Exponential Stability and Availability Analysis of a Complex Standby System

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Abstract In this paper, we first investigate the solution of the two correlated units redundant system with two types of failure. By using the method of functional analysis, especially, the c_0 semigroup theory of bounded linear operators on Banach space, we prove the well-posedness and the existence of positive solution of the system. By analyzing the spectra distribution of the system operator, we prove that the dynamic solution of the system asymptotically converges to the nonnegative steady-state solution which is the eigenfunction corresponding to eigenvalue 0 of the system operator. Furthermore, we discuss the exponential stability of the system. Finally, we analyze the reliability the system with the help of our main results and present some reliability indices of the system at the end of the paper.

Keywords redundant system; c_0 semigroups; well-posedness; stability; availability

1 Introduction

Reliability is an important concept of various complex systems. The need of obtaining highly reliable system has become more and more important with the development of the modern technology. The high degree of reliability is usually achieved by introducing redundancy. The redundant system discussed here comes from [1]. It consists of two sub-systems A and B, connected in parallel having one regular and one expert repairman. This system has four states, viz. normal, degraded, failed and waiting. The sub-system B is in standby mode and consists of further two units, viz. B_1 and B_2 , connected in parallel. It is assumed that when both the units of sub-system B start functioning simultaneously, then sub-system B works like the sub-system A. So, when some failure occurs in sub-system A, the two units of sub-system B will be in operable condition through perfect switching over device. If any one unit of sub-system B fails (after failing sub-system A), the system will work but in degraded stage. The system will be in down state only when both sub-system A and B fail or due to occurrence of environmental failure.

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In [1], the well-posedness and asymptotic stability of the redundant system have been obtained. Because it is difficult to determine the convergence rate of the dynamic solution only by asymptotic stability, the study of the exponential stability is in demand, which is a more valuable property in application. We will continue to discuss the above system and obtain the exponential stability of the system. It should be pointed out that, in recent years, though there are several papers (e.g. see [3–6]) on the exponential stability of the repairable systems, the method of [2–5] is involved in quasi-compactness and is hard to be applied here. Moreover, as the applications of the main results of [1], we will calculate the stationary availability of the system with another method different from the traditional one [6–9].

The rest of this paper is organized as follows. In section 2, we introduce the mathematical model and some notations of the system. In the meantime, some results on the well-posedness and stability are introduced in this section. In section 3, the exponential of the system is presented. In the last section, we investigate the stationary availability of the system with another method different from the traditional one and give several numerical simulation examples.

2 Mathematical model formulation and some results

Most materials of this section come from [1] and for convenience, we do not restate and introduce them freely. Let $a = \lambda_0 + \lambda_1 + \lambda_2$, $p_0(t)$ be the probability that the system is in operable state at time t , $p_1(t)$ probability that the system is in waiting state at time t , $p_i(t, x)dx$ probability that the system is in operable state at time and elapsed repair time lies between x and $x + dx$, where $j = 8, 9, 10, 11$, and $p_i(t, x)dx$ probability that the system is in failed state at time t and elapsed repair time lies between x and $x + dx$, where $i = 7, 12$. With some additional assumptions, the differential equations of the system are obtained,

$$p_0(t) = \int_0^\infty p_2(t, x)\mu_1(x)dx + \int_0^\infty p_3(t, x)\mu_2(x)dx + \int_0^\infty p_4(t, x)\mu(x)dx + \int_0^\infty \eta(x)[p_5(t, x) + p_6(t, x)]dx + \int_0^\infty p_7(t, x)\mu_3(x)dx + \int_0^\infty p_{12}(t, x)\beta(x)dx \quad (1)$$

$$\left[\frac{d}{dt} + r_1 \right] p_1(t) = \lambda_0 \int_0^\infty p_4(t, x)dx \quad (2)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_1(x) + \lambda_2 \right] p_2(t, x) = 0 \quad (3)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_2(x) + \lambda_1 \right] p_3(t, x) = 0 \quad (4)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu(x) + \lambda \right] p_4(t, x) = 0 \quad (5)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \eta(x) + \lambda_1 + \lambda_2 + r_2 \right] p_5(t, x) = 0 \quad (6)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \eta(x) + \lambda_1 + \lambda_2 \right] p_6(t, x) = 0 \quad (7)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_3(x) \right] p_7(t, x) = 0 \quad (8)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_1(x) + r_2 \right] p_8(t, x) = 0 \quad (9)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_2(x) + r_2 \right] p_9(t, x) = 0 \quad (10)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_1(x) + \lambda_2 \right] p_{10}(t, x) = 0 \quad (11)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_2(x) + \lambda_1 \right] p_{11}(t, x) = 0 \quad (12)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \beta(x) \right] p_{12}(t, x) = 0 \quad (13)$$

and with the boundary conditions

$$p_2(t, 0) = \lambda_1 p_0(t), \quad p_3(t, 0) = \lambda_2 p_0(t) \quad (14)$$

$$p_4(t, 0) = \lambda_2 \int_0^\infty p_2(t, x) dx + \lambda_1 \int_0^\infty p_3(t, x) dx \quad (15)$$

$$p_5(t, 0) = \lambda_0 p_0(t) + r_1 p_1(t) + \int_0^\infty \mu_1(x) p_8(t, x) dx + \int_0^\infty \mu_2(x) p_9(t, x) dx \quad (16)$$

$$p_6(t, 0) = r_2 \int_0^\infty p_5(t, x) dx + \int_0^\infty \mu_1(x) p_{10}(t, x) dx + \int_0^\infty \mu_2(x) p_{11}(t, x) dx \quad (17)$$

$$p_7(t, 0) = \lambda_3 p_0(t), \quad p_8(t, 0) = \lambda_1 \int_0^\infty p_5(t, x) dx, \quad p_9(t, 0) = \lambda_1 \int_0^\infty p_5(t, x) dx \quad (18)$$

$$p_{10}(t, 0) = \lambda_1 \int_0^\infty p_6(t, x) dx + r_2 \int_0^\infty p_8(t, x) dx \quad (19)$$

$$p_{11}(t, 0) = \lambda_2 \int_0^\infty p_6(t, x) dx + r_2 \int_0^\infty p_9(t, x) dx \quad (20)$$

$$p_{12}(t, 0) = \lambda_2 \int_0^\infty p_{10}(t, x) dx + \lambda_1 \int_0^\infty p_{11}(t, x) dx \quad (21)$$

and the initial conditions $p_0 = 1$; $p_1 = p_1(0, x) = \dots = p_2(0, x) = 0$. You can refer to [1] for the concrete meanings of the symbols, such as λ_0 and r_1 etc.

The equations (1)~(21) can be rewritten as an the abstract Cauchy problem (see [11–12]) in Banach space X :

$$(ACP) \quad \begin{cases} \frac{dp(t)}{dt} = \mathcal{A}p(t), & t \geq 0, \\ p(0) = (1, 0, \dots, 0)^T. \end{cases}$$

in which $\mathcal{A}$ is defined in [4] and $X = \mathbb{R}^2 \times (L^1(\mathbb{R}^+))^{11}$ is Banach space. For $p = (p_0, p_1, p_2(x), \dots, p_{12}(x)) \in X$, The norm of p is given by

$$\|p\| = |p_0| + |p_1| + \sum_{i=2}^{12} \int_0^\infty |p_i(x)| dx.$$

Let $\mu_4(x) = \mu(x)$, $\mu_5(x) = \beta(x)$, $\mu_6(x) = \eta(x)$. The main results of this paper are obtained under the following assumptions.

General Assumptions:

There exist positive constants H and c , such that for any $t \geq 0$,

$$\frac{1}{t} \int_t^\infty e^{-\int_0^u \mu_i(s) ds} du > H,$$

and

$$c = \min \left\{ \inf_{x \in \mathbb{R}^+} \mu_i(x), i = 1, 2, \dots, 6 \right\}.$$

The physical meaning of the general assumption is the mean repair rate is not too small. By the general assumptions, it is easily to obtain the following identity

$$\int_0^\infty u_i(x) e^{-\int_0^x u_i(\tau) d(\tau)} dx = 1.$$

The following theorems are main results of [4].

Theorem 1 $\mathcal{A}$ generates a positive C_0 contractive semigroup $T(t)$ on X and the unique time dependent positive solution to the equations (1)~(21) exists.

Theorem 2 Zero is an algebraically simple eigenvalue of $\mathcal{A}$ and the system is asymptotically stable. The steady-state solution is the positive unit eigenfunction corresponding eigenvalue 0 of the generator $\mathcal{A}$.

3 The exponential stability of the system

Let $T(t)$ be the semigroups generated by the operator $\mathcal{A}$ and $p(0)$ the initial value, the asymptotic stability shows that

$$T(t)p(0) = (p(0), Q)\tilde{p} + T_1(t)p(0) = \tilde{p} + T_1(t)p(0), \lim_{t \rightarrow \infty} T_1(t)p(0) = 0,$$

in which $\tilde{p}$ is the positive unit eigenfunction corresponding to eigenvalue 0 of $\mathcal{A}$ and $Q = (1, \dots, 1) \in X^*$.

In general, the above stability does not implies the exponential stability, that is to say there exist positive ω and M such that

$$\|T_1(t)p(0)\| \leq Me^{-\omega t}, \quad t \geq 0.$$

But in some special cases this implication holds. Now we will show the exponential stability of the system. First, we introduce some notations and results which come from references [3] and [14].

Let X be a Banach space and $T(t)$ be a c_0 semigroups on X and $\mathcal{A}$ its generator, for $\lambda \in \rho(\mathcal{A})$, then we have the symbols

$$\omega_1(T) = \lim_{t \rightarrow +\infty} \frac{\ln \|T(t)R(\lambda, \mathcal{A})\|}{t}, s_0(\mathcal{A}) = \sup \left\{ \omega > s(\mathcal{A}) : \sup_{Re \lambda = \omega} \|R(\lambda, \mathcal{A})\| < +\infty \right\},$$

in which $\omega_1(T)$ is the growth bound of the semigroups $T(t)$ and $s_0(\mathcal{A})$ is abscissa of uniform boundedness of the resolvent of $\mathcal{A}$. Their relationship is given by the following lemma which comes from [3].

Lemma 1 Let $T(t)$ be a c_0 semigroup on Banach space X and $\mathcal{A}$ its generator, then $\omega_1(T) < s_0(\mathcal{A})$.

The following Lemma comes from [14].

Lemma 2 For each $y \in X, b \in \mathbb{R}$ and $ib \in \mathbb{C}$, the unique solution to the equations $(ib - \mathcal{A})p = y$ exists and p is given by

$$(a + ib)p_0 = \int_0^\infty p_2(x)\mu_1(x)dx + \int_0^\infty p_3(x)\mu_2(x)dx + \int_0^\infty p_4(x)\mu(x)dx + \int_0^\infty \eta(x)[p_5(x) + p_6(x)]dx + \int_0^\infty p_7(x)\mu_3(x)dx + y_0 \quad (22)$$

$$(r_1 + ib)p_1 = \lambda_0 \int_0^\infty p_4(x)dx + y_1 \quad (23)$$

$$p_2(x) = p_2(0)e^{-\int_0^x [ib + \lambda_2 + \mu_1(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_2 + \mu_1(s)]ds} y_2(\tau) d(\tau) \quad (24)$$

$$p_3(x) = p_3(0)e^{-\int_0^x [ib + \lambda_1 + \mu_2(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_1 + \mu_2(s)]ds} y_3(\tau) d(\tau) \quad (25)$$

$$p_4(x) = p_4(0)e^{-\int_0^x [ib + \lambda_0 + \mu(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_0 + \mu(s)]ds} y_4(\tau) d(\tau) \quad (26)$$

$$p_5(x) = p_5(0)e^{-\int_0^x [ib + \lambda_1 + \lambda_2 + r_2 + \eta(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_1 + \lambda_2 + r_2 + \eta(s)]ds} y_5(\tau) d(\tau) \quad (27)$$

$$p_6(x) = p_6(0)e^{-\int_0^x [ib + \lambda_1 + \lambda_2 + \eta(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_1 + \lambda_2 + \eta(s)]ds} y_6(\tau) d(\tau) \quad (28)$$

$$p_7(x) = p_7(0)e^{-\int_0^x [ib + \mu_3(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \mu_3(s)]ds} y_7(\tau) d(\tau) \quad (29)$$

$$p_8(x) = p_8(0)e^{-\int_0^x [ib + r_2 + \mu_1(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + r_2 + \mu_1(s)]ds} y_8(\tau) d(\tau) \quad (30)$$

$$p_9(x) = p_9(0)e^{-\int_0^x [ib + r_2 + \mu_2(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + r_2 + \mu_2(s)]ds} y_9(\tau) d(\tau) \quad (31)$$

$$p_{10}(x) = p_{10}(0)e^{-\int_0^x [ib + \lambda_2 + \mu_1(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_2 + \mu_1(s)]ds} y_{10}(\tau) d(\tau) \quad (32)$$

$$p_{11}(x) = p_{11}(0)e^{-\int_0^x [ib + \lambda_1 + \mu_2(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \lambda_1 + \mu_2(s)]ds} y_{11}(\tau) d(\tau) \quad (33)$$

$$p_{12}(x) = p_{12}(0)e^{-\int_0^x [ib + \beta(s)]ds} + \int_0^\infty e^{-\int_\tau^x [ib + \beta(s)]ds} y_{12}(\tau) d(\tau) \quad (34)$$

$$p_2(0) = \lambda_1 p_0, \quad p_3(0) = \lambda_2 p_0 \quad (35)$$

$$p_4(0) = \lambda_2 \int_0^\infty p_2(x)dx + \lambda_1 \int_0^\infty p_3(x)dx \quad (36)$$

$$p_5(0) = \lambda_0 p_0 + r_1 p_1 + \int_0^\infty \mu_1(x)p_8(x)dx + \int_0^\infty \mu_2 p_9(x)dx \quad (37)$$

$$p_6(0) = r_2 \int_0^\infty p_5(x)dx + \int_0^\infty \mu_1 p_{10}(x)dx + \int_0^\infty \mu_2 p_{11}(x)dx \quad (38)$$

$$p_7(0) = \lambda_3 p_0, \quad p_8(0) = \lambda_1 \int_0^\infty p_5(x)dx, \quad p_9(0) = \lambda_2 \int_0^\infty p_5(x)dx \quad (39)$$

$$p_{10}(0) = \lambda_1 \int_0^\infty p_6(x)dx + r_2 \int_0^\infty p_8(x)dx \quad (40)$$

$$p_{11}(0) = \lambda_2 \int_0^\infty p_6(x) dx + r_2 \int_0^\infty p_9(x) dx \quad (41)$$

$$p_{12}(0) = \lambda_2 \int_0^\infty p_{10}(x) dx + \lambda_1 \int_0^\infty p_{11}(x) dx \quad (42)$$

Theorem 3 The system is exponentially stable.

Proof Let one rank projection P be the associated spectral projection of the eigenvalue 0 since 0 is an algebraically simple eigenvalue of $\mathcal{A}$. Thus X has the invariant decomposition

$$X = X_0 + Y, \quad X_0 = PX, \quad Y = (I - P)X,$$

in which X_0 is the eigenfunction subspace whose dimension is one. Y is also Banach space equipped with the norm of X . Let $T_1(t)$ and A_1 be the restrictions of $T(t)$ and A on Y respectively. Obviously, we have $\sigma(A_1) = \sigma(A) \setminus \{0\}$. Now we will show that $s_0(A_1) \leq 0$. However, 0 is a regular point of A_1 , so it is sufficient to show that

$$\sup_{\gamma=ib, |b|>\delta>0, b \in \mathbb{R}} \|R(\lambda, A_1)\| < +\infty.$$

In fact, for $r = ib, b \in \mathbb{R}$ and each given $y \in X$, from Theorem 2 the equation $(rI - A)p = y$ has solution $p \in D(A)$ and $p_i(x)$ is given by (17) and (18). Thus we have

$$\begin{aligned} & |b| \int_0^\infty |p_i(x)| dx \\ & \leq \int_0^\infty |p_i(0) r e^{-\int_0^x [r+\gamma_i+h_i(\xi)] d\xi}| dx + |b| \int_0^\infty \left| \int_0^\infty y_i(\tau) e^{-\int_\tau^x [r+\gamma_i+h_i(\xi)] d\xi} \right| dx \end{aligned} \quad (43)$$

$$\begin{aligned} & |b| \int_0^\infty |p_{2+i}(x)| dx \\ & \leq \int_0^\infty |p_{2+i}(0) r e^{-\int_0^x [r+h_i(\xi)] d\xi}| dx + \int_0^\infty |p_i(0) r (e^{-\int_0^x [r+h_i(\xi)] d\xi} - r e^{-\int_0^x [r+\gamma_i+h_i(\xi)] d\xi})| dx \\ & \quad + \frac{|b|}{c} \int_0^\infty |y_{2+i}(x)| dx + |b| \int_0^\infty \left| \int_0^x y_i(\tau) (e^{-\int_\tau^x [r+h_i(\xi)] d\xi} - e^{-\int_\tau^x [r+\gamma_i+h_i(\xi)] d\xi}) d\tau \right| dx \end{aligned} \quad (44)$$

From (13), we have the following inequality

$$(|b| + \gamma)|p_0| \leq |y_0| + \sum_{i=1}^2 \left| \int_0^\infty p_i(0) h_i(x) e^{-\int_0^\infty (r+\gamma_i+h_i(\xi)) d\xi} dx \right| + \sum_{i=1}^2 \int_0^\infty |y_i(x)| dx \quad (45)$$

Moreover, by the boundary conditions (14)~(16) we have

$$\begin{aligned} |p_1(0)| & \leq \lambda_1 |p_0| + \int_0^\infty |p_2(0)(1 - r e^{-\int_0^x [r+h_2(\xi)] d\xi} - h_2(x) e^{-\int_0^x [r+h_2(\xi)] d\xi})| dx \\ & \quad + \int_0^\infty |p_4(0)(1 - r e^{-\int_0^x [r+h_2(\xi)] d\xi})| dx + 2 \int_0^\infty |y_2(x)| dx + \int_0^\infty |y_4(x)| dx \end{aligned} \quad (46)$$

$$\begin{aligned} |p_2(0)| & \leq \lambda_2 |p_0| + \int_0^\infty |p_1(0)(1 - r e^{-\int_0^x [r+h_1(\xi)] d\xi} - h_1(x) e^{-\int_0^x [r+h_1(\xi)] d\xi})| dx \\ & \quad + \int_0^\infty |p_3(0)(1 - r e^{-\int_0^x [r+h_1(\xi)] d\xi})| dx + 2 \int_0^\infty |y_1(x)| dx + \int_0^\infty |y_3(x)| dx \end{aligned} \quad (47)$$

If adding (43)~(47) together, we would have the following estimate

$$|b| \left(|p_0| + \sum_{i=1}^4 \int_0^\infty |p_i(x)| dx \right) \leq |y_0| + \left(1 + \frac{|b|}{c} \right) \sum_{i=1}^4 \int_0^\infty |y_i(x)| dx + 2 \sum_{i=1}^2 \int_0^\infty |y_i(x)| dx \quad (48)$$

By the definition of the norm on the space X and (48), we obtain that

$$\|R(rI - A_1)\| \leq \frac{1}{|b|} \left(2 + \frac{|b|}{c} \right).$$

Therefore we know that,

$$\lim_{b \rightarrow +\infty} \|R(rI - A_1)\| = \frac{1}{c},$$

and

$$\sup_{r=ib, |b| > \delta > 0, b \in R} \|R(rI - A_1)\| < +\infty.$$

From the definition of $s_0(A)$ and Lemma 2, we obtain that $s_0(A_1) \leq 0$ and $\omega_1(T_1) < 0$ due to $\omega_1(T_1) < s_0(A_1)$. However, by [14] we know semigroups $T(t)$ is exponentially stable if and only if $\omega_1(T) < 0$. Thus we complete the proof of Theorem 3.

In fact, the exponential stability of the repairable system was firstly given in [14]. You can refer to the reference [14] in the modified version for more information. To show the effectiveness of the reliability of the repair system, we include this part again in this note. The method is mainly employed from [3].

4 Some steady-state reliability indices

According to [2],

$$p_0(t) + \sum_{i=2}^6 \int_0^\infty p_i(x, t) dx + \sum_{i=8}^{11} \int_0^\infty p_i(x, t) dx$$

and

$$AV = \lim_{t \rightarrow \infty} \left(p_0(t) + \sum_{i=2}^6 \int_0^\infty p_i(x, t) dx + \sum_{i=8}^{11} \int_0^\infty p_i(x, t) dx \right)$$

are the instantaneous availability and the stationary availability of the repairable system, respectively. The availability of the system is one of the most important reliability indices and engineers are especially interested in the steady-state availability. Now we will obtain the stationary availability of the system with another method different from the traditional one. For convenience, set

$$\begin{aligned} f_2(x) &= e^{-\int_0^x [\mu_1(s) + \lambda_2] ds}, & a_2 &= \int_0^\infty \mu_1(x) f_2(x) dx, & b_2 &= \int_0^\infty f_2(x) dx, \\ f_3(x) &= e^{-\int_0^x [\mu_2(s) + \lambda_1] ds}, & a_3 &= \int_0^\infty \mu_2(x) f_3(x) dx, & b_2 &= \int_0^\infty f_3(x) dx, \\ f_4(x) &= e^{-\int_0^x [\mu(s) + \lambda_0] ds}, & a_4 &= \int_0^\infty \mu(x) f_4(x) dx, & b_4 &= \int_0^\infty f_4(x) dx, \\ f_5(x) &= e^{-\int_0^x [\eta(s) + \lambda_1 + \lambda_2 + r_2] ds}, & a_5 &= \int_0^\infty \eta(x) f_5(x) dx, & b_5 &= \int_0^\infty f_5(x) dx, \\ f_6(x) &= e^{-\int_0^x [\eta(s) + \lambda_1 + \lambda_2] ds}, & a_6 &= \int_0^\infty \eta(x) f_6(x) dx, & b_6 &= \int_0^\infty f_6(x) dx, \\ f_7(x) &= e^{-\int_0^x \mu_3(s) ds}, & b_7 &= \int_0^\infty f_7(x) dx, \\ f_8(x) &= e^{-\int_0^x [\mu_1(s) + r_2] ds}, & a_8 &= \int_0^\infty \mu_1(x) f_8(x) dx, & b_8 &= \int_0^\infty f_8(x) dx, \end{aligned}$$

$$f_9(x) = e^{-\int_0^x [\mu_2(s) + r_2] ds}, \quad a_9 = \int_0^\infty \mu_2(x) f_9(x) dx, \quad b_9 = \int_0^\infty f_9(x) dx,$$

$$f_{12}(x) = e^{-\int_0^x \beta(s) ds}, \quad b_{12} = \int_0^\infty f_{12}(x) dx.$$

Theorem 4 The stationary availability of the system is

$$AV = \frac{1 + \sum_{i=4}^6 b_i c_i + \sum_{i=8}^9 b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2)}{1 + c_1 + \sum_{i=4}^6 b_i c_i + \lambda_3 b_7 + \sum_{i=8}^9 b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2) + b_{12} c_{12}} \quad (49)$$

in which

$$c_1 = r_1^{-1} \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3), \quad c_4 = \lambda_1 \lambda_2 (b_2 + b_3),$$

$$c_5 = [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)],$$

$$c_8 = \lambda_1 b_5 [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)],$$

$$c_9 = \lambda_2 b_5 [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)],$$

$$c_6 = r_2 (b_5 c_5 + a_2 b_8 c_8 + a_3 b_9 c_9) [1 - b_6(\lambda_1 a_2 + \lambda_2 a_3)]^{-1},$$

$$c_{10} = \lambda_1 b_6 c_6 + r_2 b_8 c_8, \quad c_{11} = \lambda_2 b_6 c_6 + r_2 b_9 c_9, \quad c_{12} = \lambda_2 b_2 c_{10} + \lambda_1 b_3 c_{11},$$

Proof In order to compute the stationary availability of the discussed system, we need to obtain the formulation of the positive unit eigenfunction $\tilde{p}$ corresponding to eigenvalue 0 of the operator $\mathcal{A}$. Now we complete this job. In fact, $\mathcal{A}p = 0$ is the following set of equations:

$$\begin{aligned} p'_2(x) &= [\mu_1(x) + \lambda_2] p_2(x), & p'_3(x) &= [\mu_2(x) + \lambda_1] p_3(x), \\ p'_4(x) &= [\mu(x) + \lambda_0] p_4(x), & p'_5(x) &= [\eta(x) + \lambda_1 + \lambda_2 + r_2] p_5(x), \\ p'_6(x) &= [\eta(x) + \lambda_1 + \lambda_2] p_6(x), & p'_7(x) &= \mu_3(x) p_7(x), \\ p'_8(x) &= [\mu_1(x) + r_2] p_8(x), & p'_9(x) &= [\mu_2(x) + r_2] p_9(x), \\ p'_{10}(x) &= [\mu_1(x) + \lambda_2] p_{10}(x), & p'_{11}(x) &= [\mu_2(x) + \lambda_1] p_{11}(x) \end{aligned} \quad (50)$$

with the initial conditions:

$$r_1 p_1 = \lambda_0 \int_0^\infty p_4(x) dx, \quad p_2(0) = \lambda_1 p_0, \quad p_0(0) = \lambda_2 p_0 \quad (51)$$

$$p_4(0) = \lambda_2 \int_0^\infty p_2(x) dx + \lambda_1 \int_0^\infty p_3(x) dx \quad (52)$$

$$p_5(0) = \lambda_0 p_0 + r_1 p_1 + \int_0^\infty \mu_1(x) p_8(x) dx + \int_0^\infty \mu_2(x) p_9(x) dx \quad (53)$$

$$p_6(0) = r_2 \int_0^\infty p_5(x) dx + \int_0^\infty \mu_1(x) p_{10}(x) dx + \int_0^\infty \mu_2(x) p_{11}(x) dx \quad (54)$$

$$p_7(0) = \lambda_3 p_0, \quad p_8(0) = \lambda_1 \int_0^\infty p_5(x) dx, \quad p_8(0) = \lambda_2 \int_0^\infty p_5(x) dx \quad (55)$$

$$p_{10}(0) = \lambda_1 \int_0^\infty p_6(x) dx + r_2 \int_0^\infty p_8(x) dx \quad (56)$$

$$p_{11}(0) = \lambda_2 \int_0^\infty p_6(x) dx + r_2 \int_0^\infty p_9(x) dx \quad (57)$$

$$p_{12}(0) = \lambda_2 \int_0^\infty p_{10}(x) dx + \lambda_1 \int_0^\infty p_{11}(x) dx \quad (58)$$

Solving equations (50), we have

$$\begin{aligned} p_i(x) &= p_i(0)f_i(x), i = 2, 3, \dots, 9, \quad p_{10}(x) = p_{10}(0)f_2(x) \\ p_{11}(x) &= p_{11}(0)f_2(x), \quad p_{12}(x) = p_{12}(0)f_{12}(x) \end{aligned} \quad (59)$$

By the Theorem 4.1 of [1], because 0 is a simple eigenvalue of $\mathcal{A}$ and there exists a corresponding positive eigenvector, we can set $p_0 = 1$. With the help of

$$\int_0^\infty u_i(x) e^{-\int_0^x u_i(\tau) d\tau} dx = 1$$

and the notations of c_i , it is easy to know that

$$p_1 = c_1, p_2(0) = \lambda_1, p_3(0) = \lambda_2, p_i(0) = c_i (i = 4, 5, 6), p_7(0) = \lambda_3,$$

and

$$p_j(0) = c_j (j = 8, 9, \dots, 12).$$

Thus,

$$P = (1, c_1, \lambda_1 f_2(x), \lambda_2 f_3(x), c_4 f_4(x), c_5 f_5(x), c_6 f_6(x), \lambda_3 f_7(x), \\ c_8 f_8(x), c_9 f_9(x), c_{10} f_{10}(x), c_{11} f_{11}(x), c_{12} f_{12}(x))$$

is the positive eigenfunction corresponding to eigenvalue 0 of the operator $\mathcal{A}$ and $\tilde{p} = \|P\|^{-1}P$. Therefore, we can get the expression $AV = 1 - (c_1 + b_7 c_7 + c_{12} c_{12}) \|P\|^{-1}$ by the definition of the stationary availability, that is the formula (49) due to

$$\|P\| = 1 + c_1 + \sum_{i=4}^6 b_i c_i + \lambda_3 b_7 + \sum_{i=8}^9 b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2) + b_{12} c_{12}.$$

We complete the proof of the Theorem 4. With the help of Theorem 4, we are able to calculate the mean up-time of the redundant system.

Theorem 5 The mean up-time (MUT) of the redundant system is equal to

$$MUT = \frac{1 + \sum_{i=4}^6 b_i c_i + \lambda_3 b_7 + \sum_{i=8}^9 b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2)}{\lambda_0(1 + b_4 c_4) + \lambda_2 b_2(c_{10} + \lambda_1) + \lambda_1 b_3(c_{11} + \lambda_2) + (\lambda_1 + \lambda_1)(b_5 c_5 + b_6 c_6) + \lambda_3^2 b_7}.$$

Proof Let M be the rate of failures of the system, then

$$M = \|p\|^{-1} [\lambda_0(1 + p_4) + \lambda_2(p_2 + p_{10}) + \lambda_1(p_3 + p_{11}) + (\lambda_1 + \lambda_1)(p_5 + p_6) + \lambda_3 b_7].$$

From the proof of the Theorem 4, we know $p_2 = \lambda_1 b_2, p_3 = \lambda_2 b_3, p_7 = \lambda_3 b_2, p_i = c_i b_i, i = 4, 5, 6, p_{10} = c_{10} b_2$, and $p_{11} = c_{11} b_3$. However, the mean up-time (MUT) of the redundant system is $\frac{AV}{M}$. Computing directly leads to the formulation of MUT .

Corollary 1 The stationary availability and MUT of the system without standby are

$$AV' = \frac{1}{1 + \lambda_3 b_7 + \lambda_0 b_{12}}, \quad MUT' = \frac{1}{\lambda_0 + \lambda_3^2 b_7}.$$

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