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On $\mathscr{M}$-supplemented subgroups of finite groups

Long Miao* and Wolfgang Lempken

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Abstract. A subgroup H is called $\mathscr{M}$-supplemented in a finite group G if there exists a subgroup B of G such that $G = HB$ and such that H_1B is a proper subgroup of G for any maximal subgroup H_1 of H. In this paper we fix a subgroup D in every non-cyclic Sylow subgroup P of G satisfying $1 < D < P$ and study the structure of G under the assumption that all subgroups H of P with $|H| = |D|$ are $\mathscr{M}$-supplemented in G or have a supersolvable supplement in G.

1 Introduction

Subgroups of prime-power order have been studied extensively in relation to the structure of a finite group. For instance, Hall [4] in 1937 proved that a group G is solvable if and only if every Sylow subgroup of G is complemented in G. Srinivasan [8] in 1980 stated that a group G is supersolvable if every maximal subgroup of every Sylow subgroup is normal in G. Wang [13] in 2000 generalized Srinivasan's result: if every maximal subgroup of every Sylow subgroup is c-supplemented in G, then G is supersolvable. Recently, Miao and Guo [7] considered $\mathscr{F}$-s-supplemented subgroups of a finite group G and obtained that G is supersolvable if and only if every maximal subgroup of every Sylow subgroup of G is supersolvable and s-supplemented in G. More recently, Skiba in [10] and [11] fixed a subgroup D of a non-cyclic Sylow subgroup P of $F^*(G)$ such that $1 < D < P$ and studied the structure of G under the assumption that all subgroups E of P of the same order as D and order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$), are c-quasinormal or weakly s-permutable in G.

Suppose that $G = HB$ with proper subgroups H and B; so H and B supplement each other in G. It is natural to consider the case where H is a minimal supplement of B in G, i.e. where $H_1B \neq G$ for every proper subgroup H_1 of H. Note that H is a minimal supplement of B in G whenever $H \cap B \leqslant \Phi(H)$; moreover, if $B \lhd G = HB$ and H is a minimal supplement of B in G, then $H \cap B \leqslant \Phi(H)$. In view of the results mentioned above we may then ask whether the fact that certain subgroups of G are

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minimal supplements in G has an effect on the structure of G. In order to deal with this question we introduce the concept of $\mathcal{M}$-supplemented subgroups.

Definition 1.1. A subgroup H is called $\mathcal{M}$-supplemented in a finite group G if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for any maximal subgroup H_1 of H; in this case, B also is said to be an $\mathcal{M}$-supplement of H in G.

Note that the trivial group 1 is $\mathcal{M}$-supplemented in any non-trivial group since the set of maximal subgroups of 1 is empty.

In this paper we investigate to what extent the presence of $\mathcal{M}$-supplements for certain subgroups of a group G controls the structure of G. As a by-product we obtain results about supersolvable and p-nilpotent groups as well as formations, one of which is the following result.

Theorem 3.6. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and let G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(H)$ has a non-trivial proper subgroup D such that every subgroup $E \leqslant P$ of order $|D|$ has a supersolvable supplement or an $\mathcal{M}$-supplement in G. Then $G \in \mathcal{F}$.*

All groups considered in this paper are finite. Most of the notation is standard and can be found in [1] and [5]. In particular, $H < G$ indicates that H is a proper subgroup of G, G_p is a Sylow p-subgroup of G and $\pi(G)$ is the set of all prime divisors of $|G|$; we write H_G for the core $\bigcap_{g \in G} H^g$ of a subgroup H. Moreover, $\Phi(G)$, $F(G)$ and $F^*(G)$ denote the Frattini subgroup, the Fitting subgroup and the generalized Fitting subgroup of G, respectively. Furthermore, $\mathcal{U}$ denotes the class of all supersolvable groups.

Let π be a set of primes. We write $G \in E_\pi$ if G has a Hall π-subgroup, $G \in C_\pi$ if $G \in E_\pi$ and any two Hall π-subgroups of G are conjugate in G, and we write $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G.

A class $\mathcal{F}$ of groups is said to be a formation if $G/H \in \mathcal{F}$ whenever $G \in \mathcal{F}$ and $H \trianglelefteq G$ and if $G/(M \cap N) \in \mathcal{F}$ whenever G/M and G/N are in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. Both the class of all supersolvable groups and the class of all p-nilpotent groups are saturated formations (e.g. see [3]).

Finally, we recall that a group G is said to possess a *supersolvable type Sylow tower* if $\pi(G) = \{p_1, \dots, p_n\}$ with $p_1 > p_2 > \cdots > p_n$ and if there exist subgroups $S_{p_i} \in \mathrm{Syl}_{p_i}(G)$ such that $S_{p_1}S_{p_2}\dots S_{p_i}$ is a normal subgroup of G for $i \in \{1, 2, \dots, n\}$.

2 Preliminaries

Here we list some basic results which will be useful in the sequel.

Lemma 2.1. (1) *If $H \leqslant M \leqslant G$ and H is $\mathcal{M}$-supplemented in G, then H is $\mathcal{M}$-supplemented in M.*

(2) *Let* $N \trianglelefteq G$ *and* $N \leqslant H \leqslant G$. *If* H *is* $\mathcal{M}$*-supplemented in* G, *then* H/N *is* $\mathcal{M}$*-supplemented in* G/N.

(3) *Let* K *be a normal* π'*-subgroup and let* H *be a* π*-subgroup of* G *for a set* π *of primes. Then* H *is* $\mathcal{M}$*-supplemented in* G *if and only if* HK/K *is* $\mathcal{M}$*-supplemented in* G/K.

Proof. (1) If H is $\mathcal{M}$-supplemented in G, then there exists a subgroup $B \leqslant G$ such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H. Set $L = M \cap B$. Then $L \leqslant M$ and $M = M \cap HB = H(M \cap B) = HL$ by Dedekind's law. Let H_1 be a maximal subgroup of H. Then $H_1L = H_1(M \cap B) = M \cap H_1B$ is a subgroup of M. If $H_1L = M$, then $H_1B = MB \supseteq HB = G$, contrary to $H_1B < G$. Hence $H_1L < M$ and so H is $\mathcal{M}$-supplemented in M.

(2) This follows from the definition of $\mathcal{M}$-supplemented subgroups.

(3) If H is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H. Clearly, $(HK/K)(BK/K) = G/K$. For any maximal subgroup T/K of HK/K, since K is a normal π'-subgroup and H is a π-subgroup of G, we have $T = H_1K$ where H_1 is a maximal subgroup of H. Therefore

$$(H_1K/K)(BK/K) = H_1BK/K < G/K.$$

Otherwise, if $H_1BK = G$, then $|G : H_1B| = |K : K \cap H_1B|$ is a π'-number. On the other hand, $|G : H_1B| = |HB : H_1B|$ is a π-number, and so $|G : H_1B| = 1$, a contradiction.

Conversely, let HK/K be $\mathcal{M}$-supplemented in G/K by the subgroup L/K and let H_1 be a maximal subgroup of H. Then H_1K/K is a maximal subgroup of HK/K and therefore we have $HL/K = (HK/K)(L/K) = G/K$ as well as $H_1L/K = (H_1K/K)(L/K) < G/K$; in particular $HL = G$ and $H_1L < G$. So H is $\mathcal{M}$-supplemented in G by L. □

Lemma 2.2. *Let* $p \in \pi(G)$ *and let* P *be a* p*-subgroup of* G *having an* $\mathcal{M}$*-supplement* B *in* G. *Then*

(1) $P \cap B = P_1 \cap B = \Phi(P) \cap B$ *and* $|G : P_1B| = p$ *for any maximal subgroup* P_1 *of* P.

(2) *If* L *is a minimal normal subgroup of* G *contained in* P, *then* $|L| = p$ *or* $L \leqslant \Phi(P)$.

Proof. (1) By hypothesis we have $G = PB$ and $P_1B < G$ for any maximal subgroup P_1 of P. Since $|P : P_1| = p$, we get

$$|G| = |PB| = p|P_1|\,|B|/|P \cap B| = (p/|(P \cap B) : (P_1 \cap B)|) \cdot |P_1B|.$$

As p is a prime and $P_1B < G$, we conclude that $P \cap B = P_1 \cap B$ and $|G : P_1B| = p$. Now the claim follows, because $\Phi(P)$ is the intersection of the maximal subgroups of P.

(2) Suppose that $L \not\leq \Phi(P)$. Then there exists a maximal subgroup Q of P with $L \not\leq Q$ and $P = LQ$ as well as $G = PB = L(QB)$; in particular, $L \not\leq QB$. Note that $L \cap (QB) \trianglelefteq G$, because L is abelian. Since L is minimal normal in G, we get $L \cap QB = 1$ and hence $|L| = |G : QB| = p$ by part (1). □

Lemma 2.3. *Let $\mathscr{F}$ be a saturated formation and suppose that $G/E \in \mathscr{F}$ for some $E \trianglelefteq G$. Then $G \in \mathscr{F}$ whenever one of the following holds*:

(1) $E \leqslant \Phi(G)$;

(2) *E is cyclic and $\mathscr{U} \subseteq \mathscr{F}$.*

Proof. (1) Since $G/\Phi(G) \cong (G/E)/(\Phi(G)/E) \in \mathscr{F}$ and $\mathscr{F}$ is saturated, $G \in \mathscr{F}$.
(2) This is shown in [9, Lemma 1.9]. □

Lemma 2.4 ([3, Theorem 1.8.17]). *Let N be a non-trivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.5 ([5, Theorem I.6.6]). *If H is a subgroup of G with $|G : H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

Lemma 2.6 ([6]). *Let $N \leqslant G$. Then the following hold.*

(1) *If N is normal in G, then $F^*(N) \leqslant F^*(G)$.*

(2) $F^*(G) \neq 1$ *if* $G \neq 1$; *in fact*, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geqslant F(G)$; *if $F^*(G)$ is solvable then $F^*(G) = F(G)$.*

(4) $C_G(F^*(G)) \leqslant F(G)$.

(5) *Let $P \trianglelefteq G$ and $P \leqslant O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

(6) *If $K \leqslant Z(G)$ then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.7. *Let $H \trianglelefteq G$, $L \trianglelefteq G$ and let $p \in \pi(G)$. Then the following hold.*

(1) $\Phi(L) \leqslant \Phi(G)$.

(2) *If $L \leqslant \Phi(G)$, then $F(G/L) = F(G)/L$.*

(3) *If $L \leqslant H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.*

(4) *If H is a p-group and $L \leqslant \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

Proof. (1) This is well known, e.g. see [5, Lemma III.3.3].
(2) Note that $F(G/\Phi(G)) = F(G)/\Phi(G)$ and $\Phi(G/L) = \Phi(G)/L$. With this we obtain

$$\begin{aligned}(F(G)/L)/\Phi(G/L) &= (F(G)/L)/(\Phi(G)/L) \cong F(G)/\Phi(G)\\ &= F(G/\Phi(G)) \cong F((G/L)/(\Phi(G)/L))\\ &= F((G/L)/\Phi(G/L)) = F(G/L)/\Phi(G/L)\end{aligned}$$

and then $F(G)/L = F(G/L)$.

(3) Note that

$$F(H/L) = H/L \cap F(G/L) = H/L \cap F(G)/L = (H \cap F(G))/L = F(H)/L.$$

(4) Since $L \leqslant \Phi(H)$, we have $\Phi(H/L) = \Phi(H)/L$. By Lemma 2.6, we get that

$$F^*((G/L)/\Phi(H/L)) = F^*(G/L)/\Phi(H/L) \cong F^*(G/\Phi(H)) = F^*(G)/\Phi(H)$$

and hence $(F^*(G)/L)/(\Phi(H)/L) = F^*(G/L)/\Phi(H/L)$. Therefore

$$F^*(G/L) = F^*(G)/L. \quad \square$$

Lemma 2.8 ([14, Theorem 3.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that H is a solvable normal subgroup of G such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leqslant M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds in the case where $\mathcal{F} = \mathcal{U}$.*

Theorem 2.9 ([2, Main theorem]). *Suppose that G has a Hall π-subgroup where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.10. *Let $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Then the following hold.*

(1) *G is p-nilpotent, whenever $N_G(P) = C_G(P)$. In particular, G is p-nilpotent whenever P is cyclic and p is the smallest prime in $\pi(G)$.*

(2) *If $N \trianglelefteq G$ with $P \cap N \leqslant \Phi(P)$, then N is p-nilpotent.*

Proof. (1) is a result of Burnside; see [5, Theorems IV.2.6 and IV.2.8], and (2) is a result of Tate [12]; see [5, Theorem IV.4.7]. $\square$

Lemma 2.11. *Let p be the smallest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P is $\mathcal{M}$-supplemented in G.*

Proof. If G is p-nilpotent, then G has a normal p-complement D. So $G = PD$ and $P_1D < G$ for every maximal subgroup P_1 of P, i.e. P is $\mathcal{M}$-supplemented in G.

Conversely, if P is $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = PB$ and $P_1B < G$ for every maximal subgroup P_1 of P. By Lemma 2.2, we have $|G : P_1B| = p$ and hence $P_1B \trianglelefteq G$ by Lemma 2.5. Observe that

$$\bigcap_{P_1 <\cdot P} (P_1B) = \left(\bigcap_{P_1 <\cdot P} P_1\right)B = \Phi(P)B$$

and therefore $\Phi(P)B \trianglelefteq G$. Since $P \cap B = \Phi(P) \cap B$ by Lemma 2.2 , we also have

$$P \cap \Phi(P)B = \Phi(P)(P \cap B) \leqslant \Phi(P).$$

Hence $\Phi(P)B$ is p-nilpotent by Lemma 2.10. Set $H = O_{p'}(\Phi(P)B)$. Then $H \leqslant O_{p'}(G)$ with $B \leqslant PH = PB = G$, and thus $H = O_{p'}(G)$; in particular, G is p-nilpotent. □

Lemma 2.12. *Let p be the smallest prime divisor of $|G|$ and let $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P has a non-trivial proper subgroup D such that every subgroup E of P with $|E| = |D|$ has a p-nilpotent supplement or an $\mathcal{M}$-supplement in G.*

Proof. As the necessity part is obvious, we only consider the sufficiency part. Assume that the lemma is false and choose G to be a counter-example of minimal order. In particular, by Lemma 2.10, P is not cyclic.

By hypothesis, P has a non-trivial proper subgroup D such that every subgroup of order $|D|$ in P has a p-nilpotent supplement or an $\mathcal{M}$-supplement in G. Let P_1 be a maximal subgroup of P with $D \leqslant P_1$ and suppose that K is p-nilpotent supplement of D in G. Then $G = DK = P_1K$ and so $K_{p'} := O_{p'}(K)$ is a Hall p'-subgroup of G with $G = P_1N_G(K_{p'})$. If $P \cap N_G(K_{p'}) = P$, then $K_{p'} \trianglelefteq G$, a contradiction. If $P \cap N_G(K_{p'}) = L$ where L is a maximal subgroup of P, then

$$|G : N_G(K_{p'})| = |P : P \cap N_G(K_{p'})| = |P : L| = p$$

and hence $N_G(K_{p'}) \trianglelefteq G$ by Lemma 2.5, a contradiction. So we may assume that $P \cap N_G(K_{p'}) \leqslant L_2 < L_1$ where L_1 is a maximal subgroup of P and L_2 is a maximal subgroup of L_1. By hypothesis, there exists a subgroup T of P with order $|D|$ such that $T \leqslant L_1$. If T has a p-nilpotent supplement H in G, then $G = TH = L_1H$. With a similar argument as above we have $G = L_1N_G(H_{p'})$ where $H_{p'} := O_{p'}(H)$ is a Hall p'-subgroup of H and of G. By Theorem 2.9, there exists an element $x \in P$ such that $N_G(K_{p'}) = (N_G(H_{p'}))^x$. Therefore

$$G = L_1N_G(H_{p'}) = (L_1N_G(H_{p'}))^x = L_1N_G(K_{p'}).$$

Furthermore,

$$P = P \cap L_1N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1,$$

a contradiction.

So we may assume that T has an $\mathcal{M}$-supplement B in G, i.e. $G = TB = L_1B$ and $T_1B < G$ for any maximal subgroup T_1 of T. If $|D| = |L_1|$, then L_2B is a proper subgroup of G. Clearly, $P \cap L_2B = L_2(P \cap B)$ is a Sylow subgroup of L_2B. If, by hypothesis, $L_2(P \cap B)$ has a p-nilpotent supplement M in G, then

$$G = L_2(P \cap B)M = L_2(P \cap B)N_G(M_{p'}).$$

By Theorem 2.9, there exists an element $g \in P$ such that $N_G(K_{p'}) = (N_G(M_{p'}))^g$. Therefore we have

$$G = L_2(P \cap B)N_G(M_{p'}) = (L_2(P \cap B)N_G(M_{p'}))^g = L_2(P \cap B)N_G(K_{p'})$$

and hence

$$P = P \cap L_2(P \cap B)N_G(K_{p'}) = L_2(P \cap B)(P \cap N_G(K_{p'})) = L_2(P \cap B),$$

a contradiction. So we may assume that $L_2(P \cap B)$ is $\mathcal{M}$-supplemented in L_2B. Then L_2B is p-nilpotent by Lemma 2.11 and hence G is p-nilpotent, a contradiction.

So we may assume that $|D| < |L_1|$. Therefore $|G : T_1B| = p$ by Lemma 2.2 and hence $T_1B \trianglelefteq G$ by Lemma 2.5. We have $G = TB = PB = PT_1B$ and $P \cap T_1B = T_1(P \cap B) \in \mathrm{Syl}_p(T_1B)$. Clearly, $T_1(P \cap B)$ is maximal in P. If a subgroup $N \leqslant T_1(P \cap B)$ of order $|D|$ has no p-nilpotent supplement in T_1B, then N also has no p-nilpotent supplement in G and hence is $\mathcal{M}$-supplemented in G by hypothesis. So N is $\mathcal{M}$-supplemented in T_1B by Lemma 2.1. Therefore T_1B satisfies the hypothesis of the lemma and hence is p-nilpotent by the minimal choice of G. Since $T_1B \trianglelefteq G$, we get that G is p-nilpotent, a contradiction which eventually proves the following:

(1) each subgroup $E \leqslant P$ of order $|D|$ has no p-nilpotent supplement in G, and thus has an $\mathcal{M}$-supplement in G.

We may assume now that D has an $\mathcal{M}$-supplement K in G. Let D_1 be a maximal subgroup of D. The same arguments as in the section preceding (1) show that D_1K is a normal subgroup of index p in G and moreover that $P_2 := P \cap D_1K = D_1(P \cap K) \in \mathrm{Syl}_p(D_1K)$ with $|P : P_2| = p$. If $|D| = |P_2|$, then D_1K is p-nilpotent by (1), Lemma 2.1 and Lemma 2.11; this implies that G is p-nilpotent, a contradiction. Therefore we have $|D| < |P_2|$. In particular, D_1K satisfies the hypothesis of the lemma because any subgroup of order $|D|$ in P_2 has an $\mathcal{M}$-supplement in D_1K by (1) and Lemma 2.1. So D_1K is p-nilpotent by the minimal choice of G. This in turn implies that G is p-nilpotent, a final contradiction. □

Lemma 2.13. *Let $p \in \pi(G)$ such that $G \in C_{p'}$ and let $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P has a subgroup D such that $|D| < |P|$ and every subgroup $L \leqslant P$ of order $|D|$ has a p-nilpotent supplement in G.*

Proof. The necessity part is obvious and we omit the proof.

For the sufficiency part, let L be a subgroup of P of order $|D|$. Clearly, G is p-nilpotent if $L = 1$, so assume without loss that $L \neq 1$. By our hypothesis, G has a p-nilpotent subgroup M such that $G = LM$. It follows that $P = L(P \cap M)$ and $P \cap M$ is a Sylow p-subgroup of M. Since M is p-nilpotent, we have $M \leqslant N_G(K)$ where $K = O_{p'}(M)$ is also a Hall p'-subgroup of G.

Since $|L| < |P|$, there exists a maximal subgroup P_1 of P such that $L \leqslant P_1$. Clearly, $G = LM = P_1M = P_1N_G(K)$. Therefore, $N_P(K)$ is a Sylow p-subgroup of $N_G(K)$. If $|G : N_G(K)| = |P : N_P(K)| \geqslant p$, then we may let P_2 be a maximal subgroup of

P such that $N_P(K) \leqslant P_2$. Since P_2 contains a subgroup R of order $|D|$, by the above arguments once again, we obtain a p-nilpotent subgroup M_1 of G such that $G = RM_1 = P_2M_1$ and $M_1 \leqslant N_G(K_1)$, where K_1 is a Hall p'-subgroup of G and hence $G = P_2N_G(K_1)$. By hypothesis, there exists $g \in P$ such that ${K_1}^g = K$ and consequently $N_G(K_1)^g = N_G(K)$. Since P_2 is normal in P and $G = P_2N_G(K_1)$, we have $G = P_2N_G(K_1) = (P_2N_G(K_1))^g = P_2N_G(K)$. It follows that $P = P_2(P \cap N_G(K)) = P_2$, a contradiction. Thus $|G : N_G(K)| = |P : N_P(K)| = 1$ and hence $K \trianglelefteq G$. This means that G is p-nilpotent. □

Lemma 2.14. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let $H \trianglelefteq G$ with $G/H \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of H has a subgroup D such that $|D| < |P|$ and all subgroups $N \leqslant P$ of order $|D|$ have a supersolvable supplement in G. Then $G \in \mathscr{F}$.*

Proof. Assume that the assertion is false and choose G to be a counter-example of smallest order.

By our hypotheses and Lemma 2.13, H has a supersolvable type Sylow tower. So $P := O_p(H) \in \mathrm{Syl}_p(H)$ for the largest prime $p \in \pi(H)$. Clearly, $P \trianglelefteq G$ and so G has a minimal normal subgroup $L \leqslant P$ and L is elementary abelian.

Assume that P is cyclic. Clearly, $H/P \trianglelefteq G/P$ and $(G/P)/(H/P) \cong G/H \in \mathscr{F}$. Now we easily verify that the pair $(G/P, H/P)$ satisfies the hypotheses of the lemma. So the minimal choice of G implies that $G/P \in \mathscr{F}$. Now Lemma 2.3 yields $G \in \mathscr{F}$, a contradiction. We have shown that P is non-cyclic.

Claim 1. $|D| \geqslant |L|$.

If $|D| < |L|$, then we may choose a subgroup $L_1 < L$ with $|L_1| = |D|$. By our hypotheses, the subgroup L_1 has a supersolvable supplement K in G. Therefore $L = L \cap L_1K = L_1(L \cap K)$. Clearly $L \cap K$ is normal in G. Since L is a minimal normal subgroup of G, we have $L \cap K \in \{1, L\}$. If $L \cap K = 1$, then $L = L_1$, a contradiction. So we have $L \cap K = L$ and hence $K = G$ is supersolvable, a contradiction which proves Claim 1.

Claim 2. $G/L \in \mathscr{F}$.

Obviously, $H/L \trianglelefteq G/L$ and $(G/L)/(H/L) \cong G/H \in \mathscr{F}$. Let QL/L be a Sylow q-subgroup of H/L, where $Q \in \mathrm{Syl}_q(H)$. If $p = q$, we may assume that $|L| \leqslant |D|$ and hence every subgroup of P/L with order $|D|/|L|$ is of the form P_1/L with P_1 a subgroup of P containing L and having order $|D|$. By the hypotheses, P_1 has a supersolvable supplement in G. It follows from Lemma 2.1 that P_1/L has a supersolvable supplement in G/L. Now we assume that $p \neq q$. Let T/L be a q-subgroup of a Sylow q-subgroup of H/L. Then we have $T = LT_q$ and $|T/L| = |T_q|$. By hypotheses, there exists a subgroup E of every non-cyclic Sylow subgroup Q of H such that $|E| < |Q|$, and all subgroups B of Q of the same order as E have a supersolvable supplement in G. By Lemma 2.1, we see that BL/L has a supersolvable supplement in G/L. This shows that G/L satisfies the hypotheses. The minimal choice of G then implies $G/L \in \mathscr{F}$.

Now observe that L is the unique minimal normal subgroup of G contained in H, since $G \notin \mathscr{F}$. Moreover, by Lemma 2.3, $L \nleqslant \Phi(G)$. Then, by Lemma 2.4 and Claim 2, we obtain $F(H) = L$ and hence $C_H(L) = L = F(H) = P$. Now Claim 1 implies that $|D| \geqslant |L| = |P|$, contrary to the choice of D. This completes the proof. □

3 The main results

We make a definition in order to simplify notation.

Definition 3.1. Let X be a finite group, let $p \in \pi(X)$ and let P be a p-subgroup of X; then $SMS(P, X)$ denotes the set of all non-trivial proper subgroups D of P such that every subgroup $E \leqslant P$ of order $|D|$ has a supersolvable or an $\mathscr{M}$-supplement in X. Note that the set $SMS(P, X)$ might be empty.

With this we can now state the next result.

Theorem 3.2. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Suppose that $SMS(P, G)$ is non-empty for every non-cyclic Sylow subgroup P of H. Then $G \in \mathscr{F}$.*

Proof. Assume that the assertion is false and choose G to be a counter-example of minimal order.

By hypothesis, there exists a subgroup $D \in SMS(P, G)$ for a non-cyclic Sylow subgroup P of H. Let $E \leqslant P$ with $|E| = |D|$. If E has no supersolvable supplement in H, it also has no supersolvable supplement in G and hence, by hypothesis, is $\mathscr{M}$-supplemented in G; thus E is $\mathscr{M}$-supplemented in H by Lemma 2.1. Therefore H has a supersolvable type Sylow tower by Lemma 2.12. Let r be the largest prime divisor of $|H|$ and $R \in \mathrm{Syl}_r(H)$. Then $R = O_r(H) \trianglelefteq G$ and so R contains a minimal normal subgroup L of G.

Claim 1. $G/R \in \mathscr{F}$ and $R \nleqslant \Phi(G)$; furthermore, R is non-cyclic and there exists a subgroup $D \in SMS(R, G)$.

Clearly G/R satisfies the hypotheses of the theorem and hence $G/R \in \mathscr{F}$. Now Lemma 2.3 shows that R is non-cyclic and that $R \nleqslant \Phi(G)$. By hypothesis we also see that $SMS(R, G)$ is non-empty.

Claim 2. $R \cap \Phi(G) = 1$; in particular $R = R_1 \times R_2 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G.

Assume that $R \cap \Phi(G) \neq 1$, and hence without loss that $L \leqslant R \cap \Phi(G)$. If $|D| \leqslant |L|$, then there exists a subgroup $E \leqslant L$ such that $|E| = |D|$. By hypothesis E has a supplement S which is supersolvable or an $\mathscr{M}$-supplement. Since $E \leqslant \Phi(G)$, we get $G = ES = S$; in particular, S cannot be an $\mathscr{M}$-supplement. Hence $G = S \in \mathscr{U}$, a contradiction. So we have $|D| > |L|$. In fact, G/L satisfies the hypotheses of the theorem and hence $G/L \in \mathscr{F}$ by the minimal choice of G. Therefore $G \in \mathscr{F}$ since $\mathscr{F}$ is a saturated formation, a contradiction. So we have $R \cap \Phi(G) = 1$. Now Lemma 2.4 yields the remaining statement in Claim 2.

Suppose now that there exists a subgroup $T \leqslant R$ of order $|D|$ such that T has an $\mathscr{M}$-supplement B in G, i.e. $G = TB$ and $T_1B < G$ for any maximal subgroup T_1 of T. By Lemma 2.2 we have $|G : T_1B| = r$ for any maximal subgroup T_1 of T. As $G = TB = RB > T_1B$ and hence $R \nleqslant T_1$, there exists $j \in \{1, \dots, t\}$ such that $R_j \nleqslant T_1B$. As $G = R_j(T_1B)$ and thus $R_j \cap T_1B \trianglelefteq G$, we get $|R_j| = |G : T_1B| = r$.

Assume that $|D| = |R_j| = r$. By hypothesis, every minimal subgroup of R having no supersolvable supplement in G, is $\mathscr{M}$-supplemented in G. Therefore every minimal subgroup of R is complemented in G. Let $\langle x\rangle$ be a minimal subgroup of R_i for some $i \in \{1, \dots, t\}$. Then $\langle x\rangle$ is complemented in G, i.e. there exists a subgroup K of G such that $G = \langle x\rangle K$ and $\langle x\rangle \cap K = 1$. Note that $R_i = R_i \cap \langle x\rangle K = \langle x\rangle(R_i \cap K)$. Since R_i is minimal normal in G we have $R_i \cap K \in \{R_i, 1\}$. If $R_i \cap K = R_i$, then $G = K$, a contradiction. So $R_i \cap K = 1$, and $R_i = \langle x\rangle$. Thus R is the direct product of some minimal normal subgroups of G of order r.

Since $R \nleqslant \Phi(G)$, there exists a maximal subgroup M of G with $R \nleqslant M$. Thus G has at least one minimal normal subgroup R_l contained in R such that $R_l \nleqslant M$. Since $G = R_lM$ and $|R_l| = r$, we get $|G : M| = |R_l| = r$. Since $G/R \in \mathscr{F}$ by Claim 1, application of Lemma 2.8 to the pair (G, R) in place of (G, H) now yields $G \in \mathscr{F}$, a contradiction.

So we may assume $|D| > r$. In fact,

$$H/R_j \trianglelefteq G/R_j \quad \text{and} \quad (G/R_j)/(H/R_j) \cong G/H \in \mathscr{F}.$$

Let QR_j/R_j be a non-cyclic Sylow q-subgroup of H/R_j, where Q is a non-cyclic Sylow q-subgroup of H. If $q = r$, we know that every subgroup of P/R_j with order $|D|/|R_j| \neq 1$ is of the form S/R_j where S a subgroup of R containing R_j with order $|D|$ since $|R_j| < |D|$. If S/R_j has no supersolvable supplement in G/R_j, then S has no supersolvable supplement in G and hence S is $\mathscr{M}$-supplemented in G by hypothesis. It follows from Lemma 2.1 that S/R_j is $\mathscr{M}$-supplemented in G/R_j. Now we may assume that $q \neq r$. Let T/R_j be a q-subgroup of H/R_j. Then we have $T = R_jT_q$ and $|T/R_j| = |T_q|$. By hypothesis, there exists a subgroup $B \in SMS(Q, G)$; by Lemma 2.1 we see that if BR_j/R_j has no supersolvable supplement in G/R_j, then BR_j/R_j is $\mathscr{M}$-supplemented in G/R_j. This shows that G/R_j satisfies the hypotheses of the theorem. The minimal choice of G implies that $G/R_j \in \mathscr{F}$ and hence $G \in \mathscr{F}$ by Lemma 2.3, a contradiction.

We have shown that any subgroup $E \leqslant P$ of order $|D|$ has a supersolvable supplement in G. Since $G/R \in \mathscr{F}$ by Claim 1, we get $G \in \mathscr{F}$ by Lemma 2.14, a final contradiction. □

As an immediate consequence of Theorem 3.2 we obtain

Corollary 3.3. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ whenever one of the following conditions is satisfied.*

(1) *Every non-cyclic Sylow subgroup P of H has a proper non-trivial subgroup D such that every subgroup $E \leqslant P$ of order $|D|$ is $\mathscr{M}$-supplemented in G.*

(2) *Every non-cyclic Sylow subgroup P of H has a proper non-trivial subgroup D such that every subgroup* $E \leqslant P$ *of order* $|D|$ *has a supersolvable supplement in G.*

(3) *Every maximal subgroup of every non-cyclic Sylow subgroup of H has a supersolvable supplement or an* $\mathcal{M}$*-supplement in G.*

(4) *Every minimal subgroup of every non-cyclic Sylow subgroup of H has a supersolvable supplement or an* $\mathcal{M}$*-supplement in G.*

Theorem 3.4. *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$ *and let G be a group with a solvable normal subgroup H such that* $G/H \in \mathcal{F}$*. If* $SMS(P, G)$ *is non-empty for every non-cyclic Sylow subgroup P of* $F(H)$*, then* $G \in \mathcal{F}$*.*

Proof. Assume that the assertion is false and choose G to be a counter-example of minimal order. The proof is divided into two cases.

Case 1. Suppose that $\Phi(G) \cap H \neq 1$.

Since $\Phi(G) \cap H \neq 1$, there exists a minimal normal subgroup L of G contained in $\Phi(G) \cap H$. Clearly, $L \leqslant O_p(H)$ for some $p \in \pi(\Phi(G) \cap H)$. Note that $F(H/L) = F(H)/L$ by Lemma 2.7. If $O_p(H)$ is cyclic, then G/L satisfies the hypotheses of the theorem; therefore $G/L \in \mathcal{F}$ by the minimal choice of G. Now Lemma 2.3 implies $G \in \mathcal{F}$, a contradiction. We have shown that $O_p(H)$ is not cyclic. By hypothesis, there exists a subgroup $D \in SMS(O_p(H), G)$. If $|D| \leqslant |L|$, then there exists a subgroup $E \leqslant L$ of order $|D|$; furthermore, E has a supersolvable supplement or an $\mathcal{M}$-supplement S in G. Since $E \leqslant L \leqslant \Phi(G)$, we get $G = ES = S$. As $G = S$, S cannot be an $\mathcal{M}$-supplement of E in G. Therefore $G = S \in \mathcal{U} \subseteq \mathcal{F}$, a contradiction. Henceforth we may assume that $|D| > |L|$ and so consider the group G/L. Since $(G/L)/(H/L) \cong G/H \in \mathcal{F}$ and $F(H/L) = F(H)/L$, we see that G/L satisfies the hypotheses of the theorem. The minimal choice of G implies $G/L \in \mathcal{F}$. Now Lemma 2.3 yields $G \in \mathcal{F}$, a contradiction.

Case 2. Suppose that $\Phi(G) \cap H = 1$.

Note that $H \neq 1$; so $F(H) = F^*(H) \neq 1$, because H is solvable. Hence $F(H) \nleqslant \Phi(G)$ and thus there exists a maximal subgroup M of G not containing $F(H)$. In particular, there exists a prime $p \in \pi(H)$ with $O_p(H) \nleqslant M$.

Next, observe that $\Phi(O_p(H)) \leqslant \Phi(G) \cap H = 1$ and so $O_p(H)$ is an elementary abelian normal subgroup of G with $G = O_p(H)M$ and $R := O_p(H) \cap M \trianglelefteq G$. Since $G \notin \mathcal{F}$, Lemma 2.8 shows that $|G : M| = |O_p(H) : R| = p^n$ with $n \geqslant 2$; in particular, $p^2 \mid |O_p(H)|$ and $O_p(H)$ is not cyclic.

Now let M_p be a Sylow p-subgroup of M; then $G_p = O_p(H)M_p$ is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of G_p containing M_p and set $P_2 = P_1 \cap O_p(H)$. Clearly, $P_1 = P_2M_p$. Moreover, since $M_p \leqslant P_1 < G_p = O_p(H)M_p$, it follows that $O_p(H) \nleqslant P_1$ and thus P_2 is a maximal subgroup of $O_p(H)$. Since $O_p(H) \trianglelefteq G$, we get $O_p(H) \cap M_p = O_p(H) \cap M = R$ and hence

$$P_2 \cap M_p = P_1 \cap O_p(H) \cap M_p = O_p(H) \cap M_p = R.$$

If $(P_2)_G \nleqslant M$, then $G = (P_2)_G M = P_2 M = O_p(H)M$ and so

$$O_p(H) = P_2(O_p(H) \cap M) = P_2 R = P_2,$$

a contradiction. Therefore $(P_2)_G \leqslant M$. Now we get

$$R \leqslant (P_2)_G \leqslant O_p(H) \cap M = R \quad \text{and thus} \quad (P_2)_G = P_2 \cap M_p = O_p(H) \cap M = R.$$

Since $O_p(H)$ is a non-cyclic Sylow subgroup of $F(H)$ and P_2 is maximal in $O_p(H)$, there exists a subgroup E of P_2 with order $|D|$ such that E has a supersolvable supplement or an $\mathscr{M}$-supplement in G.

Assume next that N is a supersolvable supplement of E in G. Clearly, $G = EN = P_2 N$. Setting $K = (P_2)_G N = RN$ we have

$$G = P_2 N = P_2 K \quad \text{and} \quad K/R = NR/R \cong N/(N \cap R) \in \mathscr{U} \subseteq \mathscr{F}.$$

Suppose next that $K < G$ and let K_1 be a maximal subgroup of G containing K. As $G = P_2 K = P_2 K_1 = O_p(H)K_1$ and as $O_p(H)$ is abelian, $O_p(H) \cap K_1 \trianglelefteq G$; in particular, $(O_p(H) \cap K_1)M$ is a subgroup of G. Note that $K_1 \geqslant K \geqslant R$ and hence

$$O_p(H) \cap K_1 \cap M = R \cap K_1 = R = O_p(H) \cap M.$$

If $(O_p(H) \cap K_1)M = G = O_p(H)M$, we get

$$O_p(H) = (O_p(H) \cap K_1)(O_p(H) \cap M) = O_p(H) \cap K_1 \leqslant K_1$$

and thus $G = O_p(H)K_1 = K_1$, a contradiction. Therefore, by maximality of M in G, $(O_p(H) \cap K_1)M = M$ and hence $O_p(H) \cap K_1 \leqslant M$; in particular,

$$O_p(H) \cap K_1 = O_p(H) \cap K_1 \cap M = R \cap K_1 = R$$

and so

$$P_2 \cap K \leqslant O_p(H) \cap K \leqslant O_p(H) \cap K_1 = R \leqslant P_2 \cap K.$$

Thus $P_2 \cap K = O_p(H) \cap K = R$; but this contradicts $G = P_2 K = O_p(H)K$ and $|O_p(H) : P_2| = p$.

We have shown that $G = K = RN$. Clearly, $R \neq 1$, because $G \notin \mathscr{F}$. Moreover, as $G/R \cong N/(N \cap R) \in \mathscr{U}$ and M/R is maximal in $G/R = O_p(H)M/R$, we get $p^2 \leqslant p^n = |G : M| = |G/R : M/R| = p$, a contradiction.

We have shown that any subgroup of P_2 of order $|D|$ is $\mathscr{M}$-supplemented in G. So there exists a subgroup B of G such that $G = EB$ and $TB = BT < G$ for every maximal subgroup T of E. Now let E_1 denote a maximal subgroup of E and put $M_1 := E_1 B = BE_1$; so $M_1 < G$ and by Lemma 2.2 also $|G : M_1| = p$.

Since $\Phi(G) \cap H = 1$, Lemma 2.4 shows that $F(H)$ is the direct product of minimal normal subgroups of G contained in H. Therefore $O_p(H) = R_1 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G.

As $O_p(H) \nleqslant M$, we may assume that $R_1 \nleqslant M$. So $G = R_1 M$ and $R_1 \cap M \trianglelefteq G$. Hence $R_1 \cap M = 1$, and thus we have $|R_1| = |G : M| = |O_p(H) : R| = p^n \geqslant p^2$ and also $O_p(H) = R_1 \times R$.

If $R_1 \nleqslant M_1$, then $G = R_1 M_1$ and $R_1 \cap M_1 \trianglelefteq G$; this implies that $R_1 \cap M_1 = 1$ and $|R_1| = |G : M_1| = p$, a contradiction. Therefore $R_1 \leqslant M_1$.

As P_2 is maximal in $O_p(H)$ and $p^2 \mid |R_1|$ we have $Q := P_2 \cap R_1 \neq 1$. Assume now that $|D| \leqslant |Q|$. Then we may assume without loss that $E_1 < E \leqslant Q$. Hence

$$G = EB = EM_1 = QM_1 \leqslant R_1 M_1 = M_1,$$

a contradiction. We have shown that $|Q| < |D|$. So we may assume that $Q < E$. As $\Phi(O_p(H)) = 1$ and hence E is elementary abelian, we may choose E_1 such that $Q \nleqslant E_1$ and hence $E = E_1 Q = QE_1$. As $Q \leqslant R_1 \leqslant M_1$, we get $M_1 = QM_1 = QE_1 B = EB = G$, a contradiction finally proving the claim of the theorem. □

Corollary 3.5. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$.*

Then $G \in \mathscr{F}$ whenever one of the following conditions is satisfied.

(1) *Every non-cyclic Sylow subgroup P of $F(H)$ has a non-trivial proper subgroup D such that all subgroups $E \leqslant P$ of order $|D|$ are $\mathscr{M}$-supplemented in G.*

(2) *Every non-cyclic Sylow subgroup P of $F(H)$ has a non-trivial proper subgroup D such that every subgroup $E \leqslant P$ of order $|D|$ has a supersolvable supplement in G.*

(3) *Every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ has a supersolvable supplement or an $\mathscr{M}$-supplement in G.*

(4) *Every minimal subgroup of every non-cyclic Sylow subgroup of $F(H)$ has a supersolvable supplement or an $\mathscr{M}$-supplement in G.*

Theorem 3.6. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Suppose that $SMS(P, G)$ is non-empty for every non-cyclic Sylow subgroup P of $F^*(H)$. Then $G \in \mathscr{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counter-example of minimal order. In view of Theorem 3.4 we know that H and hence G is non-solvable. We consider the following two cases.

Case 1. Suppose that $\mathscr{F} = \mathscr{U}$.

Claim 1. $H = G$ and $F^*(G) = F(G) \neq 1$ is supersolvable.

Note that the pair $(F^*(H), F^*(H))$ satisfies the hypotheses of Theorem 3.2 in place of (G, H); hence $F^*(H)$ is supersolvable. In particular, $F^*(H)$ is solvable and hence

$F^*(H) = F(H) \neq 1$ by Lemma 2.6. Since H satisfies the hypotheses of the theorem, the minimal choice of G implies that H is supersolvable if $H < G$, but then G is supersolvable by Theorem 3.4, a contradiction.

Claim 2. If $F^*(G) \leqslant N \trianglelefteq G$ and $N \neq G$, then $N \in \mathscr{U}$.

By Lemma 2.6, $F^*(G) = F^*(F^*(G)) \leqslant F^*(N) \leqslant F^*(G)$, so $F^*(N) = F^*(G)$. By hypothesis $SMS(P, G)$ is non-empty for every non-cyclic Sylow subgroup P of $F^*(G)$. Applying Lemma 2.1 we see that $SMS(P, N)$ is non-empty as well. Therefore N satisfies the hypotheses of the theorem and so N is supersolvable by the minimal choice of G.

Claim 3. There exists a prime $p \in \pi(F(G))$ and a maximal subgroup $M < G$ such that $O_p(G)$ is non-cyclic and $G = O_p(G)M$; in particular, $\Phi(G) < F(G)$ and there exists a subgroup $D \in SMS(O_p(G), G)$.

If every Sylow subgroup of $F(G)$ is cyclic, then $F(G) = H_1 \times \cdots \times H_r$ where $\pi(F(G)) = \{p_1, \ldots, p_r\}$ and $H_i = O_{p_i}(G) \in \mathrm{Syl}_{p_i}(F(G))$ is cyclic for all $i \in \{1, \ldots, r\}$. Since $G/C_G(H_i)$ is abelian, also $G/\bigcap_{i=1}^{r} C_G(H_i) = G/C_G(F(G))$ is abelian and hence $G/F(G)$ is abelian since $C_G(F(G)) = C_G(F^*(G)) \leqslant F(G)$. Therefore G is solvable, a contradiction. Let P be a non-cyclic Sylow subgroup of $F(G)$. By hypothesis, there exists a subgroup $D \in SMS(P, G)$; moreover, D has a supplement S in G which is supersolvable or an $\mathscr{M}$-supplement. Suppose now that $P \leqslant \Phi(G)$ and thus $G = DS = S$. Then S cannot be an $\mathscr{M}$-supplement and so we get $G = S \in \mathscr{U}$, a contradiction. Therefore we have $P \nleqslant \Phi(G)$ and hence $\Phi(G) < F(G)$; in particular, there exists a maximal subgroup $M < G$ with $G = PM$. This proves Claim 3.

Claim 4. Any subgroup $E \leqslant O_p(G)$ of order $|D|$ is $\mathscr{M}$-supplemented in G.

Suppose that S is a supersolvable supplement of E in G. Hence $G = ES = O_p(G)S$. Since $O_p(G)$ is solvable and $G/O_p(G) \cong S/(S \cap O_p(G))$, we conclude that G is solvable, a contradiction. Therefore Claim 4 follows from the hypotheses of the theorem and Claim 3.

Claim 5. $O_p(G) \cap \Phi(G) = 1$; in particular $\Phi(O_p(G)) = 1$.

Assume that $O_p(G) \cap \Phi(G) \neq 1$. Then there exists a minimal normal subgroup L of G contained in $O_p(G) \cap \Phi(G)$ and L is an elementary abelian p-group.

If $|D| \leqslant |L|$, then we may choose a subgroup $E \leqslant L$ of order $|D|$. By Claim 4, E has an $\mathscr{M}$-supplement B in G. Since $L \leqslant \Phi(G)$, we get $G = EB = LB = B$, a contradiction.

So we may assume that $|D| > |L|$ and fix $E \leqslant O_p(G)$ with $L < E$ where $|E| = |D|$. By Claim 4, E is $\mathscr{M}$-supplemented in G, i.e. there exists a subgroup $B < G$ such that $G = EB$ and $E_iB < G$ for any maximal subgroup E_i of E. By Lemma 2.2, $|G : E_iB| = p$ and $E \cap B = E_i \cap B \leqslant \Phi(E) \leqslant \Phi(O_p(G))$. Since L is a minimal normal subgroup of G and E_iB is a maximal subgroup of G for any maximal subgroup E_i of E, we have $G = LE_iB$ or $L \leqslant E_iB$. If $G = LE_iB$ for some maximal subgroup E_i of E, we have $G = E_iB$ since L is contained in $O_p(G) \cap \Phi(G)$, a contradiction. Therefore $L \leqslant E_iB$ for any maximal subgroup E_i of E. Moreover, if $L \nleqslant E_i$ for some maximal subgroup E_i of E, then $E = LE_i$ and hence $E_iB = LE_iB = EB = G$, a contradiction. Therefore we have $L \leqslant E_i$ for any maximal subgroup E_i of E and hence $L \leqslant \Phi(E) \leqslant \Phi(O_p(G))$. In fact, G/L satisfies the hypotheses of the theorem by Lemma 2.7. The minimal choice of G implies that $G/L \in \mathscr{U}$. Since $L \leqslant \Phi(G)$ and since $\mathscr{U}$ is a saturated formation, Lemma 2.3 yields $G \in \mathscr{U}$, a contradiction proving Claim 5.

By Claim 5 and Lemma 2.4 we have $O_p(G) = R_1 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G. Let L be a minimal normal subgroup of G contained in $O_p(G)$.

If $|D| \leqslant |L|$, then we may choose a subgroup $E \leqslant L$ of order $|D|$. By Claim 4, E is $\mathscr{M}$-supplemented in G, i.e. there exists a subgroup $B < G$ such that $G = EB$ and $E_iB < G$ for any maximal subgroup E_i of E. By Lemma 2.2, we have $|G : E_iB| = p$ and $E \cap B = E_i \cap B \leqslant \Phi(E)$ for any maximal subgroup E_i of E. Since E_iB is a maximal subgroup of G, it follows from $G = EB = LB = LE_iB$ that $|G : E_iB| = |L : L \cap E_iB| = |L| = p$ and thus $|D| = p$. This together with Claim 4 shows that every minimal subgroup of $O_p(G)$ is $\mathscr{M}$-supplemented in G and hence is complemented in G. If $\langle x \rangle$ is a minimal subgroup of R_i, then $\langle x \rangle$ is complemented in G. So G has a subgroup K such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = 1$ as well as $R_i = R_i \cap \langle x \rangle K = \langle x \rangle (R_i \cap K)$. Since R_i is a minimal normal in G we have $R_i \cap K \in \{R_i, 1\}$. If $R_i \cap K = R_i$, then $G = K$, a contradiction. So $R_i \cap K = 1$, and then $R_i = \langle x \rangle$. Thus $O_p(G)$ is the direct product of some minimal normal subgroups of G of order p.

Now consider the subgroups $C_G(R_i)$ for $i \in \{1, \ldots, t\}$. Clearly,

$$F(G) \leqslant C_G(R_i) \trianglelefteq G.$$

If $C_G(R_i) < G$, then $C_G(R_i)$ is solvable by Claim 2. On the other hand, since $G/C_G(R_i)$ is abelian, G is solvable, a contradiction. So we may assume $C_G(R_i) = G$. Since $R_i \leqslant Z(G)$ for any minimal normal subgroup R_i in $O_p(G)$, we have $O_p(G) \leqslant Z(G)$. Then we consider the factor group $G/O_p(G)$. By Lemma 2.6, we have

$$F^*(G/O_p(G)) = F^*(G)/O_p(G) = F(G)/O_p(G).$$

In fact, $G/O_p(G)$ satisfies the hypotheses of the theorem. Therefore the minimal choice of G implies that $G/O_p(G) \in \mathscr{U}$ and hence G is supersolvable, a contradiction.

So we may assume that $|L| < |D|$ and we may choose a subgroup $E \leqslant O_p(G)$ of order $|D|$ such that $L < E$. By Claim 4, E is $\mathscr{M}$-supplemented in G, i.e. there exists a subgroup $B < G$ such that $G = EB$ and $E_iB < G$ for any maximal subgroup E_i of E. By Lemma 2.2, we have $|L| = p$ or $L \leqslant \Phi(E)$. If $L \leqslant \Phi(E)$, then $1 < L \leqslant \Phi(O_p(G))$ since $\Phi(E) \leqslant \Phi(O_p(G))$, a contradiction. Consequently, we know that $|L| = p$. Next we consider $C_G(L)$. Clearly, $F(G) \leqslant C_G(L) \trianglelefteq G$. If $C_G(L) < G$, then $C_G(L)$ is solvable by Claim 2; since $G/C_G(L)$ is cyclic, G is then solvable, a contradiction. So we may assume $C_G(L) = G$ and hence $L \leqslant Z(G)$. By

Lemma 2.6, $F^*(G/L) = F^*(G)/L$ and every subgroup E/L of a non-cyclic Sylow subgroup of $F^*(G/L)$ with order $|D/L|$ is $\mathscr{M}$-supplemented in G/L by Lemma 2.1. Therefore the factor group G/L satisfies the hypotheses of the theorem. The minimal choice of G implies that G/L is supersolvable and hence G is supersolvable since $L \leqslant Z(G)$, a contradiction.

Case 2. Suppose that $\mathscr{F} \neq \mathscr{U}$.

By Case 1, H is supersolvable. In particular, H is solvable and hence we have $F^*(H) = F(H)$. Therefore $G \in \mathscr{F}$ by Theorem 3.4. This is a final contradiction completing the proof. □

Corollary 3.7. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ whenever one of the following conditions is satisfied.*

(1) *Every non-cyclic Sylow subgroup P of $F^*(H)$ has a non-trivial proper subgroup D such that all subgroups $E \leqslant P$ of order $|D|$ are $\mathscr{M}$-supplemented in G.*

(2) *Every non-cyclic Sylow subgroup P of $F^*(H)$ has a non-trivial proper subgroup D such that all subgroups $E \leqslant P$ of order $|D|$ have a supersolvable supplement in G.*

(3) *Every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ has a supersolvable supplement or an $\mathscr{M}$-supplement in G.*

(4) *Every minimal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ has a supersolvable supplement or an $\mathscr{M}$-supplement in G.*

Remark. Our results depend on the classification of the finite simple groups inasmuch as we rely on Theorem 2.9.

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Long Miao, School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China
E-mail: miaolong714@vip.sohu.com

Wolfgang Lempken, Institute for Experimental Mathematics, University of Duisburg-Essen, 45326 Essen, Germany
E-mail: lempken@iem.uni-due.de