On conjugacy of unipotent elements in finite groups of Lie type

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Abstract. Let **G** be a connected reductive algebraic group defined over $\mathbb{F}_q$, where q is a power of a prime p that is good for **G**. Let F be the Frobenius morphism associated with the $\mathbb{F}_q$-structure on **G** and set $G = \mathbf{G}^F$, the fixed point subgroup of F. Let **P** be an F-stable parabolic subgroup of **G** and let **U** be the unipotent radical of **P**; set $P = \mathbf{P}^F$ and $U = \mathbf{U}^F$. Let G_{uni} be the set of unipotent elements in G. In this note we show that the number of conjugacy classes of U in G_{uni} is given by a polynomial in q with integer coefficients.

1   Introduction

Let $\mathrm{U}_n(q)$ be the subgroup of $\mathrm{GL}_n(q)$ consisting of upper unitriangular matrices, where q is a power of a prime. A long-standing conjecture attributed to G. Higman (cf. [7]) states that the number of conjugacy classes of $\mathrm{U}_n(q)$ for fixed n is a polynomial in q with integer coefficients. This conjecture has been verified for $n \leqslant 13$ by computer calculation in work of Vera-Lopez and Arregi; see [21]. There has been much interest in this conjecture, for example from Robinson [17] and Thompson [20].

In [1] Alperin showed that a related question is easily answered, namely that the number of $\mathrm{U}_n(q)$-conjugacy classes in all of $\mathrm{GL}_n(q)$ is a polynomial in q with integer coefficients. In [5] the authors generalized Alperin's result twofold, by replacing $\mathrm{GL}_n(q)$ by a finite group of Lie type G and by replacing $\mathrm{U}_n(q)$ by the unipotent radical U an arbitrary parabolic subgroup P of G. Precisely, in [5, Theorem 4.5] we showed that, under the assumptions that the reductive algebraic group **G** corresponding to G has connected centre and that q is a power of a good prime for **G**, the number $k(U, G)$ of U-conjugacy classes in G is a polynomial in q with integer coefficients (if G has a simple component of type E_8, then there exist polynomials $m^i(z) \in \mathbb{Z}[z]$ for $i = \pm 1$ so that $k(U, G) = m^i(q)$, when q is congruent i modulo 3).

Using the machinery developed in [5], we discuss the following related conjugacy problem in this note: we show that the number $k(U, G_{\mathrm{uni}})$ of U-conjugacy classes in the set G_{uni} of unipotent elements of G is a polynomial in q with integer coefficients (again, if G has a simple component of type E_8, then two polynomials are required depending on the congruence class of q modulo 3); see Theorem 3.6 for a precise

statement. For this theorem we do not require the assumption that the centre of **G** is connected; this is because U and G_{uni} are 'independent' up to isomorphism of the isogeny class of **G**.

One can view Alperin's result in [1] and Theorem 3.6 for $G = \mathrm{GL}_n(q)$ and $U = \mathrm{U}_n(q)$ as evidence in support of Higman's conjecture. In [1] Alperin remarks that it is unlikely to be possible to obtain a proof of Higman's conjecture by descent from his theorem; it seems equally improbable that a proof of this conjecture can be deduced from Theorem 3.6.

As general references on finite groups of Lie type, we refer the reader to the books by Carter [2] and Digne and Michel [3].

2 Preliminaries

2.1 General notation for algebraic groups. We introduce some notation used throughout. Let q be a power of a prime p. By $\mathbb{F}_q$ we denote the field of q elements and by $\overline{\mathbb{F}}_q$ its algebraic closure. Throughout this paper, we identify algebraic groups defined over $\mathbb{F}_q$ with their group of $\overline{\mathbb{F}}_q$-rational points. So in particular, the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$ are identified with the additive group $\overline{\mathbb{F}}_q$ and multiplicative group $\overline{\mathbb{F}}_q^*$ respectively.

Let **G** be a connected reductive algebraic group defined over $\mathbb{F}_q$, where p is assumed to be good for **G**. Let F be the Frobenius morphism associated with the $\mathbb{F}_q$-structure on **G** and set $G = \mathbf{G}^F$ the finite group of fixed points of F in **G**.

Let **H** be a closed F-stable subgroup of **G**. We write $\mathbf{H}^\circ$ for the identity component of **H**, $\mathbf{H}_{\text{uni}}$ for the subset of unipotent elements in **H** and $H = \mathbf{H}^F$. By $|H|_p$ we denote the size of a Sylow p-subgroup of H and by $|H|_{p'}$ the p'-part of the order of H. Let S be an H-stable subset of G. We write $k(H, S)$ for the number of H-conjugacy classes in S. Given $x \in G$ we write $C_{\mathbf{H}}(x)$ for the centralizer of x in **H** and $C_H(x)$ for the centralizer of x in H; for $x \in H$, we write $C_S(x)$ for the set of fixed points of x in S. The H-conjugacy class of x is denoted by $H \cdot x$. We write

$$f_H^G(x) = |\{\, ^gH \mid x \in {}^gH, g \in G \,\}|$$

for the number of conjugates of H in G containing x.

2.2 Axiomatic set-up for connected reductive algebraic groups. For the statement of our main theorem (Theorem 3.6) we require the axiomatic set-up for connected reductive algebraic groups given in [5, §2.2], which we now recall for completeness and convenience. The idea is that a tuple of combinatorial objects is used to define a family of connected reductive groups indexed by prime powers. We refer the reader to [3, §§0, 3] for some of the results used below.

Let $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ be a root datum. Then given a finite field $\mathbb{F}_q$, the root datum Ψ determines a connected reductive algebraic group **G** over $\overline{\mathbb{F}}_q$ and a maximal torus **T** of **G** such that Ψ is the root datum of **G** with respect to **T**. Let Π be a base for Φ; this determines a Borel subgroup **B** of **G** containing **T**.

Let $F_0 : X \to X$ be an automorphism of finite order such that $F_0(\Phi) = \Phi$, $F_0(\Pi) = \Pi$ and $F_0^*(\check{\Phi}) = \check{\Phi}$. Then for any prime power q, the automorphism F_0 defines a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ such that the induced action of F on X is given by $q \cdot F_0$. Further, $\mathbf{B}$ and $\mathbf{T}$ are F-stable, so that $\mathbf{T}$ is a maximally split maximal torus of $\mathbf{G}$.

A subset J of Π determines the standard parabolic subgroup $\mathbf{P} = \mathbf{P}_J$ of $\mathbf{G}$. If $F_0(J) = J$, q is a prime power and F is the corresponding Frobenius morphism of $\mathbf{G}$, then $\mathbf{P}$ is F-stable.

Summing up, the discussion above implies that the quadruple $\Delta = (\Psi, \Pi, F_0, J)$, along with a prime power q determines:

· a connected reductive algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$ with corresponding Frobenius morphism F;

· a maximally split F-stable maximal torus $\mathbf{T}$;

· an F-stable Borel subgroup $\mathbf{B} \supseteq \mathbf{T}$ of $\mathbf{G}$; and

· an F-stable parabolic subgroup $\mathbf{P} \supseteq \mathbf{B}$.

The notation that we use for $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{P}$ does not reflect the fact that their $\mathbb{F}_q$-structure depends on the choice of a prime power q. Let q be a prime power and m a positive integer, and write F for the Frobenius morphism corresponding to q. Then it is not necessarily the case that the Frobenius morphism corresponding to the prime power q^m is F^m, i.e. the definition of $\mathbf{G}$ over $\mathbb{F}_{q^m}$ is not necessarily obtained from the $\mathbb{F}_q$-structure by extending scalars. These two definitions of $\mathbf{G}$ over $\mathbb{F}_{q^m}$ are not equivalent if F_0 is not the identity and there is a common divisor of m and the order of F_0. However, in order to keep the notation short, we choose not to show this dependence on q. We refer the reader to [5, Remark 2.1] for further explanation of our convention for varying q.

Given the data $\Delta = (\Psi, \Pi, F_0, J)$ and prime power q, we note that the unipotent radical $\mathbf{U} = R_{\mathrm{u}}(\mathbf{P})$ of $\mathbf{P} = \mathbf{P}_J$ and the unique Levi subgroup $\mathbf{L} = \mathbf{L}_J$ of $\mathbf{P}$ containing $\mathbf{T}$ are determined. Since both $\mathbf{P}$ and $\mathbf{T}$ are F-stable, so are $\mathbf{U}$ and $\mathbf{L}$.

2.3 Commuting varieties. Let $\mathbf{H}$ and $\mathbf{S}$ be a closed subgroup and a closed $\mathbf{H}$-stable subvariety of $\mathbf{G}$, respectively. The *commuting variety of* $\mathbf{H}$ *and* $\mathbf{S}$ is the closed subvariety of $\mathbf{H} \times \mathbf{S}$ defined by

$$\mathscr{C}(\mathbf{H}, \mathbf{S}) = \{(h, s) \in \mathbf{H} \times \mathbf{S} \mid hs = sh\}.$$

Assume that both $\mathbf{H}$ and $\mathbf{S}$ are F-stable. Then F acts on $\mathscr{C}(\mathbf{H}, \mathbf{S})$ and we have $\mathscr{C}(\mathbf{H}, \mathbf{S})^F = \mathscr{C}(\mathbf{H}^F, \mathbf{S}^F) = \mathscr{C}(H, S)$. The Burnside counting formula gives

$$|\mathscr{C}(H, S)| = \sum_{x \in H} |C_S(x)| = |H| \cdot k(H, S). \tag{2.1}$$

2.4 Kempf–Rousseau theory. We now briefly recall the theory of optimal cocharacters from geometric invariant theory. We require this in the proof of Proposition 3.2, which is key to the proof of Theorem 3.6.

Let $\check{X}(\mathbf{G})$ denote the set of cocharacters of $\mathbf{G}$, i.e., the set of homomorphisms $\mathbb{G}_m \to \mathbf{G}$. There is a left action of $\mathbf{G}$ on $\check{X}(\mathbf{G})$: for $\mu \in \check{X}(\mathbf{G})$ and $g \in \mathbf{G}$ we define $g \cdot \mu \in \check{X}(\mathbf{G})$ by $(g \cdot \mu)(t) = g\mu(t)g^{-1}$.

Let $\mathbf{X}$ be an affine variety. Let $\phi : \mathbb{G}_m \to \mathbf{X}$ be a morphism of algebraic varieties. We say that $\lim_{t \to 0} \phi(t)$ exists if there exists a morphism $\hat{\phi} : \mathbb{G}_a \to \mathbf{X}$ (necessarily unique) whose restriction to $\mathbb{G}_m$ is ϕ; if this limit exists, then we set $\lim_{t \to 0} \phi(t) = \hat{\phi}(0)$.

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, and let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$. We recall (see for example [19, Proposition 8.4.5]) that there exists $\lambda \in \check{X}(\mathbf{G})$ such that

$$\mathbf{P} = \mathbf{P}_\lambda := \left\{ g \in \mathbf{G} \,\middle|\, \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \right\};$$

$\mathbf{L} = \mathbf{L}_\lambda := C_{\mathbf{G}}(\lambda(\mathbb{G}_m))$; and $R_{\mathrm{u}}(\mathbf{P}) = \{g \in \mathbf{G} \mid \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} = 1\}$. Moreover, the map $c_\lambda : \mathbf{P}_\lambda \to \mathbf{L}_\lambda$ given by $g \mapsto \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1}$ is a homomorphism of algebraic groups with kernel $\ker c_\lambda = R_{\mathrm{u}}(\mathbf{P})$; we note that c_λ is simply the projection from $\mathbf{P}$ onto $\mathbf{L}$ along the semidirect decomposition $\mathbf{P} = \mathbf{L}R_{\mathrm{u}}(\mathbf{P})$.

Let $\mathbf{G}$ act on the affine variety $\mathbf{X}$. For $x \in \mathbf{X}$ let $\mathbf{G} \cdot x$ denote the $\mathbf{G}$-orbit of x in $\mathbf{X}$ and $C_{\mathbf{G}}(x)$ the stabilizer of x in $\mathbf{G}$. Let $x \in \mathbf{X}$ and let $\mathbf{C}$ be the unique closed orbit in the closure of $\mathbf{G} \cdot x$; we refer the reader to [15, (1.3)] for a proof that there is a unique closed $\mathbf{G}$-orbit in $\overline{\mathbf{G} \cdot x}$. The Kempf–Rousseau theory tells us that there exists a non-empty subset $\Omega(x)$ of $\check{X}(\mathbf{G})$ consisting of so called *optimal cocharacters* λ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists and belongs to $\mathbf{C}$; we refer the reader to [9] or [16] for information on the Kempf–Rousseau theory and the definition of optimal cocharacters. Moreover, there exists a parabolic subgroup $\mathbf{P}(x)$ of $\mathbf{G}$ so that $\mathbf{P}(x) = \mathbf{P}_\lambda$ for every $\lambda \in \Omega(x)$, and we have that $\Omega(x)$ is a single $\mathbf{P}(x)$-orbit. Further, for every $g \in \mathbf{G}$, we have $\Omega(g \cdot x) = g \cdot \Omega(x)$ and $\mathbf{P}(g \cdot x) = g\mathbf{P}(x)g^{-1}$. In particular, $C_{\mathbf{G}}(x) \leqslant \mathbf{P}(x)$. The parabolic subgroup $\mathbf{P}(x)$ is called the *optimal* or *destabilizing parabolic subgroup associated to* x.

Remark 2.2. Suppose that $\mathbf{G}$, $\mathbf{X}$ and the action of $\mathbf{G}$ on $\mathbf{X}$ are all defined over $\mathbb{F}_q$ and let F denote the Frobenius morphism associated with the $\mathbb{F}_q$-structures on both $\mathbf{G}$ and $\mathbf{X}$. There is an action of F on $\check{X}(\mathbf{G})$ as follows: for $\mu \in \check{X}(\mathbf{G})$ we define $F \cdot \mu \in \check{X}(\mathbf{G})$ by $(F \cdot \mu)(t) = F(\mu(F^{-1}(t)))$, where $F : \mathbb{G}_m \to \mathbb{G}_m$ is given by $F(t) = t^q$; see [9, §4]. Thanks to [9, Theorem 4.2] and [10, §2], if x is fixed by F, then both $\Omega(x)$ and $\mathbf{P}(x)$ are F-stable.

3   Polynomial behaviour of $k(U, G_{\mathbf{uni}})$

We maintain the notation and assumptions made in the previous sections. In particular, $\mathbf{G}$ is a connected reductive algebraic group defined over $\mathbb{F}_q$, where q is a power of the prime p which is good for $\mathbf{G}$.

We begin by stating a counting lemma for finite groups from [1], or also [5, Lemma 4.1]; the argument used to prove [5, Lemma 4.1], which uses the Burnside counting lemma, easily generalizes to the present situation, so we do not include it here.

Lemma 3.1. *Let* $\mathbf{P} = \mathbf{L}\mathbf{U}$ *be an F-stable parabolic subgroup of* $\mathbf{G}$. *Then the number of U-conjugacy classes in G_{uni} is given by*

$$k(U, G_{\mathrm{uni}}) = |L| \sum_{x \in \mathscr{R}} \frac{|C_G(x)_{\mathrm{uni}}|}{|C_G(x)|} f_U^G(x),$$

where $\mathscr{R}$ is a set of representatives of the unipotent G-conjugacy classes.

Armed with the theory of optimal cocharacters from Section 2.4, we are able to provide the following key result for our proof that $k(U, G_{\mathrm{uni}})$ is a polynomial in q. We note that the Levi decomposition of $C_G(u)$ stated in Proposition 3.2(i) is well known; see for example [13, Theorem A].

Proposition 3.2. *Let $u \in G_{\mathrm{uni}}$. Then*

(i) $C_{\mathbf{G}}(u)$ *admits a Levi decomposition $C_{\mathbf{G}}(u) = \mathbf{C}(u)\mathbf{R}(u)$ with $\mathbf{C}(u) \cap \mathbf{R}(u) = \{1\}$, $\mathbf{C}(u)$ reductive and $\mathbf{R}(u)$ the unipotent radical of $C_{\mathbf{G}}(u)$, such that $\mathbf{C}(u)$ is F-stable; therefore, setting $C(u) = \mathbf{C}(u)^F$ and $R(u) = \mathbf{R}(u)^F$, we obtain a Levi decomposition $C_G(u) = C(u)R(u)$ of $C_G(u)$;*

(ii) *both $C_{\mathbf{G}}(u)_{\mathrm{uni}}$ and $C_G(u)_{\mathrm{uni}}$ admit a 'Levi decomposition', $C_{\mathbf{G}}(u)_{\mathrm{uni}} = \mathbf{C}(u)_{\mathrm{uni}}\mathbf{R}(u)$ and $C_G(u)_{\mathrm{uni}} = C(u)_{\mathrm{uni}}R(u)$.*

Proof. (i) Since p is good for $\mathbf{G}$, the centralizer of u in $\mathbf{G}$ has a Levi decomposition, $C_{\mathbf{G}}(u) = \mathbf{C}(u)\mathbf{R}(u)$ with $\mathbf{C}(u)$ reductive and $\mathbf{R}(u)$ the unipotent radical of $C_{\mathbf{G}}(u)$; see for example [13, Theorem A]. More precisely, by [13, Theorem 2.1 and Proposition 2.5], let $\mathbf{P}(u)$ be the destabilizing parabolic subgroup associated to u; then as explained in Section 2.4, we have $C_{\mathbf{G}}(u) \subseteq \mathbf{P}(u)$. Let $\lambda \in \Omega(u)$ be an optimal cocharacter of $\mathbf{G}$ associated to u. We have $\mathbf{P}(u) = \mathbf{P}_\lambda = \mathbf{L}_\lambda \mathbf{U}(u)$, where $\mathbf{U}(u) = R_{\mathrm{u}}(\mathbf{P}(u))$. Setting $\mathbf{C}(u) = \mathbf{L}_\lambda \cap C_{\mathbf{G}}(u)$ and $\mathbf{R}(u) = \mathbf{U}(u) \cap C_{\mathbf{G}}(u)$, we obtain a Levi decomposition $C_{\mathbf{G}}(u) = \mathbf{C}(u)\mathbf{R}(u)$ of $C_{\mathbf{G}}(u)$.

By Remark 2.2, both $\Omega(u)$ and $\mathbf{P}(u)$ are F-stable, since $u \in G$. Further, $\Omega(u)$ is a single $\mathbf{P}(u)$-orbit. Since $\mathbf{P}(u)$ is connected and F-stable, it follows, from for example [3, Corollary 3.12], that there exists an F-stable cocharacter in $\Omega(u)$. We may therefore assume that λ is F-stable. Then $\mathbf{C}(u)$ is F-stable. Clearly, $\mathbf{R}(u)$ is also F-stable. Since $\mathbf{C}(u) \cap \mathbf{R}(u) = \{1\}$, it follows that

$$C_G(u) = C_{\mathbf{G}}(u)^F = \mathbf{C}(u)^F \mathbf{R}(u)^F = C(u)R(u).$$

(ii) Suppose that $v \in C_{\mathbf{G}}(u)$ is unipotent. Thanks to the Levi decomposition $C_{\mathbf{G}}(u) = \mathbf{C}(u)\mathbf{R}(u) \subseteq \mathbf{P}(u)$ of $C_{\mathbf{G}}(u)$ from part (i), we have $v = xy$ with $x \in \mathbf{C}(u)$ and $y \in \mathbf{R}(u)$. We have $x = c_\lambda(v)$ where $c_\lambda : \mathbf{P}(u) \to \mathbf{L}_\lambda$ is the canonical homomorphism defined in Section 2.4. Therefore, x is unipotent and we obtain the decomposition $C_{\mathbf{G}}(u)_{\mathrm{uni}} = \mathbf{C}(u)_{\mathrm{uni}}\mathbf{R}(u)$.

Since $\mathbf{C}(u)_{\mathrm{uni}} \cap \mathbf{R}(u) = \{1\}$, it follows that

$$C_G(u)_{\mathrm{uni}} = C_{\mathbf{G}}(u)_{\mathrm{uni}}^F = \mathbf{C}(u)_{\mathrm{uni}}^F \mathbf{R}(u)^F = C(u)_{\mathrm{uni}} R(u),$$

as desired.   □

Next we require a result regarding the independence in q of the orders of centralizers of unipotent elements in G; more precisely that these orders are given by a polynomial in q. We use the axiomatic set-up from Section 2.2 to achieve this. Fix (Ψ, Π, F_0), where $\Psi = (X, \Phi, \check{X}, \check{\Phi})$, and for a prime power q, let $\mathbf{G}$ and F be the connected reductive group and Frobenius morphism determined by (Ψ, Π, F_0) and q. We assume that $X/\mathbb{Z}\Phi$ is torsion-free, where $\mathbb{Z}\Phi$ denotes the root lattice of $\mathbf{G}$; this ensures that the centre of $\mathbf{G}$ is connected for all q. We also assume that q is a power of a good prime for $\mathbf{G}$.

Under these assumptions the parametrization of the unipotent conjugacy classes of G is independent of q; see for example [5, Proposition 2.5]. We let $\mathscr{R}$ be a set of representatives of the unipotent conjugacy classes of G, and we use the convention of [5, Remark 2.6] to vary $u \in \mathscr{R}$ with q. With these conventions we can state and prove the following proposition, which is crucial for our proof of Theorem 3.6.

Proposition 3.3. *Assume that $X/\mathbb{Z}\Phi$ is torsion-free and that q is a power of a good prime for $\mathbf{G}$. Let $u \in \mathscr{R}$. Then the order of $C_G(u)$ is given by a polynomial in q. Further, the order of $C_G(u)_{\mathrm{uni}}$ is given by a fixed power of q.*

Proof. That the order of $C_G(u)$ is a polynomial in q can be seen from the Lusztig–Shoji algorithm for computing Green functions; see [12] and [18]. It is straightforward to see that the order of the centralizers of unipotent elements of G can be determined from the block-diagonal matrix Λ, defined in [18, §5]; the blocks are determined by the Springer correspondence. The (unknown) matrix Λ satisfies the equation

$$^t P \Lambda P = \Pi, \tag{3.4}$$

where P is an (unknown) upper triangular block matrix with each diagonal block a matrix with entries in $\mathbb{Q}$, and Π is a known matrix with entries that are rational functions in q with coefficients independent of q; see [18, (5.6)]. As stated in *loc. cit.*, the matrix Λ is uniquely determined by (3.4); moreover, one sees that the entries of Λ are rational functions in q, with coefficients independent of q. In particular, we can deduce that $|C_G(u)|$ is a rational function in q, and then a standard argument (see for example [5, Lemma 2.12]) tells us that $|C_G(u)|$ is in fact a polynomial in q.

The second statement in the proposition now follows from Steinberg's formula (see for example [3, Corollary 9.5]) applied to the Levi factor $C(u)$ of $C_G(u)$, along with Proposition 3.2 (ii).   □

Remark 3.5. It seems likely that a stronger result than Proposition 3.3 regarding the structure of $C_G(u)$ holds, namely, that the root datum corresponding to $\mathbf{C}(u)^\circ$, the

component group $\mathbf{A}(u)$ of $\mathbf{C}(u)$, and the action of F on the root lattice for $\mathbf{C}(u)^\circ$ and $\mathbf{A}(u)$ do not depend on q. It is known that the root datum of $\mathbf{C}(u)$ does not depend on q, though this is only by a case-by-case analysis; see for example the discussion at the end of [8, (5.11)]. There is a general proof that the structure of $\mathbf{A}(u)$ does not depend on q; see [11] or [13]. One then needs to check that the action of F on $\mathbf{C}(u)$ for *split elements* u is independent of q; for G of type E_8 one other case needs to be dealt with separately. Further one needs to know that the action of $\mathbf{A}(u)$ on the set of simple roots of $\mathbf{C}(u)^\circ$ does not depend on q. We have chosen not to pursue this here.

We are now in a position to prove the principal result of this paper, which is an analogue of [5, Theorem 4.5]. We continue to use the axiomatic set-up from Section 2.2.

Theorem 3.6. *Fix the data* $\Delta = (\Psi, \Pi, F_0, J)$, *where* $\Psi = (X, \Phi, \check{X}, \check{\Phi})$. *For a prime power* q, *let* $\mathbf{G}$, F *and* $\mathbf{P}$ *be the connected reductive group, Frobenius morphism and* F-*stable parabolic subgroup of* $\mathbf{G}$ *determined by* Δ *and let* $\mathbf{U} = R_{\mathrm{u}}(\mathbf{P})$. *Assume that* q *is power of a good prime for* $\mathbf{G}$.

 (i) *Suppose that* $\mathbf{G}$ *does not have a simple component of type* E_8. *Then there exists* $m(z) \in \mathbb{Z}[z]$ *such that* $k(U, G_{\mathrm{uni}}) = m(q)$.

(ii) *Suppose that* $\mathbf{G}$ *has a simple component of type* E_8. *Then there exist* $m^i(z) \in \mathbb{Z}[z]$ $(i = \pm 1)$, *such that* $k(U, G_{\mathrm{uni}}) = m^i(q)$, *when* q *is congruent to* i *modulo* 3.

Proof. We begin by assuming that $X/\mathbb{Z}\Phi$ is torsion-free, so that the centre of $\mathbf{G}$ is connected. We write $\mathbf{L}$ for the Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$.

By Lemma 3.1, we have

$$k(U, G_{\mathrm{uni}}) = |L| \sum_{x \in \mathscr{R}} \frac{|C_G(x)_{\mathrm{uni}}|}{|C_G(x)|} f_U^G(x), \tag{3.7}$$

where $\mathscr{R}$ is a set of representatives of the unipotent G-classes. With the assumptions that $X/\mathbb{Z}\Phi$ is torsion-free and that q is power of a good prime for $\mathbf{G}$, it follows from [5, Proposition 2.5] that the set $\mathscr{R}$ is independent of q, where we use the convention of [5, Remark 2.6] to vary x with q.

Since L is a finite reductive group, the factor $|L|$ is a polynomial in q (see [2, p. 75]). Thanks to [5, Lemma 3.1(ii), Theorem 3.10], each of the factors $f_U^G(x)$ in the sum above is a polynomial in q, unless we are in case (ii) when $f_U^G(x)$ is given by two polynomials depending on q modulo 3. From Proposition 3.3 we have that $|C_G(x)|$ and $|C_G(x)_{\mathrm{uni}}|$ are polynomials in q. Hence, $k(U, G_{\mathrm{uni}})$ is a rational function in q. Now by a standard argument, see for example [5, Lemma 2.12], we can conclude that $k(U, G_{\mathrm{uni}})$ is a polynomial function in q with rational coefficients.

Assume now that $\mathbf{G}$ is split over $\mathbb{F}_q$, i.e. that F_0 is the identity. Thanks to (2.1), $|\mathscr{C}(\mathbf{U}, \mathbf{G}_{\mathrm{uni}})^F|$ is a polynomial in q with rational coefficients. The assumption that $\mathbf{G}$ is split means that $|\mathscr{C}(U, G_{\mathrm{uni}})|$ gives the number of $\mathbb{F}_q$-rational points in the variety

$\mathscr{C}(\mathbf{U}, \mathbf{G}_{\text{uni}})$ viewed as a variety defined over $\mathbb{F}_p$. Now using the Grothendieck trace formula (see [3, Theorem 10.4]), one can prove that the coefficients of this polynomial are integers; see for example [14, Proposition 6.1].

A further standard argument using the Grothendieck trace formula now tells us that the eigenvalues of F on the l-adic cohomology groups of $\mathscr{C}(\mathbf{U}, \mathbf{G}_{\text{uni}})$ are all powers of q; see for example the proof of [14, Proposition 6.1]. Now assume that $\mathbf{G}$ is not split and let d be the order of F_0. Then arguments like those used to prove [5, Proposition 3.20] imply that the eigenvalues of F on the l-adic cohomology groups of $\mathscr{C}(\mathbf{U}, \mathbf{G}_{\text{uni}})$ (viewed as a variety defined over $\mathbb{F}_q$) are of the form ζq^m, where ζ is a dth root of unity. Following the arguments to prove [5, Proposition 3.20], one can now show that the coefficients of the polynomial $|\mathscr{C}(U, G_{\text{uni}})|$ are integers. Then using (2.1) again, it follows that $k(U, G_{\text{uni}})$ is a polynomial function in q with integer coefficients.

Now remove the assumption that $X/\mathbb{Z}\Phi$ is torsion-free. Let $\sigma : \mathbf{G} \to \hat{\mathbf{G}}$ be an isogeny that is defined over $\mathbb{F}_q$, where $\hat{\mathbf{G}}$ is a reductive group defined over $\mathbb{F}_q$ with connected centre. Then σ induces an isomorphism between $\mathbf{U}$ and $\hat{\mathbf{U}}$ and between $\mathbf{G}_{\text{uni}}$ and $\hat{\mathbf{G}}_{\text{uni}}$, since $Z(\mathbf{G}) \cap \mathbf{U} = \{1\} = Z(\mathbf{G}) \cap \mathbf{G}_{\text{uni}}$, where $Z(\mathbf{G})$ is the centre of $\mathbf{G}$. It follows easily that $k(U, G_{\text{uni}}) = k(\hat{U}, \hat{G}_{\text{uni}})$ is given by a polynomial in q with integer coefficients. $\square$

Recall that two parabolic subgroups of $\mathbf{G}$ are called *associated* if they have Levi subgroups that are conjugate in $\mathbf{G}$. It was already remarked in [5, Corollary 3.5] that if $\mathbf{P}$ and $\mathbf{Q}$ are associated parabolic subgroups of $\mathbf{G}$ with unipotent radicals $\mathbf{U}$ and $\mathbf{V}$ respectively, then the functions f_U^G and f_V^G are equal on unipotent elements of G. Therefore, from (3.7) we can observe the following corollary in the same way as [5, Corollary 4.7].

Corollary 3.8. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be associated F-stable parabolic subgroups of* $\mathbf{G}$ *with unipotent radicals* $\mathbf{U}$ *and* $\mathbf{V}$ *respectively. Then*

$$k(U, G_{\text{uni}}) = k(V, G_{\text{uni}}).$$

In order to compute the polynomials given in Theorem 3.6 explicitly, we reformulate the expression for $k(U, G_{\text{uni}})$ in (3.7) in terms of Green functions. Using [5, Lemma 3.3], Proposition 3.2, and Steinberg's formula [3, Corollary 9.5], we obtain

$$\begin{aligned}
k(U, G_{\text{uni}}) &= |L| \sum_{x \in \mathscr{R}} \frac{|C_G(x)_{\text{uni}}|}{|C_G(x)|} \left(\frac{1}{|L|_p |W_{\mathbf{L}}|} \sum_{w \in W_{\mathbf{L}}} (-1)^{l(w)} Q_{\mathbf{T}_w}^{\mathbf{G}}(x) \right) \\
&= \frac{1}{|W_{\mathbf{L}}|} \frac{|L|}{|L|_p} \sum_{x \in \mathscr{R}} \frac{|C(x)|_p^2}{|C(x)|} \left(\sum_{w \in W_{\mathbf{L}}} (-1)^{l(w)} Q_{\mathbf{T}_w}^{\mathbf{G}}(x) \right) \\
&= \frac{1}{|W_{\mathbf{L}}|} |L|_{p'} \sum_{x \in \mathscr{R}} \frac{|C(x)|_p}{|C(x)|_{p'}} \left(\sum_{w \in W_{\mathbf{L}}} (-1)^{l(w)} Q_{\mathbf{T}_w}^{\mathbf{G}}(x) \right), \qquad (3.9)
\end{aligned}$$

where $C(x)$ is the reductive part of $C_G(x)$, as in Proposition 3.2 (i), $W_{\mathbf{L}}$ is the Weyl group of $\mathbf{L}$; for $w \in W_{\mathbf{L}}$ we write $\mathbf{T}_w$ for the F-stable twisted torus associated with $w \in W_{\mathbf{L}}$, and $Q_{\mathbf{T}_w}^{\mathbf{G}}$ is the Green function associated with $\mathbf{T}_w$. For more information on Green functions we refer the reader to [2, §7.6]. We note that the sum in (3.9) is effectively only over representatives of the $\mathbf{G}$-orbits that meet $\mathbf{U}$. This follows from the fact that the term $f_U^G(x)$ in (3.7) is obviously zero if $\mathbf{G} \cdot x \cap \mathbf{U} = \varnothing$.

Using the `chevie` package in GAP3 ([4]) along with some code provided by M. Geck, it is possible to explicitly calculate the polynomials $m(z)$ in Theorem 3.6 for $\mathbf{G}$ of small rank. We illustrate this with some examples for the case when $\mathbf{G} = \mathrm{GL}_n$ and $\mathbf{P} = \mathbf{B}$ is a Borel subgroup of $\mathbf{G}$.

Example 3.10. In Table 1 below we give the polynomials for $k(U, G_{\mathrm{uni}})$ for $G = \mathrm{GL}_n(q)$ and $P = B$ a Borel subgroup of G, for $n = 2, \ldots, 10$. In this case we take $U = \mathrm{U}_n(q)$ to be the group of upper unitriangular matrices.

n	$k(\mathrm{U}_n(q), \mathrm{GL}_n(q)_{\mathrm{uni}})$
2	$2q - 1$
3	$q^3 + 3q^2 - 3q$
4	$q^6 + 5q^4 - 9q^2 + 4q$
5	$q^{10} + 4q^7 + 4q^6 + 6q^5 - 20q^4 - 10q^3 + 21q^2 - 4q - 1$
6	$q^{15} + 5q^{11} - q^{10} + 13q^9 - 5q^8 + q^7 - 20q^6 - 44q^5 + 52q^4 + 25q^3 - 31q^2 + 5q$
7	$q^{21} + 6q^{16} - q^{15} - q^{14} + 13q^{13} + 22q^{12} - 41q^{11} + 37q^{10} - 49q^9 - 57q^8 - 12q^7 + 71q^6$ $+ 139q^5 - 120q^4 - 51q^3 + 50q^2 - 5q - 1$
8	$q^{28} + 7q^{22} - q^{21} - q^{20} - q^{19} + 19q^{18} + 13q^{17} + 7q^{16} - 54q^{15} + 39q^{14} + 39q^{13} - 125q^{12}$ $- 86q^{11} - 93q^{10} + 225q^9 + 160q^8 + 102q^7 - 164q^6 - 322q^5 + 207q^4 + 87q^3 - 64q^2 + 6q$
9	$q^{36} + 8q^{29} - q^{28} - q^{27} - q^{26} - q^{25} + 26q^{24} + 19q^{23} - 44q^{22} + 41q^{21} + 9q^{20} + 25q^{19}$ $- 119q^{18} + 57q^{17} - 134q^{16} + 119q^{15} - 458q^{14} + 177q^{13} + 290q^{12} - 121q^{11} + 1315q^{10}$ $- 807q^9 - 971q^8 + 60q^7 + 326q^6 + 568q^5 - 319q^4 - 145q^3 + 89q^2 - 7q$
10	$q^{45} + 9q^{37} - q^{36} - q^{35} - q^{34} - q^{33} - q^{32} + 34q^{31} + 26q^{30} - 54q^{29} - 8q^{28} + 68q^{27}$ $- 41q^{26} + 251q^{25} - 258q^{24} - 395q^{23} + 474q^{22} + 259q^{21} - 674q^{20} - 174q^{19} + 126q^{18}$ $- 1384q^{17} + 3300q^{16} - 1299q^{15} - 1227q^{14} + 4050q^{13} - 2400q^{12} - 691q^{11} - 2676q^{10}$ $+ 944q^9 + 3298q^8 - 808q^7 - 293q^6 - 1017q^5 + 455q^4 + 210q^3 - 108q^2 + 8q$

Table 1. $k(\mathrm{U}_n(q), \mathrm{GL}_n(q)_{\mathrm{uni}})$

Remark 3.11. One can easily check that each polynomial $k(\mathrm{U}_n(q), \mathrm{GL}_n(q)_{\mathrm{uni}})$ in Table 1 when expressed as a polynomial in $q - 1$ has all coefficients non-negative. One might conjecture that indeed in general each of the polynomials satisfies $k(U, G_{\mathrm{uni}}) \in \mathbb{N}[q - 1]$. As stated in [5, Remark 4.13], this is also the case for each of

the explicit examples of the polynomials $k(U, G)$ calculated in [5]. It would be interesting to know if there is a geometric explanation for these positivity phenomena.

Remark 3.12. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$. Clearly, we have

$$k(U, G_{\mathrm{uni}}) = \sum_{u \in \mathscr{R}} k(U, G \cdot u),$$

where $\mathscr{R}$ is a complete set of representatives of the unipotent G-conjugacy classes. By an analogue of Lemma 3.1, we get

$$k(U, G_{\mathrm{uni}}) = |L| \sum_{u \in \mathscr{R}} \left(\sum_{x \in \mathscr{R}} \frac{|C_G(x) \cap G \cdot u|}{|C_G(x)|} f_U^G(x) \right).$$

It would be interesting to know whether each of the summands $k(U, G \cdot u)$ is a polynomial in q; this is the case if $|C_G(x) \cap G \cdot u|$ is a polynomial in q for all x and u.

Remark 3.13. Using arguments as in [6], it is possible to show that in case $\mathbf{G} = \mathrm{GL}_n$, the number of P-conjugacy classes in G_{uni} is given by a polynomial in q. As the details are technical, we choose not to include them here. For arbitrary G and P it is not clear whether $k(P, G_{\mathrm{uni}})$ is polynomial or even given by Polynomials On Residue Classes (PORC).

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