Physical Parameter Identification of a Viscoelastic Chain from Time Response Data

Y.M. Ram* and G.Z. Voyiadjis**

**Department of Mechanical Engineering
Louisiana State University
Baton Rouge, LA 70803, USA
ram@me.lsu.edu*

***Boyd Professor
Department of Civil and Environmental Engineering
Louisiana State University
Baton Rouge, LA 70803, USA
voyiadjis@eng.lsu.edu*

ABSTRACT

It is shown that an n degree-of-freedom chain of springs and viscous dampers, which is subject to a constant unit load, may be reconstructed uniquely from $2n$ time dependent displacement data at the chain's end. This may effectively in application allow one to obtain a reduction in stiffness due to damage in structural components such as pavements and fuselages.

1. INTRODUCTION

Consider an n degree-of-freedom chain consisting of linear springs $\kappa_j > 0$ and viscous dampers $\gamma_j > 0$, $i = 1, 2, \dots, n$, arranged as shown in Figure 1(a). Let x_j denote the time dependent displacement of the j th node of the chain from its static equilibrium position.

This system may represent an elastic structural body that rests on a platform with specific frictional characteristics between the structure and the platform. These frictional effects are described in terms of the dashpot characteristics in this system.

Suppose that a unit force $f = 1$ is applied to the free end of the chain. Then the equations of motion for the nodes can be written in the following matrix form

$$\mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}, \quad (1)$$

Fig. 1: Description of system models.

where

$$\mathbf{G} = \text{diag}\{\gamma_1 \ \gamma_2 \ \cdots \ \gamma_n\}, \quad (2)$$

$$\mathbf{K} = \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_2 & & & \\ -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \kappa_{n-1} & \kappa_{n-1} + \gamma_n & -\kappa_n \\ & & & -\kappa_n & \kappa_n \end{bmatrix}, \quad (3)$$

$$\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)^T, \quad (4)$$

and $\mathbf{f}$ is the n th unit vector,

$$\mathbf{f} = \mathbf{e}_n = (0 \ 0 \ \cdots \ 1)^T. \quad (5)$$

It is shown that the system may be reconstructed uniquely from $2n$ time displacement data $x_n(t_k)$, $t_k = k\Delta t$, $k = 0, 2, \dots, 2n-1$. The reconstruction process involves two well-known procedures, one of fitting an exponential function model to the measured data, and the other of evaluating a tridiagonal system model from its spectrum. These procedures are documented in Section 3 and Section 4, respectively. A small detailed example demonstrating the entire identification process is given in Section 5, and the paper is summarized in Section 6.

2. THE SYSTEM RESPONSE

Let λ_j and $\mathbf{u}_j \neq \mathbf{0}$ be the j th eigenvalue and its associated eigenvector of the generalized eigenvalue problem

$$(\mathbf{K} - \lambda \mathbf{G})\mathbf{u} = \mathbf{0}. \quad (6)$$

It is well known, e.g. Golub and Van Loan, 1986, that the eigenvalues λ_j of (6) are positive and distinct whenever $\kappa_j > 0$ and $\gamma_j > 0$, since the eigenvalue problem (6) is in this case congruently equivalent to a non-singular Jacobi system with a weakly dominant diagonal. It thus follows that the response of the system (1) is given by

$$\mathbf{x}(t) = \mathbf{g} + \sum_{j=1}^n a_j \mathbf{u}_j e^{-\lambda_j t}, \quad (7)$$

where $\mathbf{g}$ is a constant vector,

$$\mathbf{g} = \mathbf{K}^{-1} \mathbf{e}_n, \quad (8)$$

describing the steady state displacement of the various degrees of freedom. If the system starts to move from rest then $\mathbf{x}(0) = \mathbf{0}$ and we have

$$\mathbf{a} = -\mathbf{U}^{-1} \mathbf{g} = -(\mathbf{KU})^{-1} \mathbf{e}_n, \quad (9)$$

where

$$\mathbf{a} = (a_1 \quad a_2 \quad \cdots \quad a_n)^T, \quad (10)$$

and

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]. \quad (11)$$

The response of the free end of the chain is thus

$$x_n(t) = \mathbf{e}_n^T \mathbf{x} = \mathbf{e}_n^T \mathbf{g} + \sum_{j=1}^n a_j \mathbf{e}_n^T \mathbf{u}_j e^{-\lambda_j t}, \quad (12)$$

or

$$x_n(t) = b_0 + \sum_{j=1}^n b_j e^{-\lambda_j t}, \quad (13)$$

with the obvious definitions of b_j , $j = 0, 1, 2, \dots, n$.

Knowing $x_n(t)$, the problem under consideration is of determining the physical parameters κ_j , γ_j , $j = 1, 2, \dots, n$, of the model. We first determine in the next section the mathematical parameters b_j and λ_j from the measured data, then show in Section 4 how to extract the physical parameters κ_j , γ_j from b_j and λ_j .

3. EVALUATING THE MATHEMATICAL PARAMETERS

Denote

$$h(t) \equiv x_n(t), \quad (14)$$

and assume that $x_n(t)$ has been sampled with constant sampling rate Δt . We define

$$h_k \equiv h(k\Delta t), \quad k = 0, 1, \dots, m, \quad (15)$$

so that (13) can be written in the form

$$h_k = b_0 + \sum_{j=1}^n b_j e^{-\lambda_j k \Delta t}, \quad k = 0, 1, \dots, m, \quad (16)$$

by virtue of (14) and (15). Note that the condition $x(0) = 0$ implies that

$$b_0 = -\sum_{j=1}^n b_j, \quad (17)$$

where b_0 is the steady state displacement of x_n . We further denote $h_k = h_{k+1} - h_k$ and obtain

$$h_k = \sum_{j=1}^n \xi_j e^{-\lambda_j k \Delta t}, \quad k = 0, 1, \dots, m-1 \quad (18)$$

where

$$\xi_j = b_j (e^{-\lambda_j \Delta t} - 1). \quad (19)$$

The problem of determining λ_j and ξ_j , knowing Δt and $m = 2n$ data values h_k , is thus the well known exponential fitting problem, which can be solved for example by Prony's method (Hamming, 1987). Following Hamming we determine the coefficients $c_j, j = 0, 1, \dots, n-1$ of the monic polynomial

$$L(z) = \prod_{j=1}^n (z - z_j) = \sum_{j=1}^n c_j z^j, \quad c_n = 1, \quad (20)$$

via the solution of the linear system

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{n-1} & h_n & h_{n+1} & \cdots & h_{2n-2} \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} -h_n \\ -h_{n+1} \\ -h_{n+2} \\ \vdots \\ -h_{2n-1} \end{pmatrix}. \quad (21)$$

The roots z_j of $L(z)$ generally determine the poles λ_j via

$$\lambda_j = -\frac{\ln|z_j| + i \arg z_j}{\Delta t}, \quad i = \sqrt{-1}. \quad (22)$$

In our case, whenever the data are sympathetic with the model, the roots z_j are real positive and λ_j is determined uniquely from

$$\lambda_j = -\frac{1}{\Delta t} \ln z_j. \quad (23)$$

Knowing $\lambda_j, j = 1, 2, \dots, n$, the coefficients ξ_j in (18) are determined by the solution of the following system:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-\lambda_1 \Delta t} & e^{-\lambda_2 \Delta t} & \cdots & e^{-\lambda_n \Delta t} \\ \vdots & \vdots & \cdots & \vdots \\ e^{-\lambda_1 (n-1) \Delta t} & e^{-\lambda_2 (n-1) \Delta t} & \cdots & e^{-\lambda_n (n-1) \Delta t} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}. \quad (24)$$

At this stage where the system parameters $\xi_j, \lambda_j, j = 1, 2, \dots, n$, are known, the coefficients b_j , and in particular the steady state displacement at the right end of the chain b_0 can be determined from (17) and (19). Then Laplace transform of (18) yields

$$H(s) = \frac{b_0}{s} + \sum_{j=1}^n \frac{b_j}{s + \lambda_j} = \frac{P(s)}{s \prod_{j=1}^n (s + \lambda_j)}, \quad (23)$$

where, by virtue of (17) $P(s)$ is a polynomial of degree $n - 1$

$$P(s) = \eta \prod_{j=1}^{n-1} (s + \mu_j), \quad (24)$$

where η is some constant. The root $-\mu_j$, $j = 1, 2, \dots, n - 1$ of $P(s)$ are the zeros of the transfer function $H(s)$. They are the roots of the characteristic polynomial

$$\det(\mathbf{K} - \lambda \mathbf{G}) = 0, \quad (25)$$

where $\mathbf{K}$ and $\mathbf{G}$ are the $(n - 1) \times (n - 1)$ leading principal submatrices of $\mathbf{K}$ and $\mathbf{G}$,

$$\mathbf{K} = \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_2 & & & \\ -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -\kappa_{n-2} & \kappa_{n-2} + \gamma_{n-1} & -\kappa_{n-1} \\ & & & -\kappa_{n-1} & \kappa_{n-1} + \kappa_n \end{bmatrix}, \quad (26)$$

and

$$\mathbf{G} = \text{diag}\{\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_{n-1}\}. \quad (27)$$

It is well known, see e.g., Golub and Van Loan, 1986, that the μ_j , $j = 1, 2, \dots, n - 1$, are all positive, and moreover, they interlace λ_j , $j = 1, 2, \dots, n$,

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n. \quad (28)$$

It will now be shown how to evaluate the physical dampers γ_j and the springs κ_j knowing the poles $-\lambda_j$ and zeros $-\mu_j$ of $H(s)$, and the steady state displacement b_0 of x_n .

4. EVALUATING THE PHYSICAL PARAMETERS OF THE MODEL

We are now in a position where n positive poles λ_j , $n - 1$ positive zeros μ_j , satisfying the interlacing property (28), and the steady state displacement b_0 , are known. The problem of determining the physical parameters γ_j and κ_j , $j = 1, 2, \dots, n$, from these data is similar to the problem of reconstructing a chain of mass-spring system from spectral data, which was first addressed by Gantmakher and Krein, 1961. The solution presented here is based on the Lanczos method, applied by de Boor and Golub, 1978, for reconstructing a tridiagonal matrix from spectral data.

A matrix $\mathbf{A}$ is first defined

$$\mathbf{A} = \mathbf{G}^{-1/2} \mathbf{K} \mathbf{G}^{-1/2} \quad (29)$$

where

$$\mathbf{G}^{-1/2} = \text{diag} \{ \gamma_1^{-1/2} \quad \gamma_2^{-1/2} \quad \dots \quad \gamma_n^{-1/2} \}. \quad (30)$$

The matrix $\mathbf{A}$ is symmetric tridiagonal

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix}, \quad (31)$$

with positive diagonal elements α_j and negative off diagonal β_j . Its eigenvalues are λ_j , $j = 1, 2, \dots, n$. If the last row and column of $\mathbf{A}$ are omitted, then the resulting matrix,

$$\hat{\mathbf{A}} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-3} & \alpha_{n-2} & \beta_{n-2} \\ & & & \beta_{n-2} & \alpha_{n-1} \end{bmatrix}, \quad (32)$$

has eigenvalues μ_j , $j = 1, 2, \dots, n-1$. These are the data obtained in the previous section.

Denote the spectral decomposition of $\mathbf{A}$ by

$$\mathbf{A}\mathbf{V} = \mathbf{\Lambda}\mathbf{V}, \quad (33)$$

where $\mathbf{V}$ is orthogonal

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}, \quad (34)$$

$\mathbf{I}$ is the identity matrix, and $\mathbf{\Lambda}$ is diagonal

$$\mathbf{\Lambda} = \text{diag}\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n\}. \quad (35)$$

Then the n th row of $\mathbf{V}$ is determined by (de Boor and Golub, 1978),

$$v_{nj}^2 = \frac{\prod_{k=1}^{n-1} (\lambda_j - \mu_k)}{\prod_{\substack{k=1 \\ k \neq j}}^{n-1} (\lambda_j - \lambda_k)}, \quad j = 1, 2, \dots, n. \quad (36)$$

The last row of (33) can be written as follows:

$$\begin{aligned} \beta_{n-1} v_{n-1,1} + \alpha_n v_{n1} &= \lambda_1 v_{n1} \\ \beta_{n-1} v_{n-1,2} + \alpha_n v_{n2} &= \lambda_2 v_{n2} \\ &\vdots \\ \beta_{n-1} v_{n-1,n} + \alpha_n v_{nn} &= \lambda_n v_{nn}. \end{aligned} \quad (37)$$

Multiplying the j th equation of (37) by v_{nj} and adding the n equations gives

$$\alpha_n = \sum_{j=1}^n \lambda_j v_{nj}^2, \quad (38)$$

since $\sum_{j=1}^n v_{nj}^2 = 1$ and $\sum_{j=1}^n v_{n-1,j} v_{nj} = 0$ by virtue of the orthogonality (34). We can therefore determine α_n from (38) and (36). Then we obtain the following set of equations from (37)

$$\begin{aligned} (\alpha_n - \lambda_1)^2 v_{n1}^2 &= \beta_{n-1}^2 v_{n-1,1}^2 \\ (\alpha_n - \lambda_2)^2 v_{n2}^2 &= \beta_{n-1}^2 v_{n-1,2}^2 \\ &\vdots \\ (\alpha_n - \lambda_n)^2 v_{nn}^2 &= \beta_{n-1}^2 v_{n-1,n}^2. \end{aligned} \quad (39)$$

Adding these equations we obtain

$$\beta_{n-1} = -\sqrt{\sum_{j=1}^n (\alpha_n - \lambda_j)^2 v_{nj}^2}, \quad (40)$$

since $\sum_{j=1}^n v_{n-1,j}^2 = 1$. Knowing α_n and β_{n-1} the $(n-1)$ th row of $\mathbf{V}$ is obtained from (37). The other elements of $\mathbf{A}$ can then be determined successively for $j = n-1, n-2, \dots, 1$ by following similar process in which α_j and β_{j-1} are determined by the set of equations

$$\begin{aligned} \beta_{j-1}v_{j-1,1} + \alpha_j v_{j1} + \beta_j v_{j+1,1} &= \lambda_1 v_{j1} \\ \beta_{j-1}v_{j-1,2} + \alpha_j v_{j2} + \beta_j v_{j+1,2} &= \lambda_2 v_{j2} \\ &\vdots \\ \beta_{j-1}v_{j-1,n} + \alpha_j v_{jn} + \beta_j v_{j+1,n} &= \lambda_n v_{jn}, \end{aligned} \quad (41)$$

with $\beta_0 = 0$, which are obtained from the j th row of (33).

After reconstructing $\mathbf{A}$ the matrices $\mathbf{K}$ and $\mathbf{G}$ are found as follows. Denote

$$\mathbf{p} = \sqrt{\gamma_1/\kappa_1} (1 \quad 1 \quad \dots \quad 1)^T. \quad (42)$$

Then multiplying (29) by $\mathbf{G}^{1/2}\mathbf{p}$ gives

$$\mathbf{A}\mathbf{G}^{1/2}\mathbf{p} = \mathbf{G}^{-1/2}\mathbf{K}\mathbf{p}. \quad (43)$$

It follows from (2), (3) and (42) that

$$\mathbf{G}^{-1/2}\mathbf{K}\mathbf{p} = \mathbf{e}_1, \quad (44)$$

where $\mathbf{e}_1 = (1 \ 0 \ \dots \ 0)^T$. Hence, the vector $\mathbf{y}$ determined from

$$\mathbf{A}\mathbf{y} = \mathbf{e}_1, \quad (45)$$

where, view of (2) and (42)-(44),

$$\mathbf{y} = \frac{\sqrt{\gamma_1}}{\kappa_1} (\sqrt{\gamma_1} \quad \sqrt{\gamma_2} \quad \dots \quad \sqrt{\gamma_n})^T. \quad (46)$$

The vector $\mathbf{y}$ determines $\mathbf{G}$ up to a scale factor γ_1/κ_1^2 . We now show how to determine this factor. It follows from (29) that

$$\mathbf{K} = \frac{\kappa_1^2}{\gamma_1} \mathbf{Y} \mathbf{A} \mathbf{Y} \quad (47)$$

where

$$\mathbf{Y} = \text{diag}\{y_1 \ y_2 \ \cdots \ y_n\}. \quad (48)$$

Note that at this stage $\mathbf{A}$ and $\mathbf{Y}$ are known. The steady state deflection b_0 is related to the flexibility matrix,

$$b_0 = \mathbf{e}_n^T \mathbf{K}^{-1} \mathbf{e}_n. \quad (49)$$

Therefore using (46) we have

$$b_0 = \frac{\gamma_1}{\kappa_1^2} \mathbf{e}_n^T \mathbf{Y}^{-1} \mathbf{A}^{-1} \mathbf{Y}^{-1} \mathbf{e}_n, \quad (50)$$

and the scale factor in (47) is

$$\frac{\sqrt{\gamma_1}}{\kappa_1} = \sqrt{\frac{b_0}{\mathbf{e}_n^T \mathbf{Y}^{-1} \mathbf{A}^{-1} \mathbf{Y}^{-1} \mathbf{e}_n}}. \quad (51)$$

The matrices $\mathbf{G}$ and $\mathbf{K}$ are thus determined by (46) and (47) respectively.

5. AN EXAMPLE

Let $n = 2$, and suppose that the following data have been measured using the sampling rate $\Delta t = 1$:

$$h(1) = 0.699318, \ h(2) = 1.116955, \ h(3) = 1.397648, \ h(4) = 1.588909. \quad (52)$$

Then from the definition $h_k = h_{k+1} - h_k$ with $h(0) = 0$ we obtain

$$h(0) = 0.699318, \ h(1) = 0.417637, \ h(2) = 0.280693, \ h(3) = 0.191261. \quad (53)$$

Equation (21)

$$\begin{bmatrix} 0.699317 & 0.417637 \\ 0.417637 & 0.280693 \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.280693 \\ -0.191261 \end{pmatrix}, \quad (54)$$

then yields

$$c_0 = 0.049787, \quad c_1 = -0.755464. \quad (55)$$

The characteristic polynomial (20) is

$$L(z) = z^2 - 0.755464z + 0.049787, \quad (56)$$

and its roots are

$$z_1 = 0.682518, \quad z_2 = 0.072946. \quad (57)$$

The mapping (22) from z_j to s_j gives

$$\lambda_1 = 0.381966, \quad \lambda_2 = 2.618034. \quad (58)$$

The system of equation (21)

$$\begin{bmatrix} 1 & 1 \\ 0.682518 & 0.072946 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0.699318 \\ 0.417637 \end{pmatrix}, \quad (59)$$

yields

$$\xi_1 = 0.601446, \quad \xi_2 = 0.09787. \quad (60)$$

The coefficients $b_j, j = 1, 2$, in (16) are determined by (19)

$$b_1 = \frac{\xi_1}{e^{-\lambda_1 \Delta t} - 1} = -1.894427, \quad b_2 = \frac{\xi_2}{e^{-\lambda_2 \Delta t} - 1} = -0.105573. \quad (61)$$

Hence, the steady state displacement $b_0 = x_n(t \rightarrow \infty)$ is by (17)

$$b_0 = -b_1 - b_2 = 2. \quad (62)$$

The polynomial (24) is therefore

$$\begin{aligned}
P(s) &= b_0(s + \lambda_1)(s + \lambda_2) + b_1s(s + \lambda_2) + b_2s(s + \lambda_1) \\
&= \bar{b}_0s(\bar{\lambda}_1 + \lambda_2) + b_0\lambda_1\lambda_2 + \bar{b}_1s\bar{\lambda}_2 + b_2s\lambda_1 \\
&= \eta(s + \mu_1),
\end{aligned} \tag{63}$$

by virtue of (17), and hence

$$\mu_1 = \frac{l_0\lambda_1\lambda_2}{b_0(\lambda_1 + \lambda_2) + b_1\lambda_2 + b_2\lambda_1} = 2.000000. \tag{64}$$

As expected the interlacing property $\lambda_1 < \mu_1 < \lambda_2$ holds.

At this stage the data λ_1 , λ_2 , μ_1 and b_0 have been evaluated. They determine the physical parameters γ_1 , γ_2 , κ_1 and κ_2 of the model (1)-(3) as described in Section 4. The elements in the last row of the normalized eigenvector matrix can be determined by (36)

$$v_{21} = \sqrt{\frac{\lambda_1 - \mu_1}{\lambda_1 - \lambda_2}} = 0.850651, \quad v_{22} = \sqrt{\frac{\lambda_2 - \mu_1}{\lambda_2 - \lambda_1}} = 0.525731. \tag{65}$$

Then α_2 and β_2 are obtained by (38) and (40)

$$\alpha_2 = \lambda_1 v_{21}^2 + \lambda_2 v_{22}^2 = 1.000000, \tag{66}$$

$$\beta_1 = -\sqrt{(\alpha_2 - \lambda_1)^2 v_{21}^2 + (\alpha_2 - \lambda_2)^2 v_{22}^2} = -1.000000. \tag{67}$$

The elements of the first row of $\mathbf{V}$ are then found by (37)

$$v_{11} = \frac{(\lambda_1 - \alpha_2)v_{21}}{\beta_1} = 0.525731, \quad v_{12} = \frac{(\lambda_2 - \alpha_2)v_{22}}{\beta_1} = -0.850651, \tag{68}$$

and by (41)

$$\alpha_1 = \lambda_1 v_{11}^2 + \lambda_2 v_{12}^2 = 2.000000. \tag{69}$$

The matrix $\mathbf{A}$ has thus been fully recovered,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{70}$$

In order to extract $\mathbf{K}$ and $\mathbf{G}$ from $\mathbf{A}$ we first solve (45)

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (71)$$

and obtain

$$y_1 = 1, y_2 = 1. \quad (72)$$

So by (48)

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (73)$$

We then determine from (51)

$$\frac{\sqrt{\gamma_1}}{\kappa_1} = \sqrt{\frac{2}{(0 \ 1) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} = 1, \quad (74)$$

from which, using (47), we find

$$\mathbf{K} = \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad (75)$$

and from (46)

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The unique system, which produces the data, is shown in Figure 2.

Fig. 2: The reconstructed system

Note that the strict inequalities $\kappa_j > 0$ and $\gamma_j > 0$ that we imposed on the physical parameters of the system are crucial for having a possible unique reconstruction. For example, if all damper coefficients $\gamma_j > 0$ then the response of the system is discontinuous

$$x_n = \begin{cases} 0, & t < 0 \\ d, & t > 0 \end{cases} \quad (77)$$

where $d = 1/\kappa_{eq}$ is constant and κ_{eq} is an equivalent spring constant

$$\kappa_{eq} = \frac{1}{\sum_{i=1}^n \kappa_i^{-1}}. \quad (78)$$

Clearly, there are many spring combinations, which lead to the same value of κ_{eq} , and hence unique reconstruction is not possible for this case. Similarly if $\kappa_{eq} = 0$ then the velocity is constant and the response of the right end of the chain is

$$x_n = \frac{t}{c}, \quad (79)$$

independent of the $2n - 2$ parameters $\kappa_j > 0$ and $\gamma_j > 0$, $j = 1, 2, \dots, n - 1$.

6. CONCLUDING REMARKS

We have shown that the physical elements of an n degrees of freedom chain of springs and dampers may be determined from dynamic measurement of $2n$ displacements of the chain's end, caused by a static unit load applied to it. If the data are sympathetic with the model, i.e. produced by spring-damper chain, then the reconstruction is unique. This fact follows from the two processes applied, exponential fit and Jacobi matrix reconstruction, each yielding a unique solution. Note that generally a complex exponential fit leads to a non-unique solution due to frequency aliasing. In our case the exponential coefficients are real which allows unambiguous identification.

The analysis has been done for a noise free case. In the presence of noise more than $2n$ response data could be measured and the system may be identified in a least square sense by extending the system of (21) to include $p > n$ equations. The model order can in this case be determined by considering the numerical rank of the over-determined system of equations as described in Braun and Ram, 1987.

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