Weibull Model Development for Fatigue Crack Growth

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1. ABSTRACT

This effort is an attempt to “Weibullize” fatigue crack growth rate data as a function of stress intensity range (ΔK). It is not a classical Weibull application in that it does not address a population but rather treats rate of crack growth as a fraction of the highest rate of crack growth rate in the data set (usually the final data point of the set) as the “failed fraction of a population.” All da/dN values are converted to a percentage or fraction of the 100% data value, processed into a linear relationship and plotted against the natural log of the corresponding ΔK value.

The crack growth rate data (da/dN vs. ΔK), when converted to a Weibull type plot, can be fitted with a linear curve with R^2 values in the 0.99 range and characteristic Weibull parameters β and η can be successfully determined. For the data analyzed, the mean value determined for β is 2.86 with a standard deviation of 0.36. However, data from sets of apparently similar specimens produced more closely packed results. The data sets that could not be fit with the Weibull distribution (4 of 32 sets) appeared to have been corrupted, possibly by sorting the data in the spread sheet without including all of the columns. R^2 values of > 0.99 were obtained for the linear curve fit of 15 of 28 data sets. R^2 values > 0.95 were obtained for an additional 7 while the remaining 6 were > 0.64

2. INTRODUCTION

Weibull analysis methodology has been reviewed for other applications and extended to analyze fatigue crack growth.

The Weibull distribution is defined mathematically as

$$F(t) = 1 - e^{-((t-\gamma)/\eta)^\beta} \quad (1)$$

The classical use of the Weibull distribution is to provide failure projections for a population based on observed failures in that population. In this application:

$F(t)$ = failed fraction of a population

t = time of failure

γ = starting point or origin of the distribution if other than t_0

η = characteristic life (the time or number of cycles at which 63.2% of the population is expected to have failed (Analogous to RC time constant in electronics). 63.2% is given by $F(t) = 1 - e^{-1}$ when $t = \eta$

β = slope or shape parameter

β and η are the characteristic parameters of a 2 parameter Weibull application. Adding γ makes a 3 parameter application.

By letting $\gamma = 0$ and manipulating the above expression into the form of the straight line equation

$$Y = MX + B$$

such that $\ln \ln \{1/(1 - F(t))\} = \beta \ln t - \beta \ln \eta$ (2)

and regarding $\ln \ln \{1/(1 - F(t))\}$ as Y and $\ln t$ as X

(see Appendix 1) we can now plot failure data (da/dN = crack growth rate) vs. time (cycles, ΔK , etc.). Data that approximates a straight line when processed in this fashion can be analyzed to determine the associated values of the Weibull characteristic parameters. If the data plot does not approximate a straight line, it may indicate the presence of multiple failure modes and/or an offset in the X axis initiation of the distribution (γ not = 0).

By using the Excel "Insert Trend Line" function, the data can be curve fit and the equation of the resulting line displayed. β , the shape factor, is the coefficient of X (slope of the line) from this equation. The final term in the straight line equation (the Y axis intercept) would appear to be useful in determining η , the characteristic life, but due to a Y scale shift to present the data in a "Weibull graph paper" format, a different method is used. The Y value representing 0.632 $F(t)$ is placed in the obtained equation as a known Y value and the equation is solved for the corresponding X value. This yields the natural log of the η value of ΔK .

If the data fails to plot as a straight line due to an offset in the start of the distribution, shifting the "time" scale or X axis values until the curve straightens out allows a quantification of the offset. The spreadsheet provides for use and definition of the value of the third Weibull parameter, distribution initiation offset or γ . The ΔK offset for the origin of the distribution was varied and the variation of the curve fit R^2 value was observed. The offset value at which the R^2 value peaked is suggested as an approximation of the γ value. However, human observation of the variation in the shape of the lower end of the curve seemed to provide better results than the R^2 value peak as R^2 evaluates the quality of the fit over the entire length of the curve and not just the lower end.

The β or “shape factor” values obtained were fairly consistent while there was considerable spread in the η or “characteristic life” parameter. This is believed to be due to the variety of specimens in the data (various initial crack lengths and/or specimen geometry). The Weibull parameter values obtained can then be used in various Weibull equations to provide predictions concerning the expected behavior of the population.

The rate of crack growth as a function of ΔK is known to occur in three regions (Ref. 2) as shown in the figure below.

Fig. 1: Fatigue Crack Growth Regions (Ref. 2)

The data evaluated covered primarily region II with Region I or portions of it included in some cases. Some data sets also seemed to contain the onset of Region III. In general, the data plotted was limited to Region II as the area of interest. In some cases, Regions I and II were plotted and offset varied to quantify threshold ΔK .

The methodology and resulting plots are described and provided in the following pages.

3. CRACK GROWTH RATE VS. ΔK PROCESS DEVELOPMENT

The Microsoft Excel spreadsheet program was used to process all data in this effort. da/dN data as a function of ΔK for 32 test specimens was provided by others. For development of this process, the data set for the specimen designated CT11 was used. After development of the process using CT11, data from the remaining specimens was analyzed. Table I provides a summary of the analysis results. Table 2 is the CT11 spread sheet and was used interactively with the resulting graph (Figure 2) to evaluate the quality of the resulting plot and to define the values of β and η . It was also used to explore the effect of variations of the value of γ . For specimen CT11, γ was set to the value (4) that produced the straightest line in the plot of data as indicated by the curve fit R^2 value.

Table I

Specimen	Beta	Eta	Gamma	R ²	Remarks
CT11	2.7983	42.38	0	0.9947	
CT11	2.3095	39.37	4	0.9957	
CT12	2.7613	42.91	0	0.9968	
CT12	2.5286	41.73	1.5	0.9969	
CT13	3.0065	37.3	0	0.9937	Peak R ² at 0 and -1
CT14	2.7918	45.46	0	0.991	
CT15	2.7896	42.21	0	0.9937	
CT16	3.42	30.68	0	0.9717	
CT18	3.1392	29.41	0	0.9913	Bottom 15 and Top 3 Points Censored
CT22					Corrupted Data
CT23	3.8298	27.82	0	0.9824	Does not linearize well
CT24					Corrupted Data
CT24+1	2.1814	55.9	0	0.9529	
CT2	2.7153	54.6	0	0.9984	
CT4	2.7506	40.31	0	0.9463	
CT36	2.6823	37.7	0	0.6497	
CT37	3.0598	42.12	0	0.84	
CT31	2.894	29.39	0	0.7935	
CT32					Corrupted Data
33					Corrupted Data
CC12	3.0974	30.21	0	0.9893	
CC21	2.6827	66.91	0	0.9978	
CC26	3.1276	33.75	0	0.9904	
CC16	2.9385	56.69	0	0.9903	
CC16	2.3538	53.25	5	0.9924	
CC17	2.684	67.91	0	0.9937	Lowest 7 data points censored
CC17	2.9637	64.6	0	0.9604	No data censored
CC17	1.6587	68.25	9.45	0.9858	No Data Censored
CC18	2.6813	71.23	0	0.9931	@-6
CC4	2.2727	91.99	0	0.9917	Upper 4 points censored
CC4	2.4114	87.3	0	0.9817	Uncensored
CC5	2.3883	89.57	0	0.991	Region 2
CC6	2.3848	93.17	0	0.9822	Region 2
CC23	2.7339	108.8	0	0.9946	top 4 points censored
CC23	2.8978	99.33	0	0.9908	uncensored
CC24	2.7352	69.37	0	0.9792	uncensored
CC24	2.5643	74.24	0	0.989	Top 4 points censored
CC25	3.1174	37.75	0	0.9806	uncensored
CC28	3.3399	53.23	0	0.6547	Suspect data; highly scattered
CC29	3.527	35.5	0	0.9259	Suspect data; scattered
CC30	2.8767	38.18	0	0.8198	Suspect data; scattered

Table II

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Specimen number: CT11														
Temperature: 600 degrees C														
R Ratio: 0.1														
Frequency: 0.25														
Pre-Crack (mm): 2.69														
Cycles	a*		ΔK		da/dN		Initiation	Shifted						
(N)	(mm)		(Mpa $\sqrt{m}$ 1/2)		(m/c $\sqrt{10^{-9}}$)		shift	delta K	X Axis	Y Axis	In ln (1/R)	In ln (1/R)	1/R	Normalized da/dN
									In Shifted delta K	Y Shifted In ln (1/R)				
450	9.08		15.1		77		4	11.1	2.406945	0	-2.929396	0.053429	1.054882	5.2027027
1430	9.15		15.2		78.2			11.2	2.415914	0.015888	-2.913508	0.054285	1.055785	5.2837838
2417	9.23		15.3		78.8			11.3	2.424803	0.002671	-2.932088	0.053287	1.054732	5.1891892
3366	9.3		15.4		77.9			11.4	2.433613	0.011939	-2.917458	0.054071	1.05556	5.2635135
4307	9.37		15.5		79.7			11.5	2.442347	0.035419	-2.893978	0.055356	1.056916	5.3851351
5210	9.45		15.6		79.3			11.6	2.451005	0.030246	-2.899151	0.05507	1.056615	5.3581081
6084	9.52		15.7		81.8			11.7	2.459589	0.06217	-2.867227	0.058856	1.058504	5.527027
7027	9.59		15.8		83.5			11.8	2.4681	0.083341	-2.846055	0.058073	1.059792	5.6418919
7833	9.66		16		87.1			12	2.484907	0.12683	-2.802567	0.060654	1.062531	5.8851351
8640	9.77		16.1		95.1			12.1	2.493205	0.217551	-2.711845	0.066414	1.068668	6.4256757
9352	9.87		16.3		97.2			12.3	2.509599	0.240143	-2.689253	0.067932	1.070282	6.5675676
10224	9.89		16.4		100			12.4	2.517696	0.269544	-2.659852	0.069959	1.072464	6.7567568
10730	9.98		16.5		102			12.5	2.525729	0.290064	-2.639333	0.071409	1.07402	6.8818919
12137	10.08		16.6		104			12.6	2.533697	0.310189	-2.618197	0.072861	1.075581	7.027027
12780	10.15		16.8		106			12.8	2.549445	0.329966	-2.59943	0.074316	1.077147	7.1621622
13388	10.21		16.9		109			12.9	2.557227	0.358954	-2.570442	0.078502	1.079504	7.3648649
14611	10.35		17.1		118			13.1	2.572612	0.423722	-2.505674	0.081621	1.085044	7.8378378
15659	10.48		17.3		122			13.3	2.587764	0.476326	-2.45307	0.086029	1.089838	8.2432432
16168	10.54		17.4		127			13.4	2.595255	0.518309	-2.411087	0.089718	1.093865	8.5810811
16731	10.61		17.6		131			13.6	2.61007	0.550778	-2.378819	0.092679	1.097109	8.8513514
17682	10.74		17.8		134			13.8	2.624669	0.574516	-2.35488	0.094905	1.099554	9.0540541
18587	10.87		18		142			14	2.639057	0.635435	-2.293961	0.100866	1.106129	9.5945946
19476	10.99		18.3		147			14.3	2.66026	0.671881	-2.257516	0.10461	1.110278	9.9324324
20289	11.12		18.5		147			14.5	2.674149	0.671881	-2.257516	0.10461	1.110278	9.9324324
21093	11.24		18.7		152			14.7	2.687847	0.707174	-2.222222	0.108368	1.114458	10.27027
21947	11.36		19		157			15	2.70805	0.741391	-2.188005	0.11214	1.11867	10.608108
22689	11.46		19.2		163			15.2	2.721295	0.781125	-2.148272	0.116686	1.123766	11.013514
24015	11.72		19.7		174			15.7	2.753661	0.850539	-2.078857	0.125073	1.133231	11.756757
24689	11.83		20		175			16	2.772589	0.856645	-2.072752	0.125839	1.1341	11.824324
25357	11.95		20.2		176			16.2	2.785011	0.862718	-2.066678	0.126606	1.134969	11.891892
26590	12.17		20.7		186			16.7	2.815409	0.921746	-2.00765	0.134304	1.14374	12.567568
27159	12.28		21		194			17	2.833213	0.966887	-1.962508	0.140505	1.150855	13.108108
28253	12.5		21.5		200			17.5	2.862201	0.999629	-1.929787	0.145182	1.15625	13.513514
28822	12.61		21.8		211			17.8	2.879198	1.057378	-1.872018	0.153813	1.166273	14.256757
29754	12.82		22.3		231			18.3	2.906801	1.155666	-1.77373	0.169699	1.184948	15.608108
30620	13.02		22.9		246			18.9	2.938162	1.224445	-1.704952	0.181781	1.199352	16.621622
31410	13.22		23.4		273			19.4	2.965273	1.339291	-1.590105	0.203904	1.226181	18.445946
32398	13.52		24.3		296			20.3	3.010621	1.429456	-1.49994	0.223144	1.25	20
33381	13.8		25.2		298			21.2	3.054001	1.437004	-1.482392	0.224834	1.252115	20.135135
33968	13.99		25.6		318			21.8	3.08191	1.510163	-1.419233	0.241899	1.273686	21.486486
35045	14.35		27		368			23	3.135494	1.67722	-1.252177	0.285882	1.330935	24.864865
35735	14.62		28		390			24	3.178054	1.744763	-1.184613	0.305864	1.357798	26.351351
36585	14.97		29.4		419			25.4	3.234749	1.829273	-1.100123	0.33283	1.39491	28.310811
37481	15.37		31.2		497			27.2	3.303217	2.035816	-0.89358	0.409188	1.505595	33.581081
38370	15.84		33.5		574			29.5	3.38439	2.217592	-0.711804	0.490758	1.633554	38.783764
39208	16.37		36.4		677			32.4	3.478158	2.437462	-0.491934	0.611443	1.843088	45.743243
40043	17		40.5		845			36.5	3.597312	2.762364	-0.167032	0.846172	2.330709	57.094595
40871	17.79		46.5		1100			42.5	3.749504	3.236606	0.30721	1.359626	3.894737	74.324324
41711	18.95		80		1480			56	4.025352					100
E10 = 207 GPa 39.36941														
ln E10 = 3.672986														
2.22505														
2.22505														

The source data for specimen CT11 appears in columns A through F of Table II. The data processed is ΔK in column D and da/dN in column F.

The initial part of the process is to develop the Weibull distribution values representing ΔK to be plotted on the X axis. To do this, a distribution initiation adjustment (γ) is provided in column H. Column I, labeled "Shifted delta K" is then the value from the same row in column D minus the value of γ entered in column H. Column J contains the natural logs of the values from the same row in column I and which are the values plotted on the X axis. The scale values for ln ΔK were suppressed by changing them to background color. A ΔK scale was created by plotting the natural log of values of interest as data with corresponding ΔK values typed in as the scale.

Fig. 2: Weibull plot da/dN vs. ΔK

Crack growth rate data is then processed for plotting on the Y axis. The bottom row of columns D & F contains the final set of data for the test specimen. The highest da/dN value (1480) is used as the final member (100%) of this pseudo population. This column of data is expressed in column O as a decimal fraction or percentage of the highest da/dN value. This provides a pseudo “failed fraction of a population” value.

The data (except for the final point) is then further processed from right to left through columns N, M, L, and K to J which contains the Y axis values. Since column O is the failed fraction column ($F(t)$), the complement of column O is the surviving fraction or a “Reliability” number (R). Column N is $100/(100-F(t))$ from the adjacent column O and is thus labeled $1/R$. The 100% point cannot be processed here because it results in dividing by zero. The next two columns to the left successively provide the natural log of the column to their immediate right. Column K, which is plotted as the “Y” value simply shifts the data scale for presentation as the Weibull Failure Fraction $F(t)$. The Y axis scale construction is shown in Table III. As with the X axis, the scale values for $f(da/dN)$ are suppressed and the Weibull $F(t)$ scale is inserted as data. The scale is then manually labeled.

3.1 Plotting and Weibull Parameter Determination

Columns J and K are plotted (Figure 2) as an X-Y scatter plot. The Excel “Insert Trend Line” function is

Table III
Weibull Paper Y Axis Parameters

F(t)	$\ln \ln \{1/(1-F(t))\}$	Shifted Column 2 Value for Weibull F(t) plot (Column 2 minus minimum Column 2 value (-6.91))
0.001	-6.91	0.00
0.01	-4.60	2.31
0.1	-2.25	4.66
0.5	-0.37	6.54
0.9	0.83	7.74
0.99	1.53	8.44
0.999	1.93	8.84
(Extracted from Ref. 1)		

then used to obtain a linear trend line and the associated equation. At this point the γ value is varied to produce the straightest line of plotted data as indicated by the R^2 value. The R^2 value peaked at 0.9957 with a γ value of 4 as compared to an initial value ($\gamma = 0$) of 0.9947. In the resulting equation with $\gamma = 4$, the coefficient of X (2.3095) is the value of the Weibull parameter β . The β value with $\gamma = 0$ was 2.8

Using the definition of η as $F(t) = 63.2\%$, this value is inserted into the last row of the spreadsheet in column O and processed to the left to column K just as the actual da/dN data to obtain the Y axis value of the 63.2% point. Given this Y value, the straight line equation provided by the trend line is used in the bottom cell of column J to solve for the corresponding X value. This X value represents the natural log of η . Raising epsilon to this power then provides the ΔK value of η (39.37) displayed in the bottom row of column D.

The Weibull PDF and CDF plots produced by the Excel Weibull function using these values are included as Figure 3.

3.2 γ or Offset Determination

The data for specimen CC6 covers both Regions I and II. Figure 4 was produced by processing CC6 data with varying values of offset to demonstrate the effect of offset on the shape of the curve. The value of offset providing the straightest lower section of the curve is speculated to be the threshold value of ΔK (ΔK_{th}) for initiation of crack growth. In instances where the data was sufficient to define threshold ΔK , this would not be of interest. However, in instances where the data has Region I content but not enough to clearly define ΔK_{th} , this technique may be of interest.

Fig. 3: Weibull da/dN PDF/CDF

Fig. 4: Offset effect

3.3 Composite Results Plot

Figure 5 is a composite plot of all Weibull plots produced for the 28 sets of usable data. A consistent slope (β parameter) is apparent while a large range of ΔK for the same da/dN is evident. This results in a range of the η parameter from mid twenties to around 80. Groupings of trends from closely numbered data sets were noted during plotting. Specimen geometry groupings were also noted with “corner crack” specimens generally lying below and to the right of “compact tensile” specimens. Figures 6 and 7 present the same information as Figure 5 but separated by specimen type.

Fig. 5: Weibull composite chart

Fig. 6: Compact tensile specimen Weibull plots

Fig. 7: Corner crack specimen Weibull plots

CONCLUSIONS:

1. The methodology was successfully applied to describe the fatigue crack growth rates within Mode I stress intensity factor range (ΔK).
2. The Weibull distribution predicted crack growth rate probability as a function of Mode I ΔK . In most cases, a straight line fit was observed.
3. From the limited data assessed with the methodology developed in this paper, mixed results were achieved.
4. Additional work is recommended to refine the concepts to develop a more general approach applicable to crack growth rate predictions.

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APPENDIX I**Weibull Distribution Failed Fraction Equation****Conversion to Straight Line Expression**

$$F(t) = 1 - e^{-(t/\eta)^\beta}$$

Let $\gamma = 0$

$$F(t) = 1 - e^{-(t/\eta)^\beta}$$

Add $e^{-(t/\eta)^\beta}$ to each side, Subtract $F(t)$ from each side

$$1 - F(t) = e^{-(t/\eta)^\beta}$$

Eliminate the negative power of e by taking the reciprocal of the left side

$$1/(1 - F(t)) = e^{(t/\eta)^\beta}$$

Take natural log of each side

$$\ln \{1/(1 - F(t))\} = (t/\eta)^\beta$$

Take natural log again

$$\ln \ln \{1/(1 - F(t))\} = \beta (\ln t) - \beta \ln \eta$$

$$Y = M(X) + B$$

 η Overview

$$F(t) = 1 - e^{-(t/\eta)^\beta}$$

When $t = \eta$

$$F(t) = 1 - e^{-(1)^\beta}$$

$$F(t) = 1 - 0.368 \text{ (independent of } \beta \text{)}$$

$$F(t) = 0.632$$

Thus η represents the time at which 63.2% of the population is expected to have failed and is known as the “Characteristic Life.”