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Pseudo-tori and subtame groups of finite Morley rank

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1 Introduction

The *tameness hypothesis* has been very important in the study of the Cherlin–Zil'ber conjecture. Most of the deep theorems about the conjecture were initially proven under this hypothesis.

Definition 1.1. A *tame group* is a group G of finite Morley rank which does not interpret a *bad field* (i.e. a structure $\langle K, +, \cdot, T\rangle$ where $\langle K, +, \cdot\rangle$ is a field and T is an infinite proper subgroup of $K^{\times}$) and which does not have a definable *bad section* (see Definition 3.4).

The tameness hypothesis has now become too restrictive in view of the following.

Fact 1.2 (see [2]). There exists a bad field (of characteristic 0).

The main purpose of the tameness hypothesis was to provide a skeleton for the proofs of the most difficult theorems. However, this hypothesis immediately yields an involution in non-nilpotent connected groups (see [3, Theorem B.1, p. 353]), and so it cannot be a skeleton for a proof of a 'Feit–Thompson theorem' for groups of finite Morley rank, i.e. a proof that simple groups of finite Morley rank have involutions. A hypothesis that could also provide a skeleton for the proof of results such as a Feit–Thompson theorem would be very useful. The aim of this article is to propose such a hypothesis. Thus we introduce *subtame groups*, and we begin the study of these groups (see Theorems 3.8 and 4.3).

We preserve only the core of the tameness hypothesis and we show that it is sufficient to imply some well-known properties of tame groups. The center of the tame notion lies in the Zil'ber field theorem [3, Theorem 9.1] and in the presence of tori in all tame non-nilpotent connected groups. We weaken this property as far as we can to obtain the definition of a subtame group.

Definition 1.3. A group G of finite Morley rank is *subtame* if each non-nilpotent connected definable section of G has a minimal divisible infinite definable section that is not definably isomorphic to K_+ for any interpretable field K.

Example 1.4. Every definable section of a subtame group of finite Morley rank is subtame.

Every tame group of finite Morley rank is subtame; in particular, every algebraic group over a pure algebraically closed field K is subtame.

However, some non-tame groups are subtame. Indeed, any definable subgroup of an algebraic group over a (not necessarily pure) field of finite Morley rank and of non-zero characteristic is subtame (by [10, Théorème 2.8] and [12]).

Moreover, every locally finite group of finite Morley rank is subtame (see [11, Corollaire 3.32] and [3, Theorem 9.21]).

The main conjecture about subtame groups is the following. We show that this conjecture reduces to a question about bad fields and bad groups (Corollary 3.6).

Conjecture 1.5. Every group of finite Morley rank is subtame.

We will prove that subtame groups have some similar properties to those of tame groups; for example Jaligot's lemma extends (Theorem 4.3), as well as the conjugacy of *Carter subgroups* (Theorem 3.8), a *Carter subgroup* of a group G of finite Morley rank being a definable nilpotent subgroup C such that $N_G(C)^\circ = C$. Our study of the subtame groups is based on an analysis of *pseudo-tori*. The notion of a pseudo-torus is new and it generalizes the notion of a decent torus (see Definition 2.1).

Definition 1.6. An abelian divisible group T of finite Morley rank is a *pseudo-torus* if no definable quotient of T is definably isomorphic to K_+ for an interpretable field K.

Strictly speaking we should call T an $\mathscr{M}$-*pseudo-torus*, where $\mathscr{M}$ is the ambient structure, since the notion of a pseudo-torus depends on $\mathscr{M}$. However, we will only ever work inside one model, and so we can slightly simplify the terminology.

The main result concerning pseudo-tori is the following (see Section 2).

Theorem 1.7. *The maximal pseudo-tori in a group G of finite Morley rank are conjugate in G.*

2 Pseudo-tori

In this section we prove the conjugacy of maximal pseudo-tori (Theorem 1.7); we do not use the subtameness hypothesis. We use the notation of [3], which is also our basic reference.

2.1 Indecomposable subgroups. We need to consider the *indecomposable subgroups*. We recall that an abelian connected group A of finite Morley rank is *indecomposable* if it is not the sum of two proper definable subgroups; see [4]. If $A \neq 1$, then A has a unique maximal proper definable connected subgroup $J(A)$, and if $A = 1$ we define $J(1) = 1$.

First we recall the definition of some special pseudo-tori called *decent tori*. As in [3], for each subset X of a group G of finite Morley rank, the intersection of all definable subgroups containing X is denoted by $d(X)$.

Definition 2.1. A divisible abelian group G of finite Morley rank is said to be a *decent torus* if $G = d(T)$ where T is its (divisible) torsion subgroup, and G is said to be a *good torus* if each definable connected subgroup is a decent torus.

Lemma 2.2. *Let G be a group of finite Morley rank and H a definable normal subgroup of G. If $\bar{B}$ is a divisible indecomposable subgroup of G/H, then there is an indecomposable subgroup A of G such that $\bar{B} = AH/H$.*

Proof. By [4, Lemma 2.10], we may assume that $\bar{B}/J(\bar{B})$ has a non-trivial torsion part $\bar{T}$. Since $\bar{B}$ is divisible and indecomposable, we have $\bar{B} = d(\bar{T})$. So there exists a decent torus R of G such that $RH/H = \bar{B}$ by [6, Lemma 3.1]. We may assume R of minimal Morley rank.

We show that R is indecomposable. Let A_1 and A_2 be two connected definable subgroups of R such that $R = A_1A_2$. Since $\bar{B} = (A_1H/H)(A_2H/H)$ is indecomposable, we have $\bar{B} = A_iH/H$ for an index $i \in \{1,2\}$. Then $A_i/(A_i \cap H)$ is a decent torus and there exists a decent torus S_i such that $A_i = S_i(A_i \cap H)$ by [6, Lemma 3.1]. Hence we have $\bar{B} = S_iH/H$. By the minimality of R we obtain $R = S_i = A_i$, so R is indecomposable. □

2.2 Rigidity of the pseudo-tori. In any group G of finite Morley rank, we denote by $T(G)$ the subgroup generated by the indecomposable pseudo-tori of G.

Lemma 2.3. *Let G be an abelian group of finite Morley rank. Then G is a pseudo-torus if and only if $T(G) = G$.*

Proof. First we suppose that $T(G) \neq G$ and show that G is not a pseudo-torus. We may assume that G is divisible. Let M be a maximal proper definable connected subgroup of G containing $T(G)$. Then G/M is indecomposable and Lemma 2.2 shows that G has an indecomposable subgroup A such that $G = AM$. In particular A is not a peudo-torus and it is divisible, so it has a definable quotient A/B definably isomorphic to K_+ for an interpretable field K of characteristic zero. By the maximality of M in G and [11, Corollaire 3.3], we have $B = J(A) = M \cap A$; therefore $G/M \cong A/J(A)$ is definably isomorphic to K_+ and G is not a pseudo-torus.

Now we suppose that G is not a pseudo-torus, and show that $T(G) \neq G$. Since G is abelian, $T(G)$ is divisible, and we may assume that G is divisible. Therefore G has a definable subgroup H such that G/H is definably isomorphic to K_+ for an interpretable field K of characteristic zero. Thus G/H has no non-trivial proper definable subgroup, by [11, Corollaire 3.3]. Hence, if A is an indecomposable subgroup of G not contained in H, then $G = AH$. In particular $A/(A \cap H)$ is definably isomorphic to $G/H \cong K_+$, and A is not a pseudo-torus. This proves that $T(G)$ is contained in H, so $T(G) \neq G$. □

Corollary 2.4. *Let G be a group of finite Morley rank. Then $T(G)$ is the subgroup generated by its pseudo-tori.*

Lemma 2.5. *Let G be a group of finite Morley rank, and N be a normal definable subgroup of G. Then $T(G/N) = T(G)N/N$.*

Proof. By the definition of a pseudo-torus, a definable quotient of a pseudo-torus is a pseudo-torus, so Corollary 2.4 gives $T(G)N/N \leqslant T(G/N)$.

We show that $T(G)N/N$ contains $T(G/N)$. Let A/N be a non-trivial indecomposable pseudo-torus of G/N. By Lemma 2.2, there is an indecomposable subgroup B of G such that $A = BN$. Since $B/(B \cap N) \cong A/N$ is divisible, B is divisible. Hence, if B is not a pseudo-torus, then $B/J(B)$ is definably isomorphic to K_+ for an interpretable field K of characteristic zero; see [11, Corollaire 3.3]. In particular $B/J(B)$ is torsion-free and we obtain $B \cap N \leqslant J(B)$, so $A/J(B)N$ is definably isomorphic to $B/J(B) \cong K_+$, contradicting the choice of A/N. Thus B is a pseudo-torus and $A/N = BN/N \leqslant T(G)N/N$, proving that $T(G)N/N$ contains $T(G/N)$. □

Fact 2.6 (see [9]). Let A be an abelian torsion-free group of finite Morley rank. Suppose that A has an infinite uniformly definable family S of automorphisms such that A is S-minimal. Then there is a subgroup A_1 of A and a field K such that A_1 is definably isomorphic to K_+.

Proposition 2.7 (Rigidity). *Let G be a connected group of finite Morley rank acting by conjugation on a pseudo-torus T. Then G centralizes T.*

Proof. We consider a minimal counter-example $L = T \rtimes G$. We may assume that $C_G(T) = 1$. By minimality of L and by [3, Theorem 6.4], G is abelian. Let A be an L-minimal subgroup of T. Since T is a pseudo-torus, T/A is a pseudo-torus too and the minimality of L yields $[G, T] \leqslant A$. By L-minimality of A and since $C_G(T) = 1$, we obtain $[g, T] = A$ for each $g \in G \backslash \{1\}$. Now, for each $g \in G \backslash \{1\}$, the map $t \mapsto [g, t]$ is a surjective homomorphism from T to A, and so A is a pseudo-torus. Hence the Zil'ber field theorem [3, Theorem 9.1] shows that A is central in L, and therefore L is nilpotent.

Since T is divisible, the torsion part of L centralizes T by [3, Corollary 6.12]. Then, since $C_G(T) = 1$, G is torsion-free and, by the minimality of L, we have $C_T(g) = C_T(G)$ for each $g \in G \backslash \{1\}$. Thus each $g \in G \backslash \{1\}$ defines an isomorphism $f_g : \bar{t} \mapsto [g, t]$ from $T/C_T(G)$ to A, and

$$\mathscr{F} = \{f_g \circ f_h^{-1} \mid (g, h) \in G \times G, g \neq 1, h \neq 1\}$$

is an infinite uniformly definable family of automorphisms of A. Now Fact 2.6 says that A is definably isomorphic to K_+ for an interpretable field K, contradicting that A is a pseudo-torus. □

Corollary 2.8. *Let G be a nilpotent group of finite Morley rank. Then $T(G)$ is the unique maximal pseudo-torus of G.*

Proof. We prove this by induction on the Morley rank of G. We may assume that G is not a pseudo-torus. Let T be a maximal pseudo-torus of G, and H a maximal proper definable connected subgroup of G° containing T. By the induction hypothesis, we have $T = T(H)$ and T is normal in G°. By Proposition 2.7, G° centralizes T, and Lemma 2.3 shows that TA is a pseudo-torus for each indecomposable pseudo-torus A. Hence, by the maximality of T, we obtain $T = T(G)$. □

We recall that the *Fitting subgroup* $F(G)$ of a group G is the subgroup generated by all normal nilpotent subgroups of G. Nesin proved that in any group of finite Morley rank the Fitting subgroup is definable and nilpotent; see [3, Theorem 7.3].

Corollary 2.9. *In any connected group G of finite Morley rank, $T(F(G))$ is a pseudo-torus central in G.*

Corollary 2.10. *Let G be a group of finite Morley rank. Then each pseudo-torus of G is contained in a Carter subgroup of G.*

Proof. Let T be a maximal pseudo-torus of G. By [8], $N_G(T)/T$ has a Carter subgroup C/T. Then C is nilpotent by Proposition 2.7, and $N_G(C)$ normalizes T by Corollary 2.8. Hence C is a Carter subgroup of G. □

2.3 Conjugacy theorem. Wagner observed that because of Fact 2.6 the proof of [5, Nongenericity] works as soon as there is no field in the subgroup T. By using his argument and our notion of pseudo-tori, we can prove Theorem 2.11. Furthermore, we note that a minor change in [5, Nongenericity] yields a non-connected version of this result (see the fourth paragraph of our proof).

Theorem 2.11 (cf. [5, Nongenericity]). *Let G be a group of finite Morley rank and $\mathscr{F}$ be a uniformly definable family of subgroups of G. If $\bigcup \mathscr{F}$ is generic in G, then there exists $F \in \mathscr{F}$ containing $T(Z(G))$.*

Proof. We assume towards a contradiction that G is a counter-example of minimal Morley rank. In particular, $T(Z(G))$ is non-trivial and has a minimal non-trivial pseudo-torus T, which is indecomposable (by Lemma 2.3). Let

$$\mathscr{F}_T = \{F \in \mathscr{F} \mid T(Z(G/T)) \leqslant FT/T\}.$$

Since $\bigcup \mathcal{F}$ is generic in G, either $\bigcup(\mathcal{F} \setminus \mathcal{F}_T)$ or $\bigcup \mathcal{F}_T$ is generic in G. In the first case, $(\bigcup(\mathcal{F} \setminus \mathcal{F}_T))T/T$ is generic in G/T and, by the minimality of $\mathrm{rk}(G)$, there exists $F \in \mathcal{F} \setminus \mathcal{F}_T$ such that FT/T contains $T(Z(G/T))$, contradicting the choice of $\mathcal{F}_T$. Hence $\bigcup \mathcal{F}_T$ is generic in G, and we may assume that $\mathcal{F} = \mathcal{F}_T$. Thus FT contains $T(Z(G))$ for each $F \in \mathcal{F}$ by Lemma 2.5, and no $F \in \mathcal{F}$ contains T.

Since T is indecomposable, we have $(F \cap T)^\circ \leqslant J(T)$ for each $F \in \mathcal{F}$ and, since Lemma 2.5 gives $T(Z(G/J(T))) \geqslant T(Z(G))/J(T)$, the minimality of $\mathrm{rk}(G)$ yields $J(T) = 1$. Consequently T is a G-minimal subgroup and $F \cap T$ is finite for each $F \in \mathcal{F}$. Now either T is torsion-free and $F \cap T = 1$ for each $F \in \mathcal{F}$, or T is a good torus. In the second case, each uniformly definable family of subgroups of T is finite, by [5, Rigidity II], so we may assume that $F \cap T = F_T$ for a fixed finite subgroup F_T of T and for each $F \in \mathcal{F}$. Hence, by replacing G by G/F_T, we may assume that $F \cap T = 1$ for each $F \in \mathcal{F}$.

Since $\bigcup \mathcal{F}$ is generic in G, there is a coset V of T in G such that $V \cap (\bigcup \mathcal{F})$ is generic in V, in other words $\mathcal{V} = \{g_0 \in \bigcup \mathcal{F} \mid g_0 T \cap (\bigcup \mathcal{F}) \text{ is generic in } g_0 T\}$ is non-empty. We fix $g \in \mathcal{V}$ such that g minimizes the Morley rank and the Morley degree of $d(g)$. Then, for every $gt \in gT \cap (\bigcup \mathcal{F})$, we have $d(gt) \leqslant d(g) \times T$ and $d(gt) \cap T = 1$, so $d(g) \times T = d(gt) \times T$. Let B be a connected definable subgroup of $d(g) \times T$ maximal among the ones contained in infinitely many subgroups $d(gt)$ for $gt \in gT \cap (\bigcup \mathcal{F})$. We may assume that $B \leqslant d(g)$.

Suppose that B has finite index in $d(g)$. Let $\mathcal{F}_0 = \{F \in \mathcal{F} \mid B \leqslant F, g \in FT\}$. Then $F/B \cap (d(g)T/B)$ is finite of order $|d(g)/B|$ for each $F \in \mathcal{F}_0$. By the choice of B, we obtain an infinite subgroup of bounded exponent in $d(g)T/B$, and so T has an infinite subgroup of bounded exponent, contradicting that T is an indecomposable pseudo-torus. Hence B has infinite index in $d(g)$.

Let A/B be a minimal infinite definable subgroup of $d(g)/B$. In particular A/B is abelian and connected. We consider the uniformly definable family

$$\mathcal{G} = \{(F \cap AT)/B \mid F \in \mathcal{F}, B \leqslant F, g \in FT\} \setminus \{A/B\}$$

of subgroups of AT/B. By the choice of A and B, $\mathcal{G}$ is infinite. We note that each element of $\mathcal{G}$ is the graph of a surjective homomorphism from A/B to T with a finite kernel. If T is a good torus, then every uniformly definable family of homomorphisms from A/B to T is finite, by [5, Rigidity III], contradicting that $\mathcal{G}$ is infinite. Hence T is not a good torus and, by minimality of T, it is torsion-free. Then A/B is torsion-free too, and $\mathcal{G}$ yields an infinite uniformly definable family of isomorphisms from A/B to T. Thus T is definably isomorphic to K_+ for an interpretable field K, by Fact 2.6, contradicting our choice of T. □

We say that a definable subset of a group G of finite Morley rank is *generous* if its conjugates cover G generically.

The connectedness of the centralizers of decent tori has very recently been proven by Altınel and Burdges [1, Theorem 1]. The following result generalizes their result to pseudo-tori, with a very different proof.

Corollary 2.12 (cf. [1, Theorem 1]). *Let T be a pseudo-torus of a connected group G of finite Morley rank. Then $C_G(T)$ is connected and generous in G, and $N_G(C_G(T))^\circ = C_G(T)$.*

Proof. Let $H = C_G(T)$. By Corollary 2.4, we have $T \leqslant T(F(H))$ and $T(F(H))$ is normal in $N_G(H)$. Therefore Corollary 2.9 gives

$$N_G(H)^\circ \leqslant C_G(T(F(H)))^\circ \leqslant C_G(T)^\circ \leqslant H.$$

We show that H° is generous in G. We may suppose without loss of generality that T is a maximal pseudo-torus of G; in particular we have $T = T(F(H))$. If T is contained in H^g for $g \in G$, then TT^g is a pseudo-torus containing T by Corollary 2.9, so $T = T^g$ by the maximality of T and we obtain $g \in N_G(T) = N_G(H)$. This shows that no element of $\mathscr{F} = \{(H^\circ) \cap (H^\circ)^g \mid g \in G \backslash N_G(H)\}$ contains T. So $\bigcup \mathscr{F}$ is not generic in H° by Theorem 2.11, and

$$\mathrm{rk}\left(\bigcup_{g \in G} (H^\circ)^g\right) = \mathrm{rk}(H) + \mathrm{rk}(G/N_G(H)) = \mathrm{rk}(G).$$

Now we prove that H is connected. Assume that H is not connected and that, for every pseudo-torus $T_0 > T$, the subgroup $C_G(T_0)$ is connected; in particular we have $T = T(Z(H))$. Let $h \in H \backslash H^\circ$. If the subset

$$hH^\circ \cap \left(\bigcup_{g \notin N_G(H)} H^g\right)$$

is generic in hH°, then Theorem 2.11 shows that there exists $g \notin N_G(H) = N_G(T)$ such that H^g contains T and $hH^\circ \cap H^g \neq \varnothing$. In particular TT^g is a pseudo-torus such that $T < TT^g$, so $C_G(TT^g) = H \cap H^g$ is connected. Thus we have $H \cap H^g \leqslant H^\circ$, contradicting $hH^\circ \cap H^g \neq \varnothing$. Hence $hH^\circ \cap (\bigcup_{g \notin N_G(H)} H^g)$ is not generic in hH° and we obtain

$$\mathrm{rk}\left(\bigcup_{g \in G} (hH^\circ)^g\right) = \mathrm{rk}(H) + \mathrm{rk}(G/N_G(H)) = \mathrm{rk}(G).$$

Consequently, by the generosity of H°, the subset $hH^\circ \cap (\bigcup_{g \in G} (H^\circ)^g)$ is generic in hH°, contradicting the non-genericity of $hH^\circ \cap (\bigcup_{g \notin N_G(H)} H^g)$ in hH°. $\square$

We now obtain Theorem 1.7 exactly as [5, Conjugacy Theorem]. Moreover, we have the following result as a corollary of Theorem 1.7.

Corollary 2.13. *Let G be a group of finite Morley rank, N a normal definable subgroup of G and T a maximal pseudo-torus of G. Then TN/N is a maximal pseudo-torus of G/N and every maximal pseudo-torus of G/N has this form.*

Proof. By Theorem 1.7, we just have to prove that TN/N is a maximal pseudo-torus of G/N. Let R/N be a maximal pseudo-torus of G/N containing TN/N. By the conjugacy of maximal pseudo-tori in R (Theorem 1.7), we have $T(R) \leqslant TN$, and Lemma 2.5 gives $R/N = T(R/N) = T(R)N/N \leqslant TN/N$. $\square$

3 Subtame groups

We show that the subtame groups are connected to *loosely exponential fields* (Definition 3.1 and Corollary 3.6), and we prove the conjugacy of the Carter subgroups in subtame groups (Theorem 3.8). The following notion is similar to the notion of a bad field.

Definition 3.1. A structure $\mathscr{M} = \langle K, +, \cdot, *, L \rangle$ of finite Morley rank is a *loosely exponential field* if $\langle K, +, \cdot, L \rangle$ is a bad field and if $\langle L, \cdot, * \rangle$ is a field.

Conjecture 3.2. There exists no loosely exponential field.

Lemma 3.3. *Let G be a connected subtame group of finite Morley rank. If G has no non-trivial pseudo-torus, then G is nilpotent.*

Proof. If G is non-nilpotent, then it has a minimal divisible infinite definable section U/V not definably isomorphic to K_+ for any interpretable field K. Now U/V is a non-trivial pseudo-torus, which contradicts Corollary 2.13. $\square$

We recall the definition of a *bad group*, a hypothetical counter-example to the Cherlin–Zil'ber conjecture.

Definition 3.4. A *bad group* is a non-solvable connected group G of finite Morley rank all of whose proper definable connected subgroups are nilpotent.

Proposition 3.5. *Let G be a group of finite Morley rank. Suppose that G interprets no loosely exponential field and that each bad section of G has a non-trivial pseudo-torus. Then G is subtame.*

Proof. We assume towards a contradiction that G is a counter-example of minimal Morley rank and Morley degree. Then G is connected, non-nilpotent, and has a non-nilpotent connected definable section U/V such that every minimal divisible infinite definable section of U/V is definably isomorphic to K_+ for an interpretable field K. By the minimality of $\mathrm{rk}(G)$, we have $U = G$ and V is finite. If G has a non-trivial pseudo-torus T, then T has a maximal proper connected definable subgroup H, and $(TV/V)/(HV/V)$ is a minimal divisible infinite definable section of U/V. Hence $T/(T \cap HV)$ is definably isomorphic to K_+ for an interpretable field K, contradicting that T is a pseudo-torus. Thus G has no non-trivial pseudo-torus, and in particular G is not a bad group.

Let $\overline{H}$ be a connected definable section of G such that $\mathrm{rk}(\overline{H}) < \mathrm{rk}(G)$. Then the previous paragraph and Lemma 3.3 show that $\overline{H}$ is nilpotent. In particular, since G is not a bad group, G is solvable.

Let A be a G-minimal subgroup of G. By the previous paragraph, G/A is nilpotent, and so A is not central. If M is a maximal proper connected definable subgroup containing A, then M nilpotent and, by G-minimality of A, we have $M \leqslant C_G(A)$. Hence $G/C_G(A)$ is a minimal infinite definable group. We consider $A \rtimes G/C_G(A)$ where $G/C_G(A)$ acts by conjugation on A. By the Zil'ber field theorem [3, Theorem 9.1], there is an interpretable field K such that $G/C_G(A)$ is definably isomorphic to a subgroup L of $K^{\times}$. Thus $G/C_G(A)$ is divisible and, since G has no non-trivial pseudo-torus, $G/C_G(A)$ is not a pseudo-torus (by Corollary 2.13). So, by the minimality of $G/C_G(A)$, it is definably isomorphic to F_+ for an interpretable field F of characteristic zero.

Thus there is a definable isomorphism f from $L \cong G/C_G(A)$ to F_+. For each $(x, y) \in K \times K$, we define $x * y$ by $x * y = 0$ if $(x, y) \notin L \times L$ and by $x * y = f^{-1}(f(x) \cdot f(y))$ if $(x, y) \in L \times L$. Now the structure $\langle K, +, \cdot, *, L\rangle$ is a loosely exponential field, contradicting our hypothesis on G. □

Corollary 3.6. *All groups of finite Morley rank are subtame if and only if there exists no loosely exponential field and each bad group has a non-trivial pseudo-torus.*

We can now prove the conjugacy and the generosity of Carter subgroups in subtame groups of finite Morley rank.

Lemma 3.7. *Let G be a connected subtame group of finite Morley rank. Then $C_G(T)$ is a generous Carter subgroup of G for each maximal pseudo-torus T of G.*

Proof. Let T be a maximal pseudo-torus of G. We obtain the nilpotence of $C_G(T)^{\circ}/T$ by Corollary 2.13 and Lemma 3.3, so $C_G(T)^{\circ}$ is nilpotent Consequently, $C_G(T)$ is a generous Carter subgroup of G (Corollary 2.12). □

Theorem 3.8. *In any subtame group of finite Morley rank, Carter subgroups are conjugate and generous.*

Proof. This is similar to the proof of [6, Theorem 1.1]. □

Corollary 3.9. *In any connected subtame group G of finite Morley rank, Carter subgroups are maximal nilpotent subgroups.*

Proof. Let C be a Carter subgroup contained in a nilpotent subgroup N. By Lemma 3.7 and Theorem 3.8, we have $C = C_G(T)$ for a pseudo-torus T. But C has finite index in N and each element of finite order of N centralizes T by [3, Corollary 6.12], so $N \leqslant C_G(T) = C$. □

4 A subtame version of Jaligot's lemma

We consider the *subtame* minimal connected simple groups of finite Morley rank, and we generalize Jaligot's lemma from tame groups to subtame groups.

Proposition 4.1. *If G is a solvable connected group G of finite Morley rank, then G' has no non-trivial pseudo-torus.*

Proof. By Corollary 2.4, we have just to prove that $T(G') = 1$. First we show that, if G is nilpotent, then $T(G') = 1$. We proceed by induction on $\mathrm{rk}(G)$. We may assume that G is not abelian. We fix $g \in Z_2(G)\backslash Z(G)$, and we consider the map γ from G to $Z(G)$ defined by $\gamma(x) = [g, x]$. Then γ is a definable homomorphism and $\gamma(G)$ is an infinite normal subgroup of G definably isomorphic to $G/C_G(g)$. But g centralizes $T(G)$ by Corollary 2.9, and so $T(\gamma(G)) \cong T(G/C_G(g))$ is trivial by Lemma 2.5. Since the induction hypothesis yields $T((G/\gamma(G))') = 1$, Lemma 2.5 shows that $T(G') \leqslant T(\gamma(G)) = 1$.

Let $\overline{T}$ be a maximal pseudo-torus of $\overline{G} = G/G''$, and $\overline{C}$ be a Carter subgroup of $\overline{G}$ containing $\overline{T}$ (cf. Corollary 2.10). Since $\overline{G}$ is 2-solvable, $\overline{G}$ has a definable subgroup $\bar{A} = A/G''$ containing G'/G'' and such that $\overline{G} = \bar{A} \rtimes \overline{C}$ by [7, Corollaire 7.7]. The previous paragraph shows that $T((G/A)') = 1$, so Lemma 2.5 gives $T(G') = T(A)$. By the conjugacy of the maximal pseudo-tori in $\overline{G}$ (Theorem 1.7) and since $\bar{A} \cap \overline{T} = 1$, $\bar{A}$ has no non-trivial pseudo-torus, so that $T(\bar{A}) = 1$ and Lemma 2.5 yields $T(A) = T(G'')$. But G' is nilpotent by [3, Theorem 9.21], and the previous paragraph gives $T(G'') = 1$, hence $T(G') = 1$. □

Corollary 4.2. *Every connected solvable group G of finite Morley rank has a unique definable connected subgroup, denoted by $P(G)$, maximal among the ones without a non-trivial pseudo-torus.*

Proof. Let H be a definable connected subgroup containing G', and maximal among the ones such that $T(H) = 1$. Let A be a definable connected subgroup of G such that $T(A) = 1$. By Lemma 2.5, we have

$$T(AH)H/H = T(AH/H) \cong T(A/(A \cap H)) = 1 \quad \text{and} \quad T(AH) = T(H) = 1.$$

Hence H contains A, and we obtain the conclusion by Corollary 2.4. □

Theorem 4.3 (Jaligot's lemma, subtame version). *Let G be a subtame minimal connected simple group of finite Morley rank. Let B_1 and B_2 be two distinct Borel subgroups of G. Then $F(B_1) \cap F(B_2) = 1$.*

Proof. Let H be a proper definable connected subgroup of G containing no non-trivial pseudo-torus. We show that H is contained in a unique Borel subgroup. We may assume H maximal among the connected definable subgroups of G contained in two distinct Borel subgroups E_1 and E_2, and containing no non-trivial pseudo-torus.

Let $N_i = N_{P(E_i)}(H)^\circ$ for $i = 1, 2$. Since $P(E_1)$ is nilpotent by Lemma 3.3 and contains $H = P(H)$, we have either $H = P(E_1)$ or $N_1 > H$. In the first case, we have $E_1 = N_G(H)^\circ$, and N_2 is contained in $P(E_1) = H$, hence $H = N_2$ and $H = P(E_2)$. This gives $E_2 = N_G(H)^\circ = E_1$, contradicting $E_1 \neq E_2$. Consequently, $N_1 > H$ and E_1 is the unique Borel subgroup containing N_1 and $N_G(H)^\circ$. But, in the same way, E_2 is the unique Borel subgroup containing $N_G(H)^\circ$, contradicting $E_1 \neq E_2$.

We suppose towards a contradiction that there exists $f \in (F(B_1) \cap F(B_2)) \setminus \{1\}$. For $i = 1, 2$, if $P(B_i) \neq 1$, then $C_{P(B_i)}(f)^\circ$ is infinite and, by the previous paragraph, B_i is the unique Borel subgroup containing $C_G(f)^\circ$. Thus we may assume that $P(B_1) = 1$ and B_1 is abelian by Proposition 4.1; in particular $C_G(f)^\circ = B_1$ and $C_G(f)^\circ \not\leq B_2$. Consequently $P(B_2) = 1$ too and, in the same way, $C_G(f)^\circ = B_2$, contradicting $B_1 \neq B_2$. $\square$

References

[1] T. Altınel and J. Burdges. On analogies between algebraic groups and groups of finite Morley rank. *J. London Math. Soc.*, to appear.

[2] A. Baudisch, M. Hils, A. Martin-Pizarro and F. O. Wagner. Die böse Farbe. *J. Inst. Math. Jussieu*, to appear.

[3] A. V. Borovik and A. Nesin. *Groups of finite Morley rank* (Oxford University Press, 1994).

[4] J. Burdges. A signalizer functor theorem for groups of finite Morley rank. *J. Algebra* **274** (2003), 215–229.

[5] G. Cherlin. Good tori in groups of finite Morley rank. *J. Group Theory* **8** (2005), 613–622.

[6] O. Frécon. Carter subgroups in tame groups of finite Morley rank. *J. Group Theory* **9** (2006), 361–367.

[7] O. Frécon. Sous-groupes anormaux dans les groupes de rang de Morley fini résolubles. *J. Algebra* **229** (2000), 118–152.

[8] O. Frécon and E. Jaligot. The existence of Carter subgroups in groups of finite Morley rank. *J. Group Theory* **8** (2005), 623–633.

[9] J. Loveys and F. O. Wagner. Le Canada semi-dry. *Proc. Amer. Math. Soc.* **118** (1993), 217–221.

[10] Y. Mustafin. Structure des groupes linéaires définissables dans un corps de rang de Morley fini. *J. Algebra* **281** (2004), 753–773.

[11] B. Poizat. *Groupes stables* (Nur Al-mantiq Wal-Ma'rifah, 1987).

[12] F. O. Wagner. Fields of finite Morley rank. *J. Symbolic Logic* **66** (2001), 703–706.

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