DYNAMICS OF COLLAPSE OF A CONFINED BOSE GAS

L.P. Pitaevskii^{1,2}

¹*Department of physics, Technion, 32000 Haifa, Israel*

²*Kapitza Institute for Physical Problems, 117454 Moscow, Russia*

Abstract

Rigorous results on the nonlinear dynamics of a dilute Bose gas in an harmonic magnetic trap are presented and sufficient conditions for the collapse of the system are formulated. By using the virial theorem for the Gross-Pitaevskii equation in an external field we analyze the temporal evolution of the mean square radius $I = \int r^2 |\psi|^2 dV$ of the gas cloud. In the 2D case this quantity undergoes harmonic oscillations with frequency $2\omega_0$. For a negative scattering length and for a negative value of the energy of the system, the gas cloud will collapse after a finite time interval. In the 3D case the system also collapses after a finite time for a state with negative energy. A stringent condition for the collapse is also derived.

1. Extended Summary

In 1925 A. Einstein published a paper devoted to the generalization of Bose's work on the statistical mechanics of photons to an ideal gas of atoms. In this paper he predicted a remarkable phenomenon - the condensation of the atoms in a single quantum state. Such a condensation takes place below some critical temperature, which depends on the density of the gas. Below this temperature the number of the atoms N_0 , which occupy the lowest quantum state $\mathbf{p} = 0$, is proportional to the total number of atoms N , i.e. is macroscopically large:

$$N_0 = N[1 - (T/T_c)^{3/2}]. \quad (1)$$

The remaining atoms are distributed in the momentum space according to the usual Bose distribution with chemical potential $\mu = 0$. The recent observation of Bose-Einstein condensation of atoms of ^{87}Rb [1], ^{23}Na [2] and ^7Li [3] confined in magnetic traps has opened a new important field of investigation of quantum phenomena. In particular, a new feature of these systems is that they are significantly inhomogeneous so one can directly observe effects of quantum uncertainty on a macroscopical scale. The formal evidence of such a situation is that dynamics of these non-uniform gases at low temperatures can be properly described by a non-linear equation for the "condensate wave function" $\psi(\mathbf{r}, t)$ [4], [5]:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + U(\mathbf{r}) + g |\psi(\mathbf{r}, t)|^2 \right) \psi(\mathbf{r}, t). \quad (2)$$

Here U is the confining potential and $g = \frac{4\pi\hbar^2 a}{m}$, where a is the s-wave scattering length. This equation is quite classical in its merit but contains explicitly the Plank's constant $\hbar$. (We will put $\hbar = m = 1$ below.)

The physical situation is very different for ^{87}Rb , ^{23}Na on one side and ^7Li on the other. In the first case $a > 0$ and the homogeneous state of the gas is stable. For ^7Li the scattering length a is negative, which corresponds to an effective attraction, and the gas possesses negative compressibility. The uniform state in this case is unstable and the gas can undergo a "collapse" which is an unrestricted contraction. Fetter however has recently shown [7], using a variational approach, first employed by Baym and Pethick [6], that a system of a finite number N of atoms in a trap can be in a metastable state for N below some critical number N_c .

In this paper we will present some rigorous results relative to the behaviour of the solution of Eq.

(2). [8] We will analyze the temporal evolution of the quantity $I = \int r^2 |\psi|^2 dV$, which is proportional to the average square of the radius of the gas cloud.

Repeating differentiation of I with respect to t , excluding $\partial\psi/\partial t$ from (2) and integrating by parts one gets for the two-dimensional motion of the gas in an isotropic cylindrical trap with a confining potential $U = \frac{\omega_0^2 r^2}{2}$, $r^2 = x^2 + y^2$:

$$\frac{d^2 I}{dt^2} = -4\omega_0^2 I + 4E, \quad I = A \cos(2\omega_0 t + \gamma) + E/\omega_0^2, \quad (3)$$

where E is the total energy of the gas. Eq. (3) describes harmonic pulsations of I with constant frequency $2\omega_0$ in spite of the nonlinear character of Eq. (2). Furthermore, one can see that for a negative value of the energy the positive quantity I approaches zero after a finite time smaller than $\pi/2\omega_0$. This means complete contraction of the cloud.

Results for the 3D case are not so conclusive. Consider an anisotropic harmonic trap with a confining potential of the form $U = \frac{\alpha_{ik} x_i x_k}{2}$. One gets:

$$\frac{d^2 I}{dt^2} = 6E - 5 \int \alpha_{ik} x_i x_k dV - 2K, \quad (4)$$

where $K = \frac{1}{2} \int |\nabla \psi|^2 dV$ is the kinetic energy. Eq. (4) gives the inequality:

$$\frac{d^2 I}{dt^2} + 5\omega_m^2 I - 6E < 0, \quad (5)$$

where ω_m^2 is the smallest eigenvalue of the α_{ik} matrix. We can prove again that the system collapses after a finite time for a negative E .

One can get more rigorous sufficient conditions of the collapse using the method developed in [9]-[10]. In this method an important part is played by two inequalities imposed on the kinetic energy:

$$IK \geq \frac{9}{8} N^2, \quad \frac{1}{2} \int |\psi|^4 dV \leq \beta N^{1/2} K^{3/2}, \quad \beta=0.0575 \quad (6)$$

Combining the inequalities, one immediately gets an inequality for the energy E as a function of K :

$$E(K) > K + \frac{9}{16} \omega_m^2 N^2 K^{-1} - \beta |g| K^{3/2} N^{1/2}. \quad (7)$$

Introducing dimensionless variables ε and κ defined by $E = \frac{3}{4} N \omega_m \varepsilon$, $K = \frac{3}{4} N \omega_m \kappa$, we can rewrite this equation as:

$$\varepsilon > \kappa + \kappa^{-1} - 1.18 \frac{2}{3} \sigma \equiv f(\kappa, \sigma), \quad (8)$$

where $\sigma = \sqrt{2/\pi} N |a| \omega_m^{1/2}$.

Behaviour of the function $f(\kappa)$ depends on the value of σ . For $\sigma \geq \sigma_c = 0.454$ the curve is a monotonous one. Let the point ε^* , κ^* to be an intersection of the curve $\varepsilon = f(\kappa)$ and the straight line $\varepsilon = \kappa/3$. Let us be

$$E < E^* = \frac{3}{4} N \omega_m \varepsilon^* > 0. \quad (9)$$

Then the kinetic energy $K > K^* = 3E^*$ for any t . This means that equation (4) leads to the inequality $\frac{d^2 I}{dt^2} + 5\omega_m^2 I - 6(E - E^*) < 0$. This formula obviously demonstrates that at condition (9) we get the collapse.

If $\sigma < \sigma_c$, the curve $\varepsilon = f(\kappa)$ has this case a minimum ε_m and a maximum ε_M . Let us consider an initial state, where

$$\varepsilon_m < \varepsilon < \varepsilon_M, \quad \kappa_1 < \kappa < \kappa_2. \quad (10)$$

Since any solution of (2) evolves continuously, the kinetic energy κ will stay in the same range at every time. The uncertainty relation (6) gives then that $I > \frac{3N}{2\omega_m \kappa_2}$. Thus the mean square radius stays restricted and collapse of the cloud as a whole is impossible.

If $E < E^*$ the cloud will collapse as in the previous case. The evolution of the cloud for $E > E^*$ and K out the interval (10) demands additional investigation.

I believe that suggested method can be useful for investigation of developing instabilities in different unstable media.

References

- [1] M.N. Anderson, J.R. Ensher, M. R. Matthews, C.E. Wieman, and E.A. Cornell, *Science* **269**, 198 (1995).
- [2] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. Van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterly, *Phys. Lett.* **75**, 3969 (1995).
- [3] C.C. Bradley, C.A. Sackett, and R.G. Hulet, *Phys. Rev. Lett.* **78**, 985 (1997).
- [4] E.P Gross, *Nuovo Cimento* **20**, 454 (1961).
- [5] L.P. Pitaevskii, *Sov. Phys. JETP* **13**, 451 (1961).
- [6] G. Baym and C. Pethick, *Phys. Rev. Lett.* **76**, 6 (1996).
- [7] A. Fetter, *Journ. Low Temp. Phys.* **106**, 643 (1997).
- [8] L.P. Pitaevskii *Physics Letters A* **221**, 14 (1996).
- [9] E.A. Kuznetov, A.M. Rubenchik, and V.E. Zakharov, *Phys. Rep.*, **142**, 103 (1986).
- [10] S.K. Turitsyn, *Phys. Rev. E* **47**, R13 (1993).