Exact Solution of the Axisymmetric Thermoelasticity Problem for a Long Cylinder Subjected to Varying With-Respect-To-Length Loads

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1. SUMMARY

A method for constructing exact analytical solutions for axisymmetric elasticity and thermoelasticity problems is proposed for the case of a full infinite cylinder subjected to loads varying along its length. The method consists in applying the technique of direct integration to corresponding equilibrium and compatibility equations in terms of stresses. Such an approach avoids using any auxiliary potential functions.

2. INTRODUCTION

Traditional methods employed in constructing analytical solutions for the main elasticity and thermoelasticity boundary value problems make use the methods of functions for stresses or displacements. In particular, to solve axisymmetric thermoelasticity problems, the Love, Boussinesq, Michell stress functions and the Goodier potential have been used /1–4/. However, sometimes their usage leads to the need of determining quantities that have no physical sense. In our opinion, this makes the process to obtain a solution extremely difficult. Moreover, application of the methods orientated towards solving the potential function of the corresponding thermoelasticity problems in bounded regions /4,5/ does not make it possible to describe the stressed state adequately in the small neighborhoods of corner points. This is connected with the fact that in order to obtain the solution one should solve infinite sets of linear algebraic or differential equations.

The method of direct integration of equilibrium and compatibility equations in terms of stresses has turned out to be more efficient for solving thermoelasticity problems, e.g. the problems for infinite regions /6,7/, for inhomogeneous, thermosensitive /8/ and bounded solids having corner points /9,10/. The solutions

found have the form of functional dependencies on loading factors, which enables their usage for solving the problems of optimal control as well as the inverse problems /4/. Here is the outline of the method. The so-called governing stresses are selected, for which the key equations are derived on the basis of the system of the original equilibrium and compatibility equations. If the boundary conditions are available for different stress tensor components, they are equivalently replaced by the boundary and integral conditions for governing stresses. The solution of the formulated boundary value problem is constructed. By integration of the equilibrium equations the relations between the stresses are established, hence the other stress tensor components are expressed through the governing ones.

The paper presents application of the method of direct integration to solving the axisymmetric thermoelasticity problem for a full long cylinder. The solution is constructed in the form of functional dependencies of the stress tensor components on force and thermal loads. On the basis of the solution the numerical analysis of the stressed state of a cylinder is carried out.

3. FORMULATION OF THE PROBLEM

Let us consider the axisymmetric quasi-static thermoelasticity problem for a long full cylinder provided that the material is isotropic and homogeneous and there are no body forces, the latter being not essential for implementation of the method. In a cylindrical coordinate system (ρ, φ, x) the problem is governed – see /1–4/ for details – by the equilibrium equations

$$\frac{\partial}{\partial \rho}(\rho \sigma_r) + \rho \frac{\partial \sigma_{rz}}{\partial x} = \sigma_\varphi, \quad \frac{\partial}{\partial \rho}(\rho \sigma_{rz}) + \rho \frac{\partial \sigma_z}{\partial x} = 0, \quad (1)$$

compatibility equations in terms of strains

$$\rho \frac{\partial e_\varphi}{\partial \rho} = e_r - e_\varphi, \quad \rho \frac{\partial^2 e_\varphi}{\partial x^2} - \frac{\partial e_{rz}}{\partial x} + \frac{\partial e_z}{\partial \rho} = 0, \quad (2)$$

and physical relations

$$E e_r = \sigma_r - \nu(\sigma_\varphi + \sigma_z) + \alpha E T, \quad E e_z = \sigma_z - \nu(\sigma_r + \sigma_\varphi) + \alpha E T, \quad E e_\varphi = \sigma_\varphi - \nu(\sigma_r + \sigma_z) + \alpha E T, \quad G e_{rz} = \sigma_{rz}. \quad (3)$$

Equations (1)–(3) should be solved in terms of the stress tensor components σ_j ($j = r, \varphi, z$), σ_{rz} under the prescribed tractions on the boundary

$$\sigma_r|_{\rho=1} = -p(x, \tau), \quad \sigma_{rz}|_{\rho=1} = q(x, \tau). \quad (4)$$

Here $(\rho, \varphi, x) \in D$; $D = \{(\rho, \varphi, x) \in [0, 1] \times [0, 2\pi] \times (-\infty, \infty)\}$ is the region occupied by the cylinder; $\rho = r/R$

and $x = z/R$ are dimensionless radial and axial coordinates, R is the radius of the cylinder; e_j ($j = r, \varphi, z$), e_{rz} are strain tensor components; E, G, ν, α are the elastic and shear moduli, Poisson's ratio and coefficient of linear thermal expansion respectively; $T(\rho, x, \tau)$ is a temperature field in the cylinder; τ is the time parameter omitted in what follows for the sake of brevity in writing.

4. CONSTRUCTION OF SOLUTION

Using the physical relations (3), we present the compatibility equations (2) in terms of stresses

$$\rho \frac{\partial}{\partial \rho} [\sigma_\varphi - \nu(\sigma_r + \sigma_z) + \alpha ET] + (1 + \nu)(\sigma_\varphi - \sigma_r) = 0, \quad (5)$$

$$\rho \frac{\partial^2}{\partial x^2} [\sigma_\varphi - \nu(\sigma_r + \sigma_z) + \alpha ET] - 2(1 + \nu) \frac{\partial \sigma_{rz}}{\partial x} + \frac{\partial}{\partial \rho} [\sigma_z - \nu(\sigma_r + \sigma_\varphi) + \alpha ET] = 0. \quad (6)$$

To solve the axisymmetric thermoelasticity problem (1), (4)–(6) in the region D , we apply the method of direct integration [6], selecting the shear stress σ_{rz} and the first invariant of stress $\sigma = \sigma_r + \sigma_\varphi + \sigma_z$ as the governing ones. From the equations in (1) the relations

$$\rho^2 \sigma_r = \int_0^\rho \eta \left(\sigma_* - \eta \frac{\partial \sigma_{rz}}{\partial x} \right) d\eta, \quad \sigma_* = \sigma_r + \sigma_\varphi, \quad (7)$$

$$2\sigma_z = - \int_{-\infty}^{\infty} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \sigma_{rz}) \operatorname{sign}(x - \xi) d\xi \quad (8)$$

follow. By making use of the first equation (1) and relation (8), we reduce equation (5) to the form

$$(1 + \nu) \int_{-\infty}^{\infty} \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \sigma_{rz}) \right) \operatorname{sign}(x - \xi) d\xi + 2(1 + \nu) \frac{\partial \sigma_{rz}}{\partial x} + 2 \frac{\partial}{\partial \rho} (\sigma + \alpha ET) = 0. \quad (9)$$

Differentiating equation (9) with respect to x yields the equivalent differential equation

$$\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \sigma_{rz}) \right) + \frac{\partial^2 \sigma_{rz}}{\partial x^2} + \frac{1}{1 + \nu} \frac{\partial^2}{\partial \rho \partial x} (\sigma + \alpha ET) = 0, \quad (10)$$

which is the first key equation for determining shear stress σ_{rz} through the first invariant σ .

By taking relation (8) into account, we rewrite equation (6) in the form

$$\rho \frac{\partial^2}{\partial x^2} (\sigma_* - (1+\nu)\sigma_r - \nu\sigma_z + \alpha ET) - 2(1+\nu) \frac{\partial \sigma_{rz}}{\partial x} - \frac{\partial}{\partial \rho} \left(\frac{1+\nu}{2} \int_{-\infty}^{\infty} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \sigma_{rz}) \operatorname{sign}(x-\xi) d\xi + \nu\sigma - \alpha ET \right) = 0. \quad (11)$$

If we add to the left-hand side of equation (5) the identity

$$(1+\nu) \left(\rho \frac{\partial \sigma_r}{\partial \rho} - \rho \frac{\partial \sigma_r}{\partial \rho} \right) \equiv 0,$$

it will take the form

$$(1+\nu) \frac{\partial}{\partial \rho} (\rho^2 \sigma_r) = (1+\nu) \rho \sigma_* + \rho^2 \frac{\partial}{\partial \rho} (\sigma_* - \nu\sigma_z + \alpha ET),$$

hence the relation

$$(\sigma_* - (1+\nu)\sigma_r - \nu\sigma_z + \alpha ET) = \int_0^\rho \eta ((1-\nu)\sigma_* - 2\nu\sigma_z + 2\alpha ET) d\eta$$

follows. Inserting this relation into (11) and taking the second equilibrium equation of (1) into account, we arrive at the following equation:

$$\frac{1}{\rho} \frac{\partial^2}{\partial x^2} \int_0^\rho \eta ((1-\nu)\sigma + 2\alpha ET) d\eta - (1+\nu) \frac{\partial \sigma_{rz}}{\partial x} + \frac{\partial}{\partial \rho} \left(-\frac{1+\nu}{2} \int_{-\infty}^{\infty} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \sigma_{rz}) \operatorname{sign}(x-\xi) d\xi - \nu\sigma + \alpha ET \right) = 0.$$

Adding to it equation (9) divided by 2, we obtain the integro-differential equation for the first invariant

$$\frac{\partial^2}{\partial x^2} \int_0^\rho \eta ((1-\nu)\sigma + 2\alpha ET) d\eta + \rho \frac{\partial}{\partial \rho} ((1-\nu)\sigma + 2\alpha ET) = 0. \quad (12)$$

By differentiation with respect to ρ equation (12) can be reduced to the equivalent differential equation

$$\Delta((1-\nu)\sigma + 2\alpha ET) = 0, \quad \Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial x^2}. \quad (13)$$

Equations (10) and (13) are the key ones for determining the shear stress and the first invariant. Note that the key equations coincide, naturally, with two out of the four well-known compatibility equations in terms of stresses for the axisymmetric thermoelasticity problem /4/.

If the stresses, tractions $p(x, \tau)$, $q(x, \tau)$ and the temperature field $T(x, \tau)$ tend to 0 at $|x| \rightarrow \infty$, we can apply the integral Fourier transform /12,13/ to equations (10) and (13), relations (7) and (8), and boundary conditions (4) with respect to x , obtaining thereby the equations:

$$\frac{d}{d\rho} \left[\frac{1}{\rho} \frac{d}{d\rho} (\rho \bar{\sigma}_{rz}) \right] - s^2 \bar{\sigma}_{rz} = -\frac{is}{1+\nu} \frac{d}{d\rho} (\bar{\sigma} + \alpha E \bar{T}), \quad \bar{\Delta} [(1-\nu) \bar{\sigma} + 2\alpha E \bar{T}] = 0, \quad i = \sqrt{-1}, \quad (14)$$

relations

$$\rho^2 \bar{\sigma}_r = \int_0^\rho \eta (\bar{\sigma} - \bar{\sigma}_z - is \eta \bar{\sigma}_{rz}) d\eta, \quad \bar{\sigma}_z = \frac{i}{s\rho} \frac{d}{d\rho} (\rho \bar{\sigma}_{rz}) \quad (15)$$

and boundary conditions

$$\bar{\sigma}_r|_{\rho=1} = -\bar{p}, \quad \bar{\sigma}_{rz}|_{\rho=1} = \bar{q}, \quad (16)$$

where $\bar{\Delta} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) - s^2$, s is the parameter of the integral transform; all the barred quantities are the corresponding ones in transforms. By taking the boundary condition for the radial stress (16) into account, from the first relation (15) we get the integral equilibrium condition for the stress σ_{rz} and the first invariant σ in transforms

$$\int_0^1 \rho (\bar{\sigma} - is \rho \bar{\sigma}_{rz}) d\rho = -\bar{p}. \quad (17)$$

The solution of the system of equations (14) with the boundary (16) and integral conditions (17) can be presented in the form

$$\begin{aligned} \bar{\sigma} = & \frac{2s(1+\nu)I_1(s)I_0(s\rho)}{(2\nu-2-s^2)I_1^2(s)+s^2I_0^2(s)} \left[-\bar{p} + 2\frac{\alpha E}{1-\nu} \int_0^1 \rho \bar{T}(\rho) d\rho + \right. \\ & i \frac{sI_0(s)-2I_1(s)}{sI_1(s)} \left(\bar{q} - \frac{\pi s^2}{2} \frac{\alpha E}{1-\nu} \int_0^1 \rho \bar{T}(\rho) \Delta_{01}(\rho, 1) d\rho \right) + \\ & \left. + \frac{i\pi s^3}{2} \frac{\alpha E}{1-\nu} \int_0^\rho \int_0^\rho \rho^2 \eta \bar{T}(\eta) \Delta_{01}(\eta, \rho) d\eta d\rho \right] - 2\frac{\alpha E}{1-\nu} \bar{T}, \\ \bar{\sigma}_{rz} = & \frac{is^2(I_0(s)I_1(s\rho) - \rho I_0(s\rho)I_1(s))}{(2\nu-2-s^2)I_1^2(s)+s^2I_0^2(s)} \left[-\bar{p} + 2\frac{\alpha E}{1-\nu} \int_0^1 \rho \bar{T}(\rho) d\rho \right. \\ & \left. + \frac{i\pi s^3}{2} \frac{\alpha E}{1-\nu} \int_0^\rho \int_0^\rho \rho^2 \eta \bar{T}(\eta) \Delta_{01}(\eta, \rho) d\eta d\rho \right] - \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{I_1(s)} \left[I_1(sp) - s(I_0(s) - 2I_1(s)) \frac{I_0(s)I_1(sp) - \rho I_0(sp)I_1(s)}{(2v-2-s^2)I_1^2(s) + s^2I_2^2(s)} \right] \left(\bar{q} - \frac{\pi s^2}{2} \frac{\alpha E}{1-\nu} \int_0^1 \bar{T}(\rho) \Delta_{01}(\rho, 1) d\rho \right) - \\
& - \frac{\pi s^2}{2} \frac{\alpha E}{1-\nu} \int_0^p \eta T(\eta) \Delta_{01}(\eta, \rho) d\rho.
\end{aligned} \quad (18)$$

Here $\Delta_{01}(\alpha, \beta) = I_0(s\alpha)Y_1(is\beta) - iI_1(s\beta)Y_0(is\alpha)$; $I(t) = J(it)$; J_j, Y_j ($j = 0, 1$) are Bessel functions /14/ of the first and second kind, correspondingly, of order one and two.

By applying the inverse Fourier transform /12, 13/ to equation (18), we get the shear stress σ_{rz} and the first invariant σ . The axial stress σ_z is determined from formula (8). In view of (7), the radial stress σ_r has the form

$$\rho^2 \sigma_r = \int_0^p \eta \left(\sigma - \sigma_z - \eta \frac{\partial \sigma_{rz}}{\partial x} \right) d\eta.$$

Finally we find the stress σ_φ from the formula $\sigma_\varphi = \sigma - \sigma_r - \sigma_z$.

5. NUMERICAL EXAMPLE

Let us consider the case $T(x, \rho) = 0$, and the external tractions (4) of the form

$$(A) \ p(x, \tau) = 0, \ q(x, \tau) = d \exp(-x^2), \quad (B) \ p(x, \tau) = d \exp(-x^2), \ q(x, \tau) = 0, \ d = \text{const}.$$

Figure 1

Figure 2

Figures 1 and 2 represent the distribution of dimensionless shear stress

$$\sigma_{rz} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\sigma}_{rz} \exp(isx) ds$$

divided by the parameter d , where $\bar{\sigma}_{rz}$ is determined from (18) and the Poisson's ratio $\nu = 0,25$. Fig. 1 (loading (A)) shows that the stress satisfies the boundary conditions exactly, tending to zero with the radial coordinate. Fig. 2 presents the behavior of σ_{rz} under the prescribed normal loading (B).

6. CONCLUSIONS

By making use of the method of direct integration of differential equilibrium and compatibility equations in terms of stresses, the solutions have been found for quasi-static axisymmetric elasticity and thermoelasticity problems for a long full cylinder subjected to loads varying along its length. The original problems are equivalently reduced to solving the key equations with local boundary and integral conditions for the first invariant of stress σ and shear stress σ_{rz} . With the aid of the integral Fourier transform the analytical solutions were found. Other stress tensor components were expressed in terms of the governing ones from the relations between stresses deduced by integrating the equilibrium equations. The solutions have the form of functional dependences on loading factors. Using the constructed solutions, one can easily state and solve corresponding problems in displacements as well as those with mixed boundary conditions.

In general, the possibility of determining the exact solutions for elasticity and thermoelasticity problems in infinite regions as well as bounded solids having corner points is vital for the process of solving the inverse thermoelasticity problems concerning the optimal control of a thermo-stressed state [11].

7. REFERENCES

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