Post-Buckling Problems for Thin Elastic Films

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ABSTRACT

A strain gradient fixed-fixed beam model is proposed for studying the localized symmetric buckling behavior of thin fixed-fixed films. It appears that the critical conditions are independent of the film length but they depend on the intrinsic (material) length. Invoking the multiple scales perturbation method a localized buckling solution is defined with modulated amplitude.

1. INTRODUCTION

Due to modern technology involving multiscale problems size effects is lately a topic of growing importance. Fleck *et al.* /1/ have experimentally proved that thinner copper rods are stronger in torsion, pointing out the importance of gradient elasticity. Furthermore, Aifantis and co-workers /2-5/ have interpreted size effects in yield and strength fracture localization of the deformation with the help of strain gradient theories. Likewise, Vardoulakis and his group /6/ have studied size effects in geomaterials and brittle materials with the help of strain gradient theories. Bazant and co-workers /7-9/ have been concerned with size effects of concrete and geomaterials.

Size effects in stability problems should obviously be expected, since buckling theory is mathematically very sensitive in the influence of various parameters such as imperfections. Hence the introduction of strain gradient models for studying stability problems may interpret phenomena that are difficult to be studied with standard deformation methods. Indeed, there are buckling problems where the standard beam models fail to yield a solution. Plaut *et al.* /10/ have pointed out, for the problem of a long beam on rigid foundation, that the perfect model does not accept any solution. However, the introduction of a strain gradient beam model, with specific intrinsic (material) length, yields a localized solution with the help of multiple scales method (Nayfeh /11/). In fact the material length interacts with the material constants exhibiting localized buckling that does not depend on the length of the beam, insofar as the length is not small for the development of the localization of the deformation. It is pointed out that similar localized solutions have been presented for long

structures by Lange *et al.* /12/, Potier-Ferry /13/ and Hunt *et al.* /14/, but for long beams on elastic (softening) foundations.

In the present work the localized buckling problem of a fixed-fixed thin beam will be studied introducing the strain gradient beam model, invoking Mindlin's /15/ second strain gradient energy density function. The non-linear curvature function will be considered. Buckling and post-buckling analysis will be performed for a thin beam under axial loading. Since the beam is fixed-fixed the localized deformation of the beam may be initiated not from the supports but from internal points of the elastic symmetric beam curve. Working with the multiple scales perturbation method (Nayfeh /11/), the critical conditions for the symmetric buckling will be described. It is pointed out that Fourier's spectrum in this case is not discrete but continuous. Localization of the deformation is exhibited due to the modulated amplitude. Post-critical behavior of the beam is performed through minimization of the total energy function, which yields the deformation of the beam that is not extended throughout the length of the elastic curve.

The present problem may be proved important for the practicing engineer, since apart from the obvious film applications and delamination it may be related to many applications in buried pipelines such as in water, gas or waste supply networks and generally long structures like railroad rails, road pavements etc. Similar problems of long beams supported upon rigid foundation have already been presented by Lazopoulos /16/.

2. THE STRAIN GRADIENT BEAM MODEL

Let us consider a beam with

$$x_1 \in [-0, L], \quad x_2 \in [-h/2, h/2], \quad x_3 \in [-b/2, b/2].$$

Following Bernoulli's beam assumption the strain ε_{11} equals

$$\varepsilon_{11} = k(x_1)x_2 \quad (1)$$

where (x_1, x_2, x_3) is a principal system for the cross-section moments of inertia in the plane (x_2, x_3) and $k(x_1)$ the curvature of the elastic curve. Furthermore, the other non-vanishing strains are given as $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$.

The strain energy density function, in standard (linear) elasticity theory, is assumed to depend upon the strain ε_{ij} . Nevertheless in gradient elasticity, the strain energy density function depends not only on the strain ε_{ij} but also on higher order gradients of the strain. Following Mindlin /15/, the strain energy density function for that type of materials is expressed by

$$\begin{aligned}
w = & \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + a_1 \varepsilon_{ij,j} \varepsilon_{ik,k} + a_2 \varepsilon_{ii,k} \varepsilon_{kj,j} + a_3 \varepsilon_{ii,k} \varepsilon_{jj,k} \\
& + a_4 \varepsilon_{ij,k} \varepsilon_{ij,k} + a_5 \varepsilon_{ij,k} \varepsilon_{kj,i} + b_1 \varepsilon_{ii,jj} \varepsilon_{kk,ll} + b_2 \varepsilon_{ij,kk} \varepsilon_{ij,ll} \\
& + b_3 \varepsilon_{ii,ik} \varepsilon_{jk,ll} + b_4 \varepsilon_{ii,jk} \varepsilon_{jj,ik} + b_5 \varepsilon_{ii,jk} \varepsilon_{ll,jk} + b_6 \varepsilon_{ij,kl} \varepsilon_{ij,kl} \\
& + b_7 \varepsilon_{ij,kl} \varepsilon_{jk,li} + c_1 \varepsilon_{ii} \varepsilon_{jj,kk} + c_2 \varepsilon_{ij} \varepsilon_{ij,kk} + c_3 \varepsilon_{ij} \varepsilon_{kk,jj} + b_8 \varepsilon_{ii,jj} .
\end{aligned} \quad (2)$$

Recalling that Young's modulus E and Poisson's ratio ν are equal to $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ and $\nu = \lambda/2(\mu + \lambda)$ respectively, and adjusting Mindlin's /15/ second strain gradient energy density in the beam structures we may assume that the strain energy density function includes the terms

$$w = \frac{E}{2} k^2 x_2^2 + a_1 \left((k')^2 x_2^2 - 2\nu k' k x_2 + \nu^2 k^2 \right) + c_1 k k'' x_2^2 (1 - 2\nu)^2, \quad (3)$$

with k denoting $k(x_1)$. Integrating Eq. (3) over the cross-section of the beam we get

$$w = \frac{EI}{2} k^2 + a_1 \left((k')^2 I + \nu^2 k^2 A \right) + c_1 k k'' (1 - 2\nu)^2 I, \quad (4)$$

where I and A denote the least moment of inertia with respect to x_3 and the area of the cross-section respectively. Denoting with $y(x_1)$ or y the deflection of the beam, the curvature of the inextensible elastic curve is expressed by (Thompson *et al.* /17/, p. 28)

$$k = k(x_1) = \pm \frac{y''(x_1)}{\sqrt{1 - y'^2(x_1)}} \approx \pm y''(x_1) \left(1 + \frac{1}{2} y'^2(x_1) \right) + O(y^4(x_1)). \quad (5)$$

Hence, the total potential energy W for that inextensible beam under the action of the axial compressive load p becomes

$$W = \int_0^l F(x_1, y'(x_1), y''(x_1), y'''(x_1), y^{(4)}(x_1)) dx_1, \quad (6)$$

with

$$F(x_1, y'(x_1), y''(x_1), y'''(x_1), y^{(4)}(x_1)) = w - \frac{p}{2} \left(y'^2(x_1) + \frac{y'^4(x_1)}{4} \right) + O(y'^4(x_1)). \quad (7)$$

Then, the Lagrange-Euler equation for the functional of Eq. (6) becomes (with $x = x_1$)

$$\frac{d^4}{dx^4} \frac{\partial F}{\partial y^{IV}} - \frac{d^3}{dx^3} \frac{\partial F}{\partial y^{III}} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y^{II}} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0, \quad (8)$$

with the boundary conditions corresponding to the fixed-fixed beam at $x = 0, L$

$$\frac{\partial F}{\partial y^{IV}} = 0, \quad (9a)$$

$$\frac{\partial F}{\partial y^{III}} - \frac{d}{dx} \frac{\partial F}{\partial y^{IV}} = 0, \quad (9b)$$

$$y(x) = y'(x) = 0. \quad (9c)$$

3. THE LOCALIZED SOLUTION

Following formal bifurcation theory [18] the linear equilibrium equation (kernel), controls the behavior of any stability problem and it will be studied in detail. In fact, whenever the linear problem accepts a non-trivial solution (non-trivial kernel), non-unique solutions exist for the non-linear equilibrium equation. Thus the linear problem of Eq. (8) yields

$$(c-b)y^{(6)}(x) + ay^{IV}(x) + py(x) = 0, \quad (10)$$

with $a = EI + 2a_1\nu^2 A$, $b = 2a_1 I$ and $c = 2c_1(1-2\nu)^2 I$. Furthermore, the boundary conditions for the present fixed-fixed beam, Eqs. (9a,b,c), are reduced to

$$y(x) = y''(x) = y'''(x) = y^{IV}(x) = 0 \quad x = 0, L. \quad (11)$$

The resolving equation to Eq.(10) is

$$(c-b)\omega^4 - a\omega^2 + p = 0, \quad (12)$$

with $\omega_{2,2}^2 = \left[a \pm \sqrt{a^2 - 4(c-b)p} \right] / 2(c-b) > 0$. Hence, the symmetry of the solution $y(x)$ with

$$y(x) = \xi(\cos \omega_1 x + \gamma \cos \omega_2 x), \quad (13)$$

cannot satisfy the boundary conditions (11). Thus, a solution for the linear problem does not exist. Nevertheless, when the length of the beam is long, a localized solution may be defined following Lange *et al.* /12/, suggesting continuous spectrum (and corresponding critical loading) for a thin beam-like structure supported on a rigid foundation. Using the perturbation method of multiple scales /11/, the critical loading for localized deformation is defined for a beam supported on a rigid foundation. Indeed, it is allowed for a continuous band of modes to be represented in the integration scheme of the equilibrium equation by choosing, in addition to the geometrical variable x , the slow space variable

$$X = \varepsilon x, \quad |\varepsilon| \ll 1. \quad (14)$$

Constructing a formal asymptotic expansion of the solution to the governing non-linear Eq. (8) in the form

$$y(x) = \varepsilon y_0(x, X) + \varepsilon^2 y_1(x, X) + O(\varepsilon^3), \quad (15)$$

and according to the method of multiple scales we get (see /11/)

$$\begin{aligned} \frac{dy_i}{dx} &= \frac{\partial y_i}{\partial x} + \varepsilon \frac{\partial y_i}{\partial X}, \\ \frac{d^2 y_i}{dx^2} &= \frac{\partial^2 y_i}{\partial x^2} + 2\varepsilon \frac{\partial^2 y_i}{\partial x \partial X} + \varepsilon^2 \frac{\partial^2 y_i}{\partial X^2}, \\ \frac{d^k y_i}{dx^k} &= \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X} \right)^k y_i. \end{aligned} \quad (16)$$

Substituting Eqs. (16) into the governing Eq. (8) and assuming $p = p_o(1 + \varepsilon^2 \lambda)$ with p_o the critical load and $0 \ll \lambda \ll 1$ the linearized equilibrium system defined by the vanishing coefficients of each power of ε is

$$L(y_o) = (c - b)y_o''(x) + ay_o'''(x) + py_o''(x), \quad (17a)$$

$$3(c - b)\frac{\partial^4 y_o(x)}{\partial x^4 \partial X^2} + 2a\frac{\partial^4 y_o(x)}{\partial x^2 \partial X^2} + p\frac{\partial^2 y_o(x)}{\partial X^2} = 0, \quad (17b)$$

$$\begin{aligned}
-L(y_1) = & \lambda p_o y_o''(x) + (c-b) y_1^{VI}(x) + a y_1^{IV}(x) + p y_1''(x) + \\
& 15(c-b) \frac{\partial^{VI} y_o(x)}{\partial x^4 \partial X^2} + 6a \frac{\partial^{IV} y_o(x)}{\partial x^2 \partial X^2} + p \frac{\partial^2 y_o(x)}{\partial X^2} + \\
& a y_o^{VI}(x) + 10(c-b) y_o^{IV}(x) y_o''(x) + a y_o^{IV}(x) y_o''(x) + \\
& (c-b) y_o'(x) y_o'''(x) y_o''(x) + 1.5 p y_o'^2(x) y_o''(x) + 15(c-b) y_o''(x) y_o^{IV}(x) + \\
& 4a y_o'(x) y_o''(x) y_o'''(x) + 6(c-b) y_o'(x) y_o''(x) y_o''(x) + (c-b) y_o'^2(x) y_o^{VI}(x)
\end{aligned} \quad (17c)$$

The general solution of Eq. (17a) which is bounded in the fast x -scale is

$$y_o(x, X) = A(X - X_o) \sin(\omega(x - x_o) + \theta), \quad (18)$$

where x_o is a point on the beam elastic curve where the deformation of the beam initiates. Then the homogeneous equation (17a) yields the equivalent algebraic equation

$$(c-b)\omega^4 - a\omega^2 + p_o = 0. \quad (19)$$

Proceeding to the solution of Eq. (17b) we get

$$3(c-b)\omega^4 - 2a\omega^2 + p_o = 0. \quad (20)$$

The common root of the system made by Eqs. (19,20) is given by

$$\omega^2 = \frac{a}{2(c-b)}. \quad (21)$$

Moreover, the critical load p_o equals

$$p_o = \frac{a^2}{4(c-b)}. \quad (22)$$

Proceeding to the higher order Eq. (17c), y_o is substituted by Eq. (18) and becomes

$$\begin{aligned}
-L(y_1) = & \left\{ -\lambda p_o \omega^2 A(X - X_o) + (p_o - 6a\omega^2 + 15(c-b)\omega^4) A''(X - X_o) \right. \\
& \left. - (1.5 p_o \omega^4 - 2a\omega^6 - 2(b-c)\omega^8) A^3(X) \right\} \sin(\omega(x - x_o) + \theta) \\
& + \{ 1.5 p_o \omega^4 - 6a\omega^6 + 42(c-b)\omega^8 \} A^3(X) \sin(3(\omega(x - x_o) + \theta)).
\end{aligned} \quad (23)$$

Since the operator $L(\bullet)$ admits the kernel $\{\sin(\omega x + \theta), \cos(\omega x + \theta)\}$, the existence of a bounded y_1 solution of Eq. (23) requires zero coefficient of the $\sin(\omega x + \theta)$ term, which is equal to

$$(p_o - 6a\omega^2 + 15(c-b)\omega^4)A''(X) - \lambda p_o \omega^2 A(X) + (-1.5p_o \omega^4 + 2a\omega^6 - 2(c-b)\omega^8)A^3(X) = 0. \quad (24)$$

Recalling the values for ω^2 and p_o , Eqs. (21,22), Eq. (24) is reduced to

$$A''(X) = \frac{\lambda a}{8(c-b)} A(X) - \frac{1}{32} A^3(X). \quad (25)$$

Lange *et al.* /12/ solve a differential equation similar to Eq. (28) with the use of elliptic functions that are not convenient for the present problem of the semi-infinite beam. In the present case Eq. (25) is not elliptic. The present procedure follows a different path with simpler mathematical approaches that are suitable for any boundary conditions in a long beam

$$A''(X - X_o) = \mu^2 A(X - X_o) - \frac{0.03125a^4}{(c-b)^2} A^3(X - X_o), \quad (26)$$

with $\mu^2 = \lambda a / 8(c-b)$. Assuming $A(X - X_o)$ a small amplitude function, a perturbation solution to Eq. (26) is given by

$$A(X - X_o) = \xi \sinh(\mu(X - X_o)) - \varepsilon^2 \xi^3 \left(\frac{0.03125a^4}{(c-b)^2} \right) \left(\frac{1}{8\mu^2} \sinh(3\mu(X - X_o)) \right). \quad (27)$$

Hence Eqs. (18, 27) yield

$$y_o(x, X) = \xi \sinh(\mu(X - X_o)) \sin(\omega(x - x_o) + \theta) \quad \text{for } 0 < x < L/2. \quad (28)$$

4. THE SYMMETRIC MODE

In this section the symmetric mode of the localized buckling deformation (see Fig. 1) is going to be discussed. It is evident that the solution (28) has to satisfy the symmetry conditions at the center of the up-lifted arch ($x=L/2$). The symmetry condition at the center of the beam is expressed by $[dy_o(x, X)/dx] = 0$ at

Fig.1: The symmetric mode deformation

$x=L/2$. The zero derivative of the displacement condition at that center yields after linearization with respect to ε

$$\tan(\omega(L/2 - x_o) + \theta) + \omega(L/2 - x_o) = 0 \quad (29)$$

that defines the up-lifted length of the beam if the constant θ has already been defined. In fact Eq. (29) is a relation between the parameters x_o and θ . It is clear that the solution (28) includes harmonics with modulated amplitude and is symmetric with respect to $x = L/2$. Therefore, the solution is expressed by

$$\left. \begin{aligned} y_o(x, X) &= 0 & \text{for } 0 < x < x_o \\ y_o(x, X) &= \xi_o \sinh(\varepsilon \mu(x - x_o)) \sin(\omega(x - x_o) + \theta) & \text{for } x_o < x < L/2 \end{aligned} \right\} \quad (30)$$

and is symmetric with respect to $x = L/2$. It is pointed out that the solution $y_o(x, X)$ [Eqs.(18,30)] satisfies the boundary conditions (11) since, recalling Eq.(18), $\sinh(\mu X) = 0$ at $x = x_o$ and

$$\frac{\partial^k y_o(x, X)}{\partial x^k} = 0, \quad x = x_o, \quad k = 0, 1, 2, \dots \quad (31)$$

Recalling the higher order part of Eq. (23), the solution to the third order equilibrium equation (29) has been found to be given by

$$\left. \begin{aligned} y_1 &= 0, & 0 < x < x_o \\ y_1 &= \xi_1 \sinh(\mu(X - X_o)) \sin(\omega(x - x_o) + \theta) \\ &- \xi^3 \left(\frac{0.03125a^2}{(c-b)^2} \right) \left(\frac{1}{8\mu^2} \sinh(3\mu(X - X_o)) \right) \sin(\omega(x - x_o) + \theta) \\ &+ \xi^3 \frac{0.0273a}{c-b} \sinh^3(\mu(X - X_o)) \sin(3(\omega(x - x_o) + \theta)), & x_o < x < L/2 \end{aligned} \right\} \quad (32)$$

with the parameter ξ_1 unspecified and unnecessary for the solution of the problem. In addition the symmetric solution y_1 with respect to $x = L/2$ is valid for the region $L/2 < x < L$.

5. THE ANTI-SYMMETRIC MODE

In the case of the anti-symmetric mode, see Fig. 2, the displacement should be zero at the center of the beam. Recalling Eq.(28), the zero condition displacement at $x = L/2$ yields

$$\omega(L/2 - x_o) + \theta = \pi \quad (33)$$

that defines the up-lifted length of the beam if the constant θ has already been defined. In fact, Eq. (32) is again a relation between the parameters x_o and θ . It is clear that the solution (28) in this case includes harmonics with modulated amplitude and is anti-symmetric with respect to $x = L/2$. Therefore, the solution is expressed by

$$\left. \begin{aligned} y_o(x, X) &= 0 & \text{for } 0 < x < x_o \\ y_o(x, X) &= \xi_o \sinh(\varepsilon\mu(x - x_o)) \sin(\omega(x - L/2) + \pi) & \text{for } x_o < x < L/2 \end{aligned} \right\} \quad (34)$$

and is anti-symmetric with respect to $x = L/2$. It is pointed out that the solution $y_o(x, X)$ [Eqs.(18,34)] also satisfies the boundary conditions (11) since, recalling Eq.(18), $\sinh(\mu X) = 0$ at $x = x_o$ and

$$\frac{\partial^k y_o(x, X)}{\partial x^k} = 0, \quad x = x_o, \quad k = 0, 1, 2, \dots \quad (35)$$

Recalling the higher order part of Eq.(23), the solution to the third order equilibrium equation (29) has been found to be given by

$$\left. \begin{aligned} y_1 &= 0, & 0 < x < x_o \\ y_1 &= \xi_1 \sinh(\mu(X - X_o)) \sin(\omega(x - L/2) + \pi) \\ &\quad - \xi^3 \left(\frac{0.03125a^2}{(c-b)^4} \right) \left(\frac{1}{8\mu^2} \sinh(3\mu(X - X_o)) \right) \sin(\omega(x - L/2) + \pi) \\ &\quad + \xi^3 \frac{0.0273a}{c-b} \sinh^3(\mu(X - X_o)) \sin(3(\omega(x - L/2) + \pi)), & x_o < x < L/2 \end{aligned} \right\} \quad (36)$$

with the parameter ξ_1 unspecified and unnecessary for the solution of the problem. In addition the corresponding anti-symmetric solution y_1 with respect to $x = L/2$ is valid for the region $L/2 < x < L$.

Fig.2: The anti-symmetric mode deformation

6. THE POST-CRITICAL DEFORMATION

The study of the post-critical deformation requires the definition of the up-lifted length defined by the point x_o and the amplitude parameter ξ . Those parameters will be defined through the minimization of the total potential energy function W [see Eq.(6)], after substitution of the displacement function $y(x)$ for the modulated amplitude solution $\varepsilon y_o(x, X)$ from Eq. (30) for the symmetric and Eq. (34) for the anti-symmetric case. Using computerized algebra packs, such as Mathematica, the energy function is computed integrating between $0 \leq x \leq L$. It is evident that the point x_o defining the up-lifted length is known by Eq. (29) for the symmetric mode or Eq. (33) for the anti-symmetric mode if the parameter θ has already been determined. Therefore, the total energy function has been reduced to a function of the two parameters, x_o and ξ . Likewise, the energy function has the expansion

$$W = \varepsilon^2 \xi^2 \Lambda_1(x_o) + \varepsilon^4 \Lambda_2(\xi, x_o) . \quad (37)$$

Thus the minimization of the energy function Λ defines the parameters ξ and x_o . Indeed, minimization of the total potential Λ requires

$$\Lambda_1(x_o) = 0 . \quad (38)$$

Indeed, performing the algebra, Eq. (34) defines the point x_o where the localized deformation of the beam initiates as a solution to the equation

$$\left. \begin{aligned} & (L - 2x_o)\omega \left\{ a \left(144 + 46(L - 2x_o)^2 \omega^2 + (L - 2x_o)^4 \omega^4 \right) + \right. \\ & + \omega^2 \left\{ -2c \left(240 + 82(L - 2x_o)^2 \omega^2 + (L - 2x_o)^4 \omega^4 \right) \right. \\ & + b \left(624 + 130(L - 2x_o)^2 \omega^2 + (L - 2x_o)^4 \omega^4 \right) \left. \right\} - \\ & - 24(L - 2x_o)\omega \left(-5a + (13b + 22c)\omega^2 \right) \cos((L - 2x_o)\omega) \\ & - 6(5a - (13b + 22c)\omega^2) \left(-4 + (L - 2x_o)^2 \omega^2 \right) \sin((L - 2x_o)\omega) = 0 \end{aligned} \right\} \quad (39)$$

It is evident that the solution for x_o of Eq. (35) depends on the critical conditions and on the incremental post-critical loading as well. Likewise, the parameter of the modulated amplitude depends on the incremental loading parameter λ . Indeed, minimization of the total potential energy in addition requires

$$\frac{d\Lambda_2(\xi, x_o)}{d\xi} = 0 , \quad (40)$$

which has the form of the equation

$$\xi^3 - \beta^2 \xi = 0, \quad (41)$$

and $\xi = \pm \beta$. Therefore, the first order solution is defined by

$$y_o(x, X) = 0 \quad \text{for} \quad 0 < x < x_o$$

$$y_o(x, X) = \pm \beta \sinh(\varepsilon \mu(x - x_o)) \sin(\omega(x - x_o) + \theta) \quad \text{for} \quad x_o < x < L/2. \quad (42)$$

7. ON THE IMPOSSIBILITY OF THE ANTI-SYMMETRIC DEFORMATION

Let us consider the anti-symmetric mode of deformation defined by Eq. (34). In this case, the function

$$\Lambda_1(x_o) = \frac{\sqrt{2}a^2bm}{64(b-c)^3 \sqrt{\frac{a}{c-b}}} \left[\sqrt{\frac{2a}{c-b}}(L/2 - x_o) + 3 \sin \left(\sqrt{\frac{2a}{c-b}}(L/2 - x_o) \right) \right], \quad (43)$$

is shown in Fig. 3 and never becomes zero. Therefore the anti-symmetric mode is impossible in the present problem. Hence there remains only the consideration of the symmetric mode.

8. APPLICATION

Implementing the proposed theory, we consider a non-dimensionalized application with the bending parameters

$$a = 1, \quad b = -0.8, \quad c = 0.08. \quad (44)$$

The critical load p_o derived by Eq. (22) has been found equal to $p_o = 0.2841$ and the mode number $w = 0.7538$. The next step is the definition of the up-lifted part of the beam defining the point x_o or the parameter $\theta > 0$. Eq. (29) yields

$$\theta = -\text{Arc tan}(\omega(L/2 - x_o)) - \omega(L/2 - x_o). \quad (45)$$

Fig.3: The function $\Lambda_1(x_0)$

Then Eq. (39) yields a numerical equation for x_0 with solution, $x_0 = L/2 - 0.83$. Recalling Eqs. (37) the function $\Lambda_2(\xi)$ has been calculated with the help of the computerized Mathematica pack and has been found as

$$\Lambda_2(\xi) = \xi^2 (-0.00429 + 0.00649 \xi^2), \quad (46)$$

and the modulated amplitude parameter ξ is defined by the equation

$$\frac{\partial \Lambda_2(\xi)}{\partial \xi} = \xi (-0.0086 + 0.026 \xi^2) = 0, \quad (47)$$

with solutions $\xi = \pm 0.5751$, $\xi = 0$. Checking the stability of the localized solution, the sign of the second derivative of $\Lambda_2(\xi)$ has to be defined. Indeed,

$$\frac{\partial^2 \Lambda_2(\xi)}{\partial \xi^2} = -0.0086 + 0.0779 \xi^2, \quad (48)$$

and it is positive for $\xi = \pm 0.5751$ and negative for $\xi = 0$. Therefore, the localized post-critical equilibrium path is stable whereas the trivial solution becomes unstable. It is clear that the exchange of stabilities phenomenon is shown up. Hence, the post-critical solution for the present application is defined by

$$\left. \begin{aligned} y(x) &= 0 & \text{for } 0 < x < x_0 \\ y(x) &= \xi \sinh(\varepsilon \mu(x - x_0)) \sin(\omega(x - x_0) + \theta) & \text{for } x_0 < x < L/2 \end{aligned} \right\} \quad (49)$$

with $\theta = -1.1717\text{rad}$ and $\xi = \pm -/5751$. Therefore, the solution is given by

$$\left. \begin{aligned} y(x) &= 0 & \text{for } 0 < x < x_0 \\ y(x) &= -0.5751 \sinh(\varepsilon 0.3769(x - x_0)) \sin(0.7835(x - x_0) - 1.1717) & \text{for } x_0 < x < L/2 \end{aligned} \right\} \quad (50)$$

where $x_0 = L/2 - 0.83$. It is evident that the solution is symmetric with respect to the point $x = L/2$. Fig.4 shows the localized displacement of the beam with respect to $x - x_0$.

Fig.4: The localized post-critical deformation of the application.

9. CONCLUSIONS

The localized buckling behavior of a finite fixed-fixed beam of force by an axial load is studied using strain gradient elasticity. Localized deformation is developed and analyzed with the help of method of multiple scales. The localized beam length is defined through the minimization of the total potential energy function and is independent on the length of the beam. That is feasible because the amplitude is not constant but modulated. Moreover, the amplitude parameter is defined from the higher order terms of the total potential energy function. From the two studied modes of deformation, symmetric and anti-symmetric, only the symmetric mode is possible. The stability of the localized branch of equilibrium is assessed by the minimum of the potential energy function. The present analysis may be used for the study of a large class of problems concerning localized deformation of structures such as railway track and pipe problems and generally long linear structures. In addition delamination problems of composites and thin films may be better understood through the present analysis.

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