^*(z)c^*(wG_3)) \\ &= \chi(1)c(\gamma)(c^*(wG_3))b(\beta)(b^*(z)) = \chi(1)\gamma(wG_3)\beta(z) = \chi(K). \end{aligned}$$

Suppose that $K = \{z\}$ for $z \in \mathrm{Z}(G)$. We see that

$$f(\chi)(g(K)) = \chi(1)b(\beta)(b^*(z)) = \chi(1)\beta(z) = \chi(K).$$

Finally, we suppose that $\chi \in \mathrm{Irr}(G|G_3)$. In this case, if K is either xG' for $x \in G \setminus G'\mathrm{Z}(G)$ or wzG_3 for $w \in G' \setminus G_3$ and $z \in \mathrm{Z}(G)$, then $f(\chi)(g(K)) = 0 = \chi(K)$. We know that $\chi_{\mathrm{Z}(G)} = \chi(1)\beta$ for some $\beta \in \mathrm{Irr}(\mathrm{Z}(G) \mid G_3)$, and hence

$$f(\chi)_{\mathrm{Z}(H)} = \chi(1)b(\beta).$$

It follows that

$$f(\chi)(g(K)) = \chi(1)b(\beta)(b^*(z)) = \chi(1)\beta(z) = \chi(K).$$

This implies that $f(\chi)(g(K)) = \chi(K)$ for all $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$. We conclude that G and H have identical character tables. □

We now see that Theorem 5 follows as a corollary. We believe that this result can also be obtained by appealing to [12, Theorem 5], but we include it here as a direct application of our work.

Proof of Theorem 5. We first suppose that G and H have identical character tables. It is easy to see that this implies $G/G' \cong H/H'$ and $\mathrm{Z}(G) \cong \mathrm{Z}(H)$. Since G and H are nilpotent Camina groups of nilpotence class 3, it follows from [8] that $G_3 = \mathrm{Z}(G)$ and $H_3 = \mathrm{Z}(H)$. We conclude that

$$|G : G'| = |H : H'| \quad \text{and} \quad |G_3| = |\mathrm{Z}(G)| = |\mathrm{Z}(H)| = |H_3|.$$

Conversely, suppose that $|G : G'| = |H : H'|$ and $|G_3| = |H_3|$. Since G and H are nilpotent Camina groups of nilpotence class 3, we know from [8] that $G' = Z_2(G)$, $G_3 = \mathrm{Z}(G)$, $H' = Z_2(H)$, and $H_3 = \mathrm{Z}(H)$. Also from [8] the groups G/G', G_3, H/H' and H_3 are elementary abelian p-groups. Thus, $|G : G'| = |H : H'|$ implies that $G/G' \cong H/H'$ and $|G_3| = |H_3|$ implies that $G_3 \cong H_3$. Since $\mathrm{Z}(G) \leqslant G'$, we have $\alpha_{\mathrm{Z}(G)} = 1_{\mathrm{Z}(G)}$ for all $\alpha \in \mathrm{Irr}(G/G')$. Similarly, $\beta_{\mathrm{Z}(H)} = 1_{\mathrm{Z}(H)}$ for all $\beta \in \mathrm{Irr}(H/H')$. With this in mind, the equation in Theorem 4 is trivially met, and we see that the conclusion holds from Theorem 4. □

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