Pricing Defaultable Securities under Actual Probability Measure

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Abstract In this paper, a new approach is developed to estimate the value of defaultable securities under the actual probability measure. This model gives the price framework by means of the method of backward stochastic differential equation. Such a method solves some problems in most of existing literatures with respect to pricing the credit risk and relaxes certain market limitations. We provide the price of defaultable securities in discrete time and in continuous time respectively, which is favorable to practice to manage real credit risk for finance institutes.

Keywords credit risk; market price of risk; backward stochastic differential equation; risk-adjusted discount value

1 Introduction

Credit risk, one of the most pervasive threats in today's financial markets, cannot be completely diversified away. How to manage the credit risk has become the focus in Basel II and Basel III. Researchers have developed many models to estimate default probability and price defaultable securities like [1, 4, 5, 13, 16], especially after financial crisis in 2008, most of scholars began to be concerned about correlated default^[7, 8]. Most of them discuss credit risk under risk neutral probability measure, which makes it is difficult to calibrate model parameters by market data and difficult to use these models to price the securities not traded. On the other hand, The models applied to manage credit risk by financial institution are still some simple models such as KMV based on Black-Scholes-Merton^[15] and CreditMetrics model developed by Morgan^[9] which can't satisfy the needs of investors.

Here we will give a new approach extended from [3] to price defaultable securities (like contingent securities, defaultable corporate debt or securities) under real probability measure. In

Received March 4, 2014, accepted April 29, 2014

supported by Humanity and Social Science Youth foundation of Ministry of Education of China (Grant No. 12YJCZH045); the Fundamental Research Funds for the Central Universities in UIBE (Grant No. CXTD4-03); the Key Project of the National Social Science (Grant No. 11AZD010)

[3], the author mainly presented the value of life insurance contracts without continuous market risk. We consider how to price the defaultable securities which have both diffusion risk^[13] and default risk and the price framework is more generalized. If we match this approach to existing default models, it is a kind of intensity-based model, thus the default time is completely unpredictable. In addition, this approach frames the price process of defaultable securities by the backward stochastic differential equation (BSDE). It looks the defaultable security's value as the discount value of the cash flow happened during its lifespan, here we call it Risk-Adjusted Value (RA-value).

In this paper, we reference two market price processes of risk, i.e. market price of continuous market risk and market price of default risk, to compensate the holder of defaultable securities appropriately. The two risk prices can be estimated from the financial market data and there have existed several literatures to calculate them (like Hull^[10], Giesecke^[13]). It is interesting that they also prescribe the mapping between probabilities under the physical measure and an equivalent martingale measure of it if we strengthen some conditions of the model. This conclusion is consistent with Giesecke^[13]. Given the two market prices of risk (deterministic or stochastic), the hazard rate (or intensity) of default time under real probability measure and the risk-free rate, we can estimate the price of any defaultable security under real probability measure. Further when the security is a tradable one, its price given in this study is no arbitrage. On the other hand, noticing it doesn't need the exists of risk neutral probability, the conditions model parameters must satisfy will be certainly relaxed. Finally, for easy to apply the model in practice, we will give the security's price framework in discrete time and continuous time respectively.

The price formulas given here are mainly the price of corporate securities, corporate debt, or Mortgage based securities, which is called defaultable security for simplicity. Those securities may have zero-coupon or zero recovery, which doesn't affect our results. In addition, this model can also be used to price other credit derivatives, like CDS, CDO, and others.

The paper is made as follows: In section 2, the basic concepts and notations are given. In section 3, results for the price of defaultable securities in discrete time and their proofs are presented. And the price framework in continuous time and their proofs are given in section 4. In section 3 and section 4, we will prove the prices are no arbitrage by strengthening some conditions of model's parameters. the last section we will give a summary of this research.

2 Modelling the value of defaultable securities

Model the uncertainty in the capital market with a completed filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mathbb{P})$, where T is a given positive number representing the largest possible life of defaultable securities considered. Ω denotes the state space, including all possible pathes the securities price moves as time goes on. $\mathcal{G}$ is a σ -field, representing measurable events in the state space, $\mathbb{P}$ is the actual probability measure and the information available at time t is captured by the σ -field $\mathcal{G}_t$. $\mathbb{G} = \{\mathcal{G}_t : t \in [0, T]\}$ is a standard filtration on $(\Omega, \mathcal{G}, \mathbb{P})$. All filtration are assumed to satisfy the 'usual conditions' of right-continuity and completeness.

Let $H = \{H_t\}_{0 \leq t \leq T}$ be a right-continuous nondecreasing $\mathcal{G}_t$ -adapted process valued in $\{0, 1\}$, and denote by τ a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{P})$,

$$\tau = \begin{cases} \min\{t \leq T : H_t = 1\}, & \{t \leq T : H_t = 1\} \neq \emptyset, \\ +\infty, & H_t = 0, \forall t \leq T, \end{cases}$$

satisfying: $\mathbb{P}(\tau = 0) = 0$, $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, T]$. Then $H_t = 1_{(\tau \leq t)}$, and H is a one-jump process with the form

$$H_t = M_t + A_t \quad (1)$$

Where (A_t) is the compensator of (H_t) and (M_t) is a $\mathbb{G}$ -martingale under measure $\mathbb{P}$. In fact, H describe whether the security defaults in future and M denotes the default risk. Let $\mathbb{H}$ be the associated filtration: $\mathcal{H}_t = \sigma(H_u : u \leq t)$, $0 \leq t \leq T$ and assume we are given an auxiliary filtration $\mathbb{F}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$, for any $t \in [0, T]$.

Let a standard Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$ $W = \{W_t : 0 \leq t \leq T\}$ denote continuous market risk (diffusion risk^[13]). We are common to suppose $\mathbb{F}$ is the natural filtration of W .

Let $S^1, S^2, \dots, S^n$ be traded risk securities, and the risks include the continuous market risk and default event risk, both of which can't be diversified away. It is a standard economic principle that those risks undiversified commands premiums. Generally, premiums are proportionability to the price volatility induced by the corresponding risk. We call the proportion market price of risk. Here, we denote by process $\hat{\eta} = \{\eta_t\}_{0 \leq t \leq T}$, $\tilde{\eta} = \{\tilde{\eta}_t\}_{0 \leq t \leq T}$ the relative market prices with respect to the risks M and W , respectively. It is reasonable to assume $\hat{\eta}$ is a $\mathbb{F}$ -predictable process irrelevant to default information and $\tilde{\eta}$ is a $\mathbb{G}$ -predictable process, which is affected by both of common market information and default information. Let $S = \{S_t = (S_t^i)_{n \times 1} : 0 \leq t \leq T\}$ are given by the following stochastic differential equation (SDE):

$$\begin{cases} dS_t^i = r_t S_t^i dt + \alpha^i(t) \hat{\eta}_t dt + \beta^i(t) \tilde{\eta}_t dA_t + \alpha^i(t) dW_t + \beta^i(t) dM_t - dB_t^{si} \\ S_0^i = s_0^i \end{cases} \quad (2)$$

for $i = 1, 2, \dots, n$, where $(a_i)_{n \times 1}$ represents a n -dimensional vector. B^{si} is an adapted finite variation process, which is cash flow associated with S^i . $r = \{r_t : 0 \leq t \leq T\}$, is a non-negative $\mathbb{F}$ -adapted integrable process representing default-free spot rate. $\beta = \{\beta_t = (\beta_t^i)_{n \times 1} : 0 \leq t \leq T\}$, $\alpha = \{\alpha_t = (\alpha_t^i)_{n \times 1} : 0 \leq t \leq T\}$ are all $(\mathcal{G}_t)$ -adapted measurable processes such that (1) has a unique strong solution. An element of S is called a market index if it is the price of the traded security in the sense of "securitization" (see [17]). Suppose there exists a saving account at the default-free spot rate $r = \{r_t : 0 \leq t \leq T\}$. Using those market indexes and saving account, we can calculate $\hat{\eta}$, $\tilde{\eta}$. Therefore we suppose $\hat{\eta}$, $\tilde{\eta}$ are given.

Remark The SDE(1) has obvious economic meanings. We interpret it by S^1 for an example. Because the index S^1 includes two system risks W and M comparing with saving account, the S^1 holder should be compensated for the predictable parts of the two risks except for the risk free income $r_t S_t^1 dt$ in $[t, t+dt]$. Since the compensator of quadratic variance $[M, M]$ is $\langle M, M \rangle_t = \langle M, M \rangle_t = \int_0^t (1 - \Delta A_s) dA_s$ and that of $[W, W]$ is $\langle W, W \rangle_t = \langle W, W \rangle_t = [W, W]_t = t$. If we suppose A is continuous, variances predictable induced by a unit continuous market risk and a unit default risk in $[t, t+dt]$ are $d\langle W, W \rangle_t = dt$, $d\langle M, M \rangle_t = dA_t$. Compacting with the market prices of risk $\hat{\eta}$, $\tilde{\eta}$ and the volatilities α^1 , β^1 , we should compensate

the security holders much as

$$\alpha^i(t)\hat{\eta}_t dt + \beta^i(t-)\tilde{\eta}_t dA_t$$

in $[t, t+dt]$. While $\alpha^i(t)dW_t + \beta^i(t-)dM_t$ are the stochastic part induced by W and M between t and $t+dt$. Because the price of S^1 considered at time t is the price immediately after the payment at that time is paid, so we subtract dB_t^{s1} .

Consider a new defaultable security like corporate security (it may be not tradable), which can be represented by the cash flow in its lifespan. Assume the cash flow is an $(\mathcal{G}_t)$ -adapted finite variation process B given by the following SDE,

$$dB_t = b_t dt + c(t-)\Delta H_t + X(1 - H_T)d1_{(t \geq T)} + X'H_T d1_{(t \geq T)} \quad (3)$$

where $b_t dt$ represents the coupon or dividends paid in $[t, t+dt]$, $c(t-)\Delta H_t$ represents the recovery received at default time by the security holders when default event happens prior to or at the maturity, X is the face value of the security paid to the buyers when the security dues, and X' is the recovery paid at maturity if default event happens prior to or at the maturity. In this paper, we assume the recovery will be paid at default time, i.e. $X' = 0$.

Definition 2.1 For a cash flow B given in (3), if there exists a pair of $\mathbb{G}$ -adapted measurable process $\{(V, \sigma^B) = (V(t), \hat{\sigma}(t), \tilde{\sigma}(t)) : V(t) \geq 0, 0 \leq t \leq T\}$ satisfying the following backward stochastic differential equation (BSDE),

$$\begin{cases} dV_t = V_t r_t dt + \hat{\eta}_t \hat{\sigma}_t dt + \tilde{\sigma}_t \tilde{\eta}_t dA_t + \hat{\sigma}_t dW_t + \tilde{\sigma}_t dM_t - dB_t \\ V_T = X1_{(\tau > T)} \end{cases} \quad (4)$$

then (V, σ^B) is called a Risk-Adjusted discount (RA-discount) of the cash flow B under $\hat{\eta}, \tilde{\eta}$ and r .

The integral form of BSDE(4) is as follows

$$V_{t_1} = V_{t_2} - \int_{(t_1, t_2]} (V(t)r_t + \hat{\sigma}_t \hat{\eta}_t) dt + \int_{(t_1, t_2]} \tilde{\sigma}_t \tilde{\eta}_t dA_t - \int_{(t_1, t_2]} \hat{\sigma}_t dW_t - \int_{(t_1, t_2]} \tilde{\sigma}_t dM_t + \int_{(t_1, t_2]} dB_t \quad (5)$$

for any $t_1 \leq t_2 < T$.

To ensure the existence of a unique RA-discount, we will research a particular class of securities.

Definition 2.2 For a given security with cash flow B , if there exists a unique RA-discount $\{(V, \sigma^B) = (V(t), \hat{\sigma}(t), \tilde{\sigma}(t)) : 0 \leq t \leq T\}$ of B such that $V_t = \hat{\sigma}_t = \tilde{\sigma}_t = 0, \forall t \geq \tau$, then (V, σ^B) is called to be the Risk Adjusted discount value (RA-value) of the security.

Remarks 1) The security value at time t defined above is the value at the moment immediately after the coupons or recovery at t are paid. Most of existing literature calculate the security value before the recovery is paid, which is equal to $V_t + c_{t-}\Delta H_t$ here. In addition, the recovery are commonly assumed to be proportional to the security value and this proportion is called recovery rate^[11, 12, 14]. We generalize the definition of recovery here and denote by $\{c_t\}$.

2) We will not distinguish a defaultable security from its cash flow. Given a security B , let $(V(t), \hat{\sigma}(t), \tilde{\sigma}(t))$ be the RA-value of B . Set

$$\hat{\delta}_t = \begin{cases} \frac{\hat{\sigma}_t}{V_t}, & V_t \neq 0, \\ 0, & V_t = 0, \end{cases} \quad \tilde{\delta}_t = \begin{cases} \frac{\tilde{\sigma}_t}{V_t}, & V_t \neq 0, \\ 0, & V_t = 0. \end{cases}$$

They give the fluctuation intensity of the value V_t of B and thus measure the risks of B . Let $R^B = (R_t^B : 0 \leq t \leq T)$ be a Itô process given by the following SDE,

$$dR_t^B = r_t dt + \hat{\delta}_t \hat{\eta}_t dt + \tilde{\delta}_t \tilde{\eta}_t dA_t + \hat{\delta}_t dW_t + \tilde{\delta}_t dM_t.$$

Then, formula (5) can be verified as for any $t_1 \leq t_2 \leq T$,

$$\int_{(t_1, t_2]} V_t dR_t^B = V_{t_2} - V_{t_1} + B_{t_2} - B_{t_1}.$$

That is, R^B is the instantaneous growth (return) process in the value of B . Consider two securities B^1 and B^2 , suppose that they are valued by their RA-discounts $(V_t^i, \hat{\sigma}_t^i, \tilde{\sigma}_t^i)$, $i = 1, 2$ under the same market price of risk $\hat{\eta}, \tilde{\eta}$ and default-free spot rate r . If $\hat{\delta}^1 = \hat{\delta}^2$, $\tilde{\delta}^1 = \tilde{\delta}^2$, we have $R^{B^1} = R^{B^2}$. In other words, under the RA-valuation, two securities with the same risks will have the same growth (return) rate in their value. Therefore the RA-valuation gives a fair valuation or pricing system without ambiguity.

3 Pricing discrete defaultable securities

In this section, we consider securities those pay coupons or recoveries only at the moments $0 = t_0 < t_1 < \dots < t_N = T$, where $t_i = ih$, $i = 0, 1, \dots, N$, N is a positive integer and $Nh = T$. Assume all time variables in this subsection are valued on $\{t_0, t_1, \dots, t_N\}$. Our pricing model can be restated as follows:

$\omega^1 := \{1, -1\}^N = (\omega_1^1, \omega_2^1, \dots, \omega_N^1)$ where $\omega_i^1 = 1$ or -1 ;

$\Omega^1 := \{\omega^1\} = \{(\omega_1^1, \omega_2^1, \dots, \omega_N^1)\}$;

$\mathcal{F} := 2^{\Omega^1}$;

$\mathbb{F}$: a filtration defined as $\mathcal{F}_0$ is a trivial σ -field on Ω^1 ,

$$\mathcal{F}_t = \sigma\{(\omega_1^1, \dots, \omega_{t/h}^1, \omega_{t/h+1}^1, \dots, \omega_N^1) = (x_1, \dots, x_{t/h}, \omega_{t/h+1}^1, \dots, \omega_N^1), x_i = 1 \text{ or } -1\},$$

for any $t > 0$, thus $\mathcal{F} = \mathcal{F}_N$.

$\mathbb{P}^1$: a probability measure on the space Ω^1 , such that

$$\begin{aligned} \mathbb{P}^1(\{\omega^1\}) &= \frac{1}{2^N} \\ &= \prod_{i=1}^N \mathbb{P}^1(\{(\omega_1^1, \dots, \omega_i^1, \omega_{i+1}^1, \dots, \omega_N^1) = (\omega_1^1, \dots, \omega_{i-1}^1, 1, \omega_{i+1}^1, \dots, \omega_N^1)\}), \end{aligned}$$

i.e. ω_i^1 and ω_j^1 are independent each other when $i \neq j$, and for any $1 \leq i \leq N$

$$\begin{aligned} &P^1(\{(\omega_1^1, \dots, \omega_i^1, \omega_{i+1}^1, \dots, \omega_N^1) = (\omega_1^1, \dots, \omega_{i-1}^1, 1, \omega_{i+1}^1, \dots, \omega_N^1)\}) \\ &= P^1(\{(\omega_1^1, \dots, \omega_i^1, \omega_{i+1}^1, \dots, \omega_N^1) = (\omega_1^1, \dots, \omega_{i-1}^1, -1, \omega_{i+1}^1, \dots, \omega_N^1)\}) \\ &= \frac{1}{2}. \end{aligned} \tag{6}$$

The above probability space $(\Omega^1, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^1)$ is a completed filtered probability space, and we denote by it the market information captured. ω^1 can be looked on as a path of a binomial tree with N steps, and Ω^1 is the set of all possible pathes of the binomial tree. The binomial tree represents the path where the price of a security possibly moves by as time goes on before the security defaults. For a path $\omega^1 = (\omega_1^1, \omega_2^1, \dots, \omega_N^1)$ and for any $i = 1, 2, \dots, N$,

$\omega_i^1 = 1$ implies the security price will grow up between t_{i-1} and t_i , otherwise $\omega_i^1 = -1$ implies the security price will drop down between t_{i-1} and t_i .

For any $\omega^1 = (\omega_1^1, \dots, \omega_N^1) \in \Omega^1$, denote $\omega^1(t) = \omega_{t/h}^1$, then $E^1 \omega^1(t) = 0$, $E^1(\omega^1(t))^2 = 1$. Define a process W as

$$W_t(\omega^1) = \sum_{i=1}^{t/h} (\omega^1(t_i) - E(\omega^1(t_i)))\sqrt{h}, \quad 0 < t \leq T \quad (7)$$

and $W_0 \equiv 0$, then W is a square integrable $\mathbb{F}$ -martingale under $\mathbb{P}^1$, and the sharp bracket process $\langle W, W \rangle(t)$ is the unique predictable increasing process such that $W_t^2 - \langle W, W \rangle(t)$ is a martingale, where $\langle W, W \rangle(t) = [W, W](t) = E^1(W_t^2) = t$. That is to say, t is the compensator of $[W, W]_t$. It is easy to prove when h tends to zero from positive direction, W is just a standard Brownian motion under $\mathbb{P}^1$.

In addition, let

$\{\omega_t^2(\cdot), 0 \leq t \leq T\}$: a non-decreasing function family, such that $\omega_t^2(t-h) = 0, \omega_t^2(t) = 1, \quad 0 < t \leq T$ and $\omega_0^2(t) = 0, \forall t \in \{t_0, t_1, \dots, t_N\}$;

$\Omega^2 := \{\omega^2\} = \{\omega_t^2, t = t_0, t_1, \dots, t_N\}$;

$\mathcal{H} := 2^{\Omega^2}$;

H : a nondecreasing process which is defined as: $H_t(\omega^2) := \omega^2(t) = \omega_s^2(t)$ if $\omega^2 = \omega_s^2$, then we have $H_0(\omega) \equiv 0, H_t(\omega_0^2) = 0, \forall 0 \leq t \leq T$;

$\mathbb{H}$: the associated filtration of H , i.e. $\mathcal{H}_t = \sigma(H_u, u \leq t)$. Then $\mathcal{H} = \mathcal{H}_T$.

Given a sequence number $\{q_{t_1}, q_{t_2}, \dots, q_{t_N}\}$, where $0 < q_{t_i} < \frac{1}{h}, i = 1, 2, \dots, N$, we can define a probability measure $\mathbb{P}^2$ such that for any $0 < t \leq T$, $\mathbb{P}^2(H_t = 1 | H_{t-h} = 0) / h = q_t$ and $\mathbb{P}^2(H_0 = 0) = 1$.

The above probability space $(\Omega^2, \mathcal{H}, \{\mathcal{H}_t\}_{0 \leq t \leq T}, \mathbb{P}^2)$ is another completed filtered probability space, we denote by it the default information captured.

Let

$$\tau(\omega) := \begin{cases} \min\{t : H_t = 1\}, & \omega^2 \neq \omega_0^2 \\ +\infty, & \omega^2 = \omega_0^2 \end{cases}$$

then $1_{(\tau(\omega) \leq t)} = H_t(\omega)$. If we denote by M a process : $M_t = 1_{(\tau \leq t)} - \sum_{(s \leq t)} q_{s \wedge \tau} h, \forall t > 0$ and $M_0 = 0$, it is easy to see that M is a $\mathbb{H}$ -martingale under $\mathbb{P}^2$ on the probability space $(\Omega^2, \mathcal{H}, \{\mathcal{H}_t\}_{0 \leq t \leq T}, \mathbb{P}^2)$.

Now, let $\Omega = \Omega^1 \times \Omega^2, \mathcal{G} = \mathcal{F} \otimes \mathcal{H}, \mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P} = \mathbb{P}^1 \otimes \mathbb{P}^2$, where $\mathbb{P}$ satisfies

$$\mathbb{P}(\omega^1) = \mathbb{P}^1(\omega^1), \mathbb{P}(H_t = 1 | H_{t-h} = 0) = \mathbb{P}^2(H_t = 1 | H_{t-h} = 0) = q_t h, \forall t \in (0, T],$$

and for any $\omega \in \Omega$, it has the form $\omega = (\omega^1, \omega^2)$. Obviously, W and M are also $\mathbb{G}$ -martingales under $\mathbb{P}$ and $\langle W, W \rangle(t) = t, \langle M, M \rangle(t) = \sum_{(s \leq t)} q_{s \wedge \tau} h$ on $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{0 \leq t \leq T}, \mathbb{P})$. So we have

$$\Delta \langle W, W \rangle(t) = \langle W, W \rangle(t+h) - \langle W, W \rangle(t) = h,$$

$$\Delta \langle M, M \rangle(t) = \langle M, M \rangle(t+h) - \langle M, M \rangle(t) = 1_{(\tau > t)} q_{t+h} h.$$

Now consider the following security on $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{0 \leq t \leq T}, \mathbb{P})$: at time $t = t_i, i = 1, 2, \dots, N$, the coupon of b_t is paid to a security holder in case of $H_t = 0$, otherwise the recovery of c_t is paid to the security holder at the moment t if $H_t = 1, H_{t-h} = 0$, and at maturity, if $H_T = 0$,

the face value X is received by the holder. Let $B_0 = 0$,

$$B_t = b_t 1_{(H_t=0)} + c_t 1_{(H_t=1, H_{t-h}=0)} + X 1_{(H_t=0)} 1_{(t=T)}$$

for any $t \in \{t_1, t_2, \dots, t_N\}$, define $\pi_t := b_t + X 1_{(t=T)}$, then

$$B_t = \pi_t 1_{(H_t=0)} + c_t 1_{(H_t=1, H_{t-h}=0)}.$$

The defaultable security is thus represented by the random cash flow $B = \{B_t, t = t_0, \dots, t_N\}$ and its RA-value will be calculated under a given default-free spot rate $r = \{r_t : t = t_0, t_1, \dots, t_{N-1}\}$ and two relative market price of risks $\hat{\eta} = \{\hat{\eta}_t : t = t_0, t_1, \dots, t_{N-1}\}$ and $\tilde{\eta} = \{\tilde{\eta}_t : t = t_0, t_1, \dots, t_{N-1}\}$. $r_t, \hat{\eta}_t$ and $\tilde{\eta}_t$ represent annual risk free rate and market price of risks at t respectively. Suppose $r, \hat{\eta}$ are $\mathbb{F}$ -adapted processes and $\tilde{\eta}$ is a $\mathbb{H}$ -predictable process. We first look for a RA-discount $(\tilde{B}^s, \sigma^s) = \{(\tilde{B}_t^s, \hat{\sigma}_t^s, \tilde{\sigma}_t^s) : 0 \leq t \leq s\}$ of B_s for every $s = t_1, \dots, t_N$, that is, a solution of the following backward difference equation

$$\begin{cases} \Delta \tilde{B}_t^s = \tilde{B}_t^s r_t h + \hat{\sigma}_t \hat{\eta}_t^s h + \hat{\sigma}_t^s \Delta W_t + 1_{(\tau > t)} \tilde{\sigma}_{t+h}^s \tilde{\eta}_t q_{t+h} h + \tilde{\sigma}_{t+h}^s \Delta M_t, \quad \forall t = 0, 1, \dots, s-h \\ \tilde{B}_s^s = B_s \end{cases} \quad (9)$$

where $\Delta \tilde{B}_t^s = \tilde{B}_{t+h}^s - \tilde{B}_t^s$, $\Delta M_t = M_{t+h} - M_t = H_{t+h} - H_t - 1_{(\tau > t)} q_{t+h} h$. If we define process ξ as $\xi_t(\omega) = \xi_t(\omega^1) = \omega_{t/h}^1$, $0 < t \leq T$, then

$$\Delta W_t = W_{t+h} - W_t = \omega_{t/h+1}^1 \sqrt{h} = \xi_{t+h}(\omega) \sqrt{h}.$$

Therefore BSDE(9) is verified as

$$\begin{cases} \Delta \tilde{B}_t^s = \tilde{B}_t^s r_t h + \hat{\sigma}_t \hat{\eta}_t^s h + \hat{\sigma}_t^s \xi_{t+h} \sqrt{h} + 1_{(\tau > t)} \tilde{\sigma}_{t+h}^s \tilde{\eta}_t' h + \tilde{\sigma}_{t+h}^s \Delta H_t, \quad \forall t = 0, 1, \dots, s-h \\ \tilde{B}_s^s = B_s \end{cases} \quad (10)$$

where $\tilde{\eta}_t' = q_{t+h}(\tilde{\eta}_t - 1)$, $\Delta H_t = H_{t+h} - H_t$, $\hat{\sigma}_t^s$ and $\tilde{\sigma}_{t+h}^s$ represent the viability of discount value caused by continuous market risk and default risk in the period $[t, t+h]$. We assume $\hat{\sigma}_t^s$ is determined by the information until t and $\tilde{\sigma}_t^s$ is determined by the market information until t and default information before time t . that is, $\hat{\sigma}_t^s$ is $\mathcal{G}_t$ measurable, and $\tilde{\sigma}_t^s$ is $\mathcal{F}_t \vee \mathcal{H}_{t-h}$ measurable.

For any $t \leq s$, set

$$\begin{aligned} R(t, s, h)(\omega) &= \prod_{t \leq u \leq s} (1 + r_u(\omega^1)h)^{-1}, \\ \hat{D}(t, s, h)(\omega) &= \prod_{t \leq u \leq s} (1 - \omega_{u/h+1}^1 \hat{\eta}_u(\omega^1) \sqrt{h}), \\ \tilde{D}(t, s, h) &= \prod_{t \leq u \leq s} (1 + \tilde{\eta}_u'(\omega_{u+h}^2)h), \end{aligned}$$

and let $R(t, s, h) = \hat{D}(t, s, h) = \tilde{D}(t, s, h) = 1$ if $s < t$.

From the structure of $\mathcal{F}_t$, we can see it denotes all possible pathes ω^1 walks until time t and for any $\mathcal{F}_t$ measurable random variable, its value is only determined by the path ω^1 walks before time t . We denote by a sequence $x(t) = \{x_1, \dots, x_{t/h}\}$ a path before time t , where $x_i = 1$ or -1 , $i = 1, \dots, t/h$ and denote $A(t, x) = \{\omega^1 : (\omega_1^1, \dots, \omega_N^1) = (x_1, \dots, x_{t/h}, \omega_{t/h+1}^1, \dots, \omega_N^1)\}$,

then $\mathcal{F}_t = \sigma\{A(t, x) : x \in \{1, -1\}^{t/h}\}$. For any $\omega^1, \hat{\omega}^1 \in A(t, x)$, the value of any $\mathcal{F}_t$ measurable random variable y satisfies $y(\omega^1) = y(\hat{\omega}^1)$, denoted by $y(x)$. In addition, rewrite $\mathcal{G}_t$ as $\mathcal{G}_t = \sigma\{A(t, x) \times \{\omega^2\} : x \in \{1, -1\}^{t/h}, \omega^2 \in \Omega^2\}$, and for any $\mathcal{G}_t$ measurable variable z , denote its value at any $\omega = (\omega^1, \omega^2) \in \{A(t, x) \times \{\omega^2\}\}$ by $z(x, \omega^2)$.

Proposition 3.1 Suppose $\{b_t\}, \{c_t\}$ are all non-negative bounded $\mathcal{F}_t$ -adapted processes, X are non-negative $\mathcal{F}_T$ measurable random variable and $c_t < X, \forall t \in [0, T]$. For every $t = t_0, \dots, t_N$, any sequence $x(t) = \{x_1, x_2, \dots, x_{t/h}\}$, $x_i = 1$ or -1 , $i = 1, \dots, t/h$ and any $\omega^2 \in \Omega^2$, considering each $s \geq t$ and each $\omega \in \{A(t, x) \times \{\omega^2\}\}$, let

$$\begin{aligned} \tilde{B}_t^s(\omega) &= \tilde{B}_t^s(x, \omega^2) = \begin{cases} B_s(x, \omega^2), & t = s \\ \tilde{D}(t, s-2h, h) \left(\frac{1}{2}\right)^{\left(\frac{s-t}{h}-1\right)} \left\{ \sum_{\hat{\omega}^1 \in A(t, x)} R(t, s-h, h)(\hat{\omega}^1) \hat{D}(t, s-2h, h)(\hat{\omega}^1) \right. \\ \left. (\pi_s(\hat{\omega}^1)(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s(\hat{\omega}^1)\tilde{\eta}'_{s-h}(\omega_s^2)h) \right\} (1 - H_t(\omega^2)), & t \leq s-h \end{cases} \\ \tilde{\sigma}_t^s(\omega) &= \tilde{\sigma}_t^s(x, \omega^2) = \begin{cases} (c_s(x) - \pi_s(x))1_{(\tau(\omega^2) \geq t)}, & t = s \\ -\tilde{B}_t^s(x, \omega_{t+h}^2)1_{(\tau(\omega^2) \geq t)}, & t \leq s-h \end{cases} \\ \hat{\sigma}_t^s(\omega) &= \hat{\sigma}_t^s(x, \omega^2) = \begin{cases} 0, & t = s, s-h \\ \frac{1}{2\sqrt{h}}(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h)[\tilde{B}_{t+h}^s((x(t), 1), \omega_{t+2h}^2) \\ + (-1)\tilde{B}_{t+h}^s((x(t), -1), \omega_{t+2h}^2)](1 - H_t(\omega^2)), & t \leq s-2h \end{cases} \end{aligned} \quad (11)$$

Then $(\tilde{B}^s, \hat{\sigma}^s, \tilde{\sigma}^s) = \{(\tilde{B}_t^s, \hat{\sigma}_t^s, \tilde{\sigma}_t^s) : 0 \leq t \leq T\}$ given by (11) is the RA-discount of B_s , that is, it satisfies BSDE (10). And under the condition $\hat{\sigma}_t^s = 0, \forall t \geq \tau$, and $\tilde{\sigma}_t^s = 0, \forall t > \tau$, it is the unique RA-discount of B_s . In addition, suppose $0 \leq \hat{\eta}_t < \frac{1}{\sqrt{h}}, 0 \leq \tilde{\eta}_t < 1$, for any $t \in \{t_0, \dots, t_{N-1}\}$, define a measure $\mathbb{P}^\eta$ on $\mathbb{G}$:

$$\begin{aligned} \mathbb{P}^\eta(\{\omega^1, \omega_{t_i}^2\}) &= \mathbb{P}^\eta(\{\omega_{t_i}^2\}) = -\prod_{j=0}^{i-2} (1 + \tilde{\eta}'_{t_j}(\omega_{t_{j+1}}^2)h)\tilde{\eta}'_{t_{i-1}}(\omega_{t_i}^2)h, \quad 1 < i \leq N, \\ \mathbb{P}^\eta(\{\omega^1, \omega_1^2\}) &= \mathbb{P}^\eta(\{\omega_1^2\}) = -\tilde{\eta}'_0(\omega_1^2)h, \\ \mathbb{P}^\eta(\{\omega^1, \omega_0^2\}) &= \mathbb{P}^\eta(\{\omega_0^2\}) = \prod_{j=0}^{N-1} (1 + \tilde{\eta}'_{t_j}(\omega_{t_{j+1}}^2)h), \\ \mathbb{P}^\eta(\{(\omega^1, \Omega^2)\}) &= \mathbb{P}^\eta((\omega_1^1, \dots, \omega_N^1)) = \begin{cases} \left(\frac{1}{2}\right)^N \prod_{j=0}^{N-1} (1 - \omega_{t_{j+1}}^1 \hat{\eta}_{t_j}(\omega^1)\sqrt{h}), & N \geq 2 \\ \left(\frac{1}{2}\right)^N, & N = 0, 1. \end{cases} \end{aligned}$$

Then $\mathbb{P}^\eta$ is a probability measure and for any $0 < s \leq T, 0 \leq t \leq s$

$$\tilde{B}_t^s = E^\eta(B_s R(t, s-h, h) | \mathcal{G}_t) \quad (12)$$

Proof for any $s = t_1, \dots, t_N, t \leq s$, any $x(t) \in \{1, -1\}^{t/h}$ and any $\omega^2 \in (\Omega^2)$, fixed $x(t), \omega^2$, for any $\omega = (\omega^1, \omega^2) \in \{A(t, x) \times \{\omega^2\}\}$.

If $H_t(\omega^2) = 1$, for $t < s$,

$$\hat{\sigma}_t^s(\omega) = \tilde{\sigma}_{t+h}^s(\omega) = \tilde{B}_t^s(\omega) = \tilde{B}_{t+h}^s(\omega) = 0.$$

Obviously satisfies (10).

If $H_t(\omega) = 0$, consider two cases.

1° If $H_{t+h}(\omega) = 1$, then $\omega^2 = \omega_{t+h}^2$, for $t \leq s - 2h$,

$$\begin{aligned}
& \tilde{B}_t^s(1 + r_t(\omega^1)h) + \hat{\sigma}_t^s(\omega)(\hat{\eta}_t(\omega^1)h + \xi_{t+h}(\omega^1)\sqrt{h}) + \tilde{\sigma}_{t+h}^s(\omega)(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h) \\
&= \tilde{B}_t^s(1 + r_t(\omega^1)h) + \frac{1}{2}(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h)(\hat{\eta}_t(\omega^1)\sqrt{h} + \xi_{t+h}(\omega^1)) \\
&\quad [\tilde{B}_{t+h}^s((x(t), 1), \omega_{t+2h}^2) + (-1)\tilde{B}_{t+h}^s((x(t), -1), \omega_{t+2h}^2)] \\
&\quad - \tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2)(1 + \tilde{\eta}'_t(\omega_{t+2h}^2)h) \\
&= \tilde{B}_t^s(1 + r_t(\omega^1)h) + \frac{1}{2}(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h)(\hat{\eta}_t(\omega^1)\sqrt{h} + \xi_{t+h}(\omega^1)) \\
&\quad [\xi_{t+h}(\omega^1)\tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2) - \xi_{t+h}(\omega^1)\tilde{B}_{t+h}^s((\omega_1^1, \dots, \omega_{\frac{1}{h}}^1, -\omega_{\frac{1}{h}+1}^1), \omega_{t+2h}^2)] \\
&\quad - \tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2)(1 + \tilde{\eta}'_t(\omega_{t+2h}^2)h) \\
&= \tilde{B}_t^s(1 + r_t(\omega^1)h) - (1 + \tilde{\eta}'_t(\omega_{t+h}^2)h)\frac{1}{2}[\tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2)(1 - \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h}) \\
&\quad + \tilde{B}_{t+h}^s((\omega_1^1, \dots, \omega_{\frac{1}{h}}^1, -\omega_{\frac{1}{h}+1}^1), \omega_{t+2h}^2)(1 - \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h})] \tag{13}
\end{aligned}$$

Let $A = \{\hat{\omega}^1 : \hat{\omega}^1(u) = \omega^1(u), u \leq t\}$, $A_1 = \{\hat{\omega}^1 : \hat{\omega}^1(u) = \omega^1(u), u \leq t + h\}$, $A_2 = \{\hat{\omega}^1 : \hat{\omega}^1(u) = \omega^1(u), u \leq t, \hat{\omega}^1(t + h) = -\omega^1(t + h)\}$, we have

$$\begin{aligned}
\tilde{B}_t^s(1 + r_t(\omega^1)h) &= \tilde{D}(t, s - 2h, h) \left(\frac{1}{2}\right)^{\frac{s-t}{h}-1} \left\{ \sum_{\hat{\omega}^1 \in A} R(t, s - h, h)(\hat{\omega}^1) \hat{D}(t, s - 2h, h)(\hat{\omega}^1) \right. \\
&\quad \left. [\pi_s(1 + \tilde{\eta}'_{s-h}(\omega_{s-h}^2)h) - c_s \tilde{\eta}'_{s-h}(\omega_s^2)h] \right\} (1 + r_t(\omega^1)h) \\
&= \frac{1}{2}(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h) \left\{ \right. \\
&\quad (1 - \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h}) \tilde{D}(t + h, s - 2h, h) \left(\frac{1}{2}\right)^{\frac{s-(t+h)}{h}-1} \\
&\quad \left\{ \sum_{\hat{\omega}^1 \in A_1} R(t + h, s - h, h)(\hat{\omega}^1) \hat{D}(t + h, s - 2h, h)(\hat{\omega}^1) \right. \\
&\quad \left. [\pi_s(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s \tilde{\eta}'_{s-h}(\omega_s^2)h] \right\} \\
&\quad + (1 + \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h}) \tilde{D}(t + h, s - 2h, h) \left(\frac{1}{2}\right)^{\frac{s-(t+h)}{h}-1} \\
&\quad \left\{ \sum_{\hat{\omega}^1 \in A_2} R(t + h, s - h, h)(\hat{\omega}^1) \hat{D}(t + h, s - 2h, h)(\hat{\omega}^1) \right. \\
&\quad \left. [\pi_s(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s \tilde{\eta}'_{s-h}(\omega_s^2)h] \right\} \left. \right\} \\
&= \frac{1}{2}(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h) [\tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2)(1 - \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h}) \\
&\quad + \tilde{B}_{t+h}^s((\omega_1^1, \dots, \omega_{\frac{1}{h}}^1, -\omega_{\frac{1}{h}+1}^1), \omega_{t+2h}^2)(1 + \xi_{t+h}(\omega^1)\hat{\eta}_t(\omega^1)\sqrt{h})] \tag{14}
\end{aligned}$$

Thus follows

$$\tilde{B}_t^s(1 + r_t(\omega^1)h) + \hat{\sigma}_t^s(\omega)(\hat{\eta}_t(\omega^1)h + \xi_{t+h}(\omega^1)\sqrt{h}) + \tilde{\sigma}_{t+h}^s(\omega)(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h) = 0 = \tilde{B}_{t+h}^s(\omega).$$

For $t = s - h$, $\hat{\sigma}_t^s = 0$ we have

$$\begin{aligned} & \tilde{B}_t^s(1 + r_t(\omega^1)h) + \tilde{\sigma}_t^s(\omega)(1 + \tilde{\eta}'_t(\omega_{t+h}^2)h) \\ &= (1 + r_{s-h}(\omega)h)^{-1}[\pi_s(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s\tilde{\eta}'_{s-h}(\omega_s^2)h](1 + r_{s-h}(\omega)h) \\ & \quad + (c_s - \pi_s)(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) \\ &= c_s(\omega) = B_s(\omega) = \tilde{B}_{t+h}^s(\omega). \end{aligned}$$

So (10) follows in this case.

2° If $H_{t+h}(\omega) = 0$, then $\exists k > 1$, such that $\omega^2 = \omega_{t+kh}^2$. At that time,

$$\tilde{B}_{t+h}^s(\omega^1, \omega^2) = \tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2).$$

For $t \leq s - 2h$, by equation (11) and (14), we have

$$\begin{aligned} & \tilde{B}_t^s(1 + r_t(\omega^1)h) + \hat{\sigma}_t^s(\omega)(\hat{\eta}_t(\omega^1)h + \xi_{t+h}(\omega^1)\sqrt{h}) + \tilde{\sigma}_{t+h}^s(\omega)\tilde{\eta}'_t(\omega_{t+h}^2)h \\ &= (1 + \tilde{\eta}'_t(\omega_{t+h}^2)h)\tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2) - \tilde{B}_{t+h}^s(\omega^1, \omega_{t+2h}^2)\tilde{\eta}'_t(\omega_{t+h}^2)h \\ &= \tilde{B}_{t+h}^s(\omega). \end{aligned}$$

For $t = s - h$, $\hat{\sigma}_t^s = 0$,

$$\begin{aligned} & \tilde{B}_t^s(1 + r_t(\omega^1)h) + \tilde{\sigma}_{t+h}^s(\omega)\tilde{\eta}'_t(\omega_{t+h}^2)h \\ &= (1 + r_{s-h}(\omega)h)^{-1}[\pi_s(1 + \tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s\tilde{\eta}'_{s-h}(\omega_s^2)h](1 + r_{s-h}(\omega)h) \\ & \quad + (c_s - \pi_s)\tilde{\eta}'_{s-h}(\omega_s^2)h \\ &= \pi_s = B_s(\omega) = \tilde{B}_{t+h}^s(\omega). \end{aligned}$$

and the proof of the first assertion is completed.

We now show that $\mathbb{P}^\eta$ is a probability measure on $\mathbb{G}$, for any $\omega = (\omega^1, \omega^2) = ((\omega_1^1, \dots, \omega_N^1), \omega^2) \in \Omega$,

$$\mathbb{P}^\eta(\{\omega^1\}) = \left(\frac{1}{2}\right)^N \prod_{j=0}^{N-1} (1 - \omega_{j+1}^1 \hat{\eta}_{t_j}(\omega^1)\sqrt{h}).$$

Then,

$$\sum_{\omega \in \Omega} \mathbb{P}^\eta(\{\omega\}) = \sum_{\omega \in \Omega} \mathbb{P}^\eta(\{\omega^1\})\mathbb{P}^\eta(\{\omega^2\}) = \sum_{\omega^1 \in \Omega^1} \left(\mathbb{P}^\eta(\{\omega^1\}) \sum_{i=0}^N \mathbb{P}^\eta(\{\omega_{t_i}^2\}) \right) = \sum_{\omega^1 \in \Omega^1} \mathbb{P}^\eta(\{\omega^1\}) = 1.$$

$\mathbb{P}^\eta$ is a probability measure. In addition, we can easily gain

$$\mathbb{P}^\eta(H_s = 0 | H_t = 0) = \prod_{t \leq u \leq s-h} (1 + \tilde{\eta}'_u(\omega_{u+h}^2)h) = \tilde{D}(t, s-h, h) \quad (15)$$

For any $s \leq T$ and each $\omega^1 = (\omega_1^1, \dots, \omega_N^1) \in \Omega^1$, note $A_s(\omega^1) = \{\hat{\omega}^1 : \hat{\omega}^1(t) = \omega^1(t), \forall t \leq s\}$, then

$$\begin{aligned} \mathbb{P}^\eta(A_s(\omega^1)) &= \sum_{\hat{\omega}^1 \in A_s(\omega^1)} \left(\frac{1}{2}\right)^N \prod_{j=0}^{N-1} (1 - \hat{\omega}_{j+1}^1 \hat{\eta}_j(\hat{\omega}^1)\sqrt{h}) = \left(\frac{1}{2}\right)^{s/h} \prod_{j=0}^{s/h-1} (1 - \omega_{j+1}^1 \hat{\eta}_j(\omega^1)\sqrt{h}), \\ &= \mathbb{P}^\eta(\{\omega^1\} | \mathcal{F}_t) \\ &= \mathbb{P}^\eta(\{\omega^1\} | \{\hat{\omega}^1 : \hat{\omega}^1(u) = \omega^1(u), u \leq t\}) \end{aligned}$$

$$\begin{aligned}
&= \frac{\mathbb{P}^\eta(\{\omega^1\})}{\mathbb{P}^\eta(A_t(\omega^1))} \\
&= \frac{(\frac{1}{2})^N \prod_{j=0}^{t/h-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h}) \prod_{j=t/h}^{N-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h})}{(\frac{1}{2})^{t/h} \prod_{j=0}^{t/h-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h})} \\
&= \left(\frac{1}{2}\right)^{N-t/h} \prod_{j=t/h}^{N-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h}) \\
&\triangleq \mathbb{P}_t^\eta(\{\omega^1\}),
\end{aligned}$$

that is

$$\mathbb{P}^\eta(\{\omega^1\}|\mathcal{F}_t) = \left(\frac{1}{2}\right)^{N-t/h} \prod_{j=t/h}^{N-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h}) \quad (16)$$

Now we show that the equation (12) holds. Under $\mathbb{P}^\eta$ the processes of $\{H_t : t = t_0, 1, \dots, t_N\}$ and $\{W_t, t = t_0, \dots, t_N\}$ are Markovian chains. For any $s \leq T$, $t \leq s - 2h$,

$$\begin{aligned}
&E^\eta(B_s R(t, s-h, h)|\mathcal{G}_t) \\
&= E^\eta((\pi_s 1_{(H_s=0)} + c_s 1_{(H_s=1, H_{s-h}=0)}) R(t, s-h, h)|\mathcal{G}_t) \\
&= E^\eta(\pi_s R(t, s-h, h) 1_{(H_s=0)}|\mathcal{G}_t) + E^\eta(c_s R(t, s-h, h) 1_{(H_s=1, H_{s-h}=0)}|\mathcal{G}_t).
\end{aligned}$$

Since $\mathcal{F}_t$ and $\mathcal{H}_t$ are independent, $R(t, s-h, h)$, c_s , π_s are independent with $\mathbb{H}$, and H is independent with $\mathbb{F}$. Let $x(t) = (x_1, \dots, x_{t/h}) \in \{1, -1\}^{t/h}$ and $A(t, x)$ be defined as same to the above, with $\mathcal{F}_{s-h}$ -measurable of $R(t, s-h, h)$, π_s , c_s for any $0 < s < T$ and by (15), (16), we have,

$$\begin{aligned}
&E^\eta(\pi_s R(t, s-h, h) 1_{(H_s=0)}|\mathcal{G}_t) \\
&= E^\eta(\pi_s R(t, s-h, h)|\mathcal{F}_t) E^\eta(1_{(H_s=0)}|\mathcal{H}_t) \\
&= 1_{(H_t=0)} P^\eta(H_s = 0 | H_t = 0) E^\eta(\pi_s R(t, s-h, h) | \{\omega^1 : \omega^1(u) = x_{u/h}, \forall u \leq t\})|_{x(t)} \\
&= 1_{(H_t=0)} \tilde{D}(t, s-h, h) \sum_{\omega^1 \in A(t, x)} \pi_s(\omega^1) R(t, s-h, h)(\omega^1) \mathbb{P}_t^\eta(\{\omega^1\}) \\
&= 1_{(H_t=0)} \tilde{D}(t, s-h, h) \\
&\quad \sum_{\omega^1 \in A(t, x)} \pi_s(\omega^1) R(t, s-h, h)(\omega^1) \left(\frac{1}{2}\right)^{N-t/h} \prod_{j=t/h}^{N-1} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h}) \\
&= 1_{(H_t=0)} \tilde{D}(t, s-h, h) \sum_{\omega^1 \in A(t, x)} \pi_s(\omega^1) R(t, s-h, h)(\omega^1) \\
&\quad \sum_{\tilde{\omega}^1 \in A_{s-h}(\omega^1)} \left(\frac{1}{2}\right)^{N-t/h} \prod_{j=t/h}^{N-1} (1 - \tilde{\omega}_{j+1}^1 \hat{\eta}_{jh}(\tilde{\omega}^1) \sqrt{h}) \\
&= 1_{(H_t=0)} \tilde{D}(t, s-h, h) \\
&\quad \sum_{\omega^1 \in A(t, x)} \pi_s(\omega^1) R(t, s-h, h)(\omega^1) \left(\frac{1}{2}\right)^{s/h-t/h-1} \prod_{j=t/h}^{s/h-2} (1 - \omega_{j+1}^1 \hat{\eta}_{jh}(\omega^1) \sqrt{h}) \\
&= 1_{(H_t=0)} \tilde{D}(t, s-h, h) \sum_{\omega^1 \in A(t, x)} \pi_s(\omega^1) R(t, s-h, h)(\omega^1) \left(\frac{1}{2}\right)^{s/h-t/h-1} \hat{D}(t, s-2h, h)(\omega^1),
\end{aligned}$$

and

$$\begin{aligned}
& E^\eta(c_s R(t, s-h, h) 1_{(H_s=1, H_{s-h}=0)} | \mathcal{G}_t) \\
&= E^\eta(c_s R(t, s-h, h) | \mathcal{F}_t) E^\eta(1_{(H_s=1, H_{s-h}=0)} | \mathcal{H}_t) \\
&= 1_{(H_t=0)} \mathbb{P}^\eta(H_s=1, H_{s-h}=0 | H_t=0) E^\eta(c_s R(t, s-h, h) | \{\omega^1 : \omega^1(u) = x_{u/h}, \forall u \leq t\}) \Big|_{x(t)} \\
&= -1_{(H_t=0)} \tilde{D}(t, s-2h, h) \tilde{\eta}'_{s-h}(\omega_s^2) h \\
&\quad \sum_{\omega^1 \in A(t, x)} c_s(\omega^1) R(t, s-h, h)(\omega^1) \left(\frac{1}{2}\right)^{s/h-t/h-1} \hat{D}(t, s-2h, h)(\omega^1).
\end{aligned}$$

So for $t \leq s-2h$,

$$E^\eta(B_s R(t, s-h, h) | \mathcal{G}_t) = \tilde{B}_t^s.$$

If $t = s-h$, with the prediction of π_s, c_s and the adaptation of $R(t, s-h, h)$,

$$\begin{aligned}
& E^\eta(B_s R(t, s-h, h) | \mathcal{G}_{s-h}) \\
&= E^\eta(\pi_s R(t, s-h, h) 1_{(H_s=0)} | \mathcal{G}_{s-h}) + E^\eta(c_s R(t, s-h, h) 1_{(H_s=1, H_{s-h}=0)} | \mathcal{G}_{s-h}) \\
&= 1_{(H_{s-h}=0)} \pi_s R(t, s-h, h) \mathbb{P}^\eta(H_s=0 | H_{s-h}=0) + 1_{(H_{s-h}=0)} c_s R(t, s-h, h) \mathbb{P}^\eta(H_s=1 | H_{s-h}=0) \\
&= 1_{(H_{s-h}=0)} [\pi_s(\omega^1)(1+r_{s-h}(\omega^1)h)^{-1}(1+\tilde{\eta}'_{s-h}(\omega_s^2)h) - c_s(\omega^1)(1+r_{s-h}(\omega^1)h)^{-1}\tilde{\eta}'_{s-h}(\omega_s^2)h] \\
&= \tilde{B}_{s-h}^s.
\end{aligned}$$

If $t = s$, $R(t, s-h, h) = 1$ and

$$E^\eta(B_s R(t, s-h, h) | \mathcal{G}_t) = E^\eta(B_s | \mathcal{G}_s) = B_s.$$

equation (12) holds.

It is easy to gain the uniqueness of $(\tilde{B}_t^s, \hat{\sigma}_t^s, \tilde{\sigma}_{t+h}^s)_{0 \leq t \leq t_{N-1}}$ by the solve processes of BSDE(10).

Remark 1. Formula (12) shows us the RA-discount value of B_s is just the discount value of B_s with the discount rate of r_t between time t and $t+h$ under probability measure $\mathbb{P}^\eta$. It is clearly that the price formula of B_s given in (11) is no arbitrage. By analyzing $\tilde{B}_t^s$, for any $t < s-h$,

$$\begin{aligned}
\tilde{B}_t^s &= (1 - H_t(\omega^2)) \sum_{\hat{\omega}^1 \in A(t, x)} \left\{ \right. \\
&\quad \pi_s R(t, s-h, h)(\hat{\omega}^1) \tilde{D}(t, s-h, h) \left(\frac{1}{2}\right)^{\left(\frac{s-t}{h}-1\right)} \hat{D}(t, s-2h, h)(\hat{\omega}^1) \\
&\quad \left. + c_s R(t, s-h, h) [-\tilde{D}(t, s-2h, h) \tilde{\eta}'_{s-h}(\omega_s^2) h] \left[\left(\frac{1}{2}\right)^{\left(\frac{s-t}{h}-1\right)} \hat{D}(t, s-2h, h)(\hat{\omega}^1) \right] \right\},
\end{aligned}$$

$(1 - H_t(\omega^2))$ makes the value of B_s at time t be zero if the security defaults before time t . Conditioned on the information captured by $\mathcal{G}_t$ and under $\mathbb{P}^\eta$, the probability that the security's price moves by any way of $\hat{\omega}^1 \in A(t, x)$ is $(\frac{1}{2})^{\left(\frac{s-t}{h}-1\right)} \hat{D}(t, s-2h, h)(\hat{\omega}^1)$, the probability that the security won't default before or at time s is $\hat{D}(t, s-2h, h)$, and the probability that the security will default at time s is $-\tilde{D}(t, s-2h, h) \tilde{\eta}'_{s-h}(\omega_s^2)$. As we know, if the security doesn't

default before or at time s , π_s will be paid at s , otherwise if the security defaults just at time s , the recovery c_s will be paid out and the security contract ends. Let the discount rate between t and $t+h$ be r_t , with the independence of W and H , the discount value of B_s at time t should be

$$(1 - H_t(\omega^2)) \sum_{\hat{\omega}^1 \in A(t,x)} \{ \pi_s R(t, s-h, h) \mathbb{P}^\eta(\{\hat{\omega}^1\} | \mathcal{F}_t) \mathbb{P}^\eta(H_s = 0 | H_t = 0) \\ + c_s R(t, s-h, h) \mathbb{P}^\eta(\{\hat{\omega}^1\} | \mathcal{F}_t) \mathbb{P}^\eta(H_s = 1, H_{s-h} = 0 | H_t = 0) \}.$$

It is just $\tilde{B}_t^s$ given in (11). Here, we calling $\mathbb{P}^\eta$ the pricing measure.

2. There gives the mapping between probabilities under the actual probability measure and under the pricing measure. It is just the market price of risks that prescribes the mapping. This conclusion is consistent with [13].

3. From the definition of $\mathbb{P}^\eta$, we can easily prove that $M^* := \{M_t^* = H_t + \sum_{h \leq u \leq t} \tilde{\eta}'_{(u \wedge \tau) - h} h, 0 < t \leq T, M_0^* = 0\}$, $W^* := \{W_t^* = W_t + \sum_{0 \leq u < t} \hat{\eta}_u h, 0 < t \leq T, W_0^* = 0\}$ are $\mathbb{G}$ -martingale under $\mathbb{P}^\eta$.

For the above security $\{B_s, s = t_0, \dots, t_N\}$, let $\tilde{B}_t^s$ given in (11) and (12). Define three processes $V = (V(t))_{0 \leq t \leq T}$, $\hat{\sigma} = (\hat{\sigma}(t))_{0 \leq t \leq T}$, $\tilde{\sigma} = (\tilde{\sigma}(t))_{0 \leq t \leq T}$ as follows, for $t = t_0, \dots, t_{N-1}$,

$$V(t) = \sum_{t+h \leq s \leq T} \tilde{B}_t^s, \quad \hat{\sigma}(t) = \sum_{t+h \leq s \leq T} \hat{\sigma}_t^s, \quad \tilde{\sigma}(t) = \sum_{t \leq s \leq T} \tilde{\sigma}_t^s \quad (17)$$

$$V_T = X 1_{(\tau > T)}, \quad \hat{\sigma}(T) = 0, \quad \tilde{\sigma}(T) = \tilde{\sigma}_T^T \quad (18)$$

Theorem 3.2 *The process family $(V, \hat{\sigma}, \tilde{\sigma})$ given in (17) and (18) is an adapted solution of the following BSDE,*

$$\begin{cases} \Delta V_t(\omega) = V_t(\omega) r_t(\omega) h + \hat{\sigma}_t(\omega) \hat{\eta}_t(\omega) h + \hat{\sigma}_t(\omega) \Delta W_t \\ \quad + 1_{(\tau > t)} \tilde{\sigma}_{t+h}(\omega) \tilde{\eta}_t(\omega) q_{t+h}(\omega) h + \tilde{\sigma}_{t+h}(\omega) \Delta M_t(\omega) - B_{t+h}, \quad 0 \leq t \leq T - 2h \\ V_T(\omega) = X 1_{\tau > T} \end{cases} \quad (19)$$

where $\Delta V_t = V_{t+h} - V_t$, $\Delta M_t = M_{t+h} - M_t$, $\Delta W_t = W_{t+h} - W_t = \xi_{t+h}(\omega) \sqrt{h}$. And the solution is unique with the condition $V(t) = \hat{\sigma}(t) = \tilde{\sigma}(t+h) = 0, \forall t \geq \tau$.

Proof for any $0 \leq t \leq T - 2h$

$$\begin{aligned} V(t+h) - V(t) &= \sum_{t+2h \leq s \leq T} \tilde{B}_{t+h}^s - \sum_{t+h \leq s \leq T} \tilde{B}_t^s \\ &= \sum_{t+2h \leq s \leq T} (\tilde{B}_{t+h}^s - \tilde{B}_t^s) - \tilde{B}_t^{t+h} \\ &= \sum_{t+2h \leq s \leq T} (\tilde{B}_t^s r_t + \hat{\sigma}_t^s \hat{\eta}_t h + \hat{\sigma}_t^s \Delta W_t \\ &\quad + 1_{(\tau > t)} \tilde{\sigma}_{t+h}^s \tilde{\eta}_t q_{t+h} h + \tilde{\sigma}_{t+h}^s \Delta M_t) - \tilde{B}_t^{t+h} \\ &= \left(\sum_{t+2h \leq s \leq T} \tilde{B}_t^s \right) r_t h + \left(\sum_{t+2h \leq s \leq T} \hat{\sigma}_t^s \right) \hat{\eta}_t h + \left(\sum_{t+2h \leq s \leq T} \hat{\sigma}_t^s \right) \Delta W_t \\ &\quad + \left(\sum_{t+2h \leq s \leq T} \tilde{\sigma}_{t+h}^s \right) 1_{(\tau > t)} \tilde{\eta}_t q_{t+h} h + \left(\sum_{t+2h \leq s \leq T} \tilde{\sigma}_{t+h}^s \right) \Delta M_t - \tilde{B}_t^{t+h} \end{aligned}$$

$$\begin{aligned}
&= (V_t - \tilde{B}_t^{t+h})r_th + (\hat{\sigma}_t - \hat{\sigma}_t^{t+h})\hat{\eta}_th + (\hat{\sigma}_t - \hat{\sigma}_t^{t+h})\Delta W_t \\
&\quad + (\tilde{\sigma}_{t+h} - \tilde{\sigma}_{t+h}^{t+h})1_{(\tau>t)}\tilde{\eta}_tq_{t+h}h + (\tilde{\sigma}_{t+h} - \tilde{\sigma}_{t+h}^{t+h})\Delta M_t - \tilde{B}_t^{t+h} \\
&= V_tr_th + \hat{\sigma}_t\hat{\eta}_th + \hat{\sigma}_t\Delta W_t + \tilde{\sigma}_{t+h}1_{(\tau>t)}\tilde{\eta}_tq_{t+h}h + \tilde{\sigma}_{t+h}\Delta M_t - B_{t+h}.
\end{aligned}$$

That follows (19). The uniqueness of the solution of BSDE(19) is easily to see by the uniqueness of $(\tilde{B}_t^s, \hat{\sigma}_t^s, \tilde{\sigma}_{t+h}^s)_{0 \leq t \leq t_{N-1}}$ for every $s = t_1, \dots, t_N$.

It is obviously that the RA-discount value V of B is an non-arbitrage value.

4 Continuous defaultable security

In this section, we discuss the price of defaultable securities in continuous time for general cases. The time horizon is still $[0, T]$. First, We structure the underlying probability space as following.

Let $(\Omega^1, \mathcal{F}, \mathbb{P}^1)$ be a completed probability space and $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion on it. Denote by $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the natural filtration of W and specially, let $\mathcal{F} = \mathcal{F}_T$. It is obvious that $\langle W, W \rangle_t = [W, W]_t = t$. The completed filtered probability $(\Omega^1, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}^1)$ represents the continuous market information captured.

Similar to discrete model, define another sample space as following:

$\{\omega_t^2\}_{0 \leq t \leq T}$: a family of nondecreasing function on $[0, T]$, $\omega_t^2 : \omega_t^2(u) = 0, \forall u < t; \omega_t^2(u) = 1, \forall u \geq t$, for any $0 < t \leq T$;

$\omega_0^2(u) = 0, \forall u \in [0, T]$;

$\Omega^2 := \{\omega^2\} = \{\omega_t^2 : 0 \leq t \leq T\}$;

H : a nondecreasing right continuous process, $H_t(\omega^2) := \omega^2(t)$;

Let

$$\tau(\omega^2) = \begin{cases} \min\{t : H_t = 1\}, & \omega^2 \neq \omega_T^2, \\ +\infty, & \omega = \omega_T^2, \end{cases}$$

it is easy to see that $H_t(\omega^2) = 1_{(\tau(\omega^2) \leq t)}$, and (H_t) is a pure jump process. After that, define $\mathcal{H}_t = \sigma(H_s, s \leq t)$ for every $t \in [0, T]$, then $\{\tau \leq t\} \in \mathcal{H}_t$. Suppose $\mathbb{P}^2$ is a probability measure on $(\Omega^2, \mathcal{H})$, satisfying $\mathbb{P}^2(\tau = 0) = 0, \mathbb{P}^2(\tau > 0) > 0, \forall t \in [0, T]$.

Then let $\Omega = \Omega^1 \times \Omega^2, \mathcal{G} = \mathcal{F} \otimes \mathcal{H}, \mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P} = \mathbb{P}^1 \otimes \mathbb{P}^2$ and $\mathbb{P}$ satisfies:

$$\mathbb{P}(\omega^1) = \mathbb{P}^1(\omega^1), \mathbb{P}(\tau \leq t) = \mathbb{P}^2(\tau \leq t).$$

For any $t \in [0, T]$, we write $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$, then F is a bounded non-negative continuous $\mathcal{F}_t$ -submartingale under $\mathbb{P}$. Let $F_t < 1$, then there exists $\mathbb{F}$ -adapted continuous process under $\mathbb{P}$, denoted by $\lambda = (\lambda_t)_{0 \leq t \leq T}$, such that $F_t = 1 - \exp\{-\int_0^t \lambda_s ds\}$. λ is called the intensity process of τ under $\mathbb{P}$.

Now we consider the security B , its coupons will be paid continuously, the recovery will be paid at default time when default happens prior to or at the maturity, and a claim of X will be paid at maturity if default event doesn't happen during the lifespan of the security. Its accumulated cash flows can be described as following,

$$B_t(\omega) = \int_0^t b_s 1_{(\tau>s)} ds + \int_{(0,t]} c_s - dH_s + X 1_{(\tau>T)} 1_{(t \geq T)}, \quad \forall t \in [0, T] \quad (20)$$

where $\{b_t : 0 \leq t \leq T\}$ is a bounded non-negative $\mathbb{F}$ -predictable process, $\{c_t : 0 \leq t \leq T\}$ is a

non-negative $\mathbb{F}$ -adapted process, X is a non-negative $\mathcal{F}_T$ -measurable variable, and $c_t < X$ for any $t \in [0, T]$. Then $B = (B_t)_{0 \leq t \leq T}$ is a finite variable process.

Let risk-free interest rate $(r_t)_{0 \leq t \leq T}$ is a $\mathbb{F}$ -adapted process, the market price process of the continuous market risk $(\hat{\eta}_t)_{0 \leq t \leq T}$ is a $\mathbb{F}$ -predictable process and the market price of the default risk $(\tilde{\eta}_t)_{0 \leq t \leq T}$ is a $\mathbb{G}$ -predictable process.

Denote $\tilde{\eta}'_s = \lambda_s(\tilde{\eta}_s - 1)$.

Proposition 4.1 *suppose $\tilde{\eta}_t < 1$, $\lambda_t > 0$, and let*

$$\begin{cases} V_t = 1_{(\tau > t)} E \left[\int_{(t, T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} ds \middle| \mathcal{F}_t \right] \\ \quad + 1_{(\tau > t)} E \left[X \exp \left\{ - \int_t^T r_u du - \int_t^T \hat{\eta}_u dW_u - \frac{1}{2} \int_t^T \hat{\eta}_u^2 du + \int_t^T \tilde{\eta}'_u du \right\} \middle| \mathcal{F}_t \right] \\ \tilde{\sigma}_t = (c_t - V_t) 1_{(\tau > t)} \end{cases} \quad (21)$$

then there must exist a $\mathbb{G}$ -adapted process $\hat{\sigma} = (\hat{\sigma}_t)_{0 \leq t \leq T}$, such that $(V_t, \hat{\sigma}_t, \tilde{\sigma}_t)_{0 \leq t \leq T}$ is a $\mathbb{G}$ -adapted solution of the following BSDE,

$$\begin{cases} dV_t = V_t r_t dt + \hat{\sigma}_t dW_t + \hat{\sigma}_t \hat{\eta}_t dt + 1_{(\tau > t)} \tilde{\sigma}_t \tilde{\eta}_t \lambda_t dt + \tilde{\sigma}_t dM_t - dB_t, 0 \leq t < T \\ V_T = X 1_{(\tau > T)} \end{cases} \quad (22)$$

or in its integral form,

$$V_{t_1} = V_{t_2} - \int_{(t_1, t_2]} (V(t) r_t + \hat{\sigma}_t \hat{\eta}_t + 1_{(\tau > t)} \tilde{\sigma}_t \tilde{\eta}_t \lambda_t) dt - \int_{(t_1, t_2]} \hat{\sigma}_t dW_t - \int_{(t_1, t_2]} \tilde{\sigma}_t dH_t + \int_{(t_1, t_2]} dB_t$$

for any $t_1 \leq t_2 \leq T$, that is, $V(t)$ given in (21) is a RA-discount of B .

Proof For any $t < T$, $dB_t = b_t(1 - H_t)dt + c_t - dH_t$,

$$\begin{aligned} V_t &= 1_{(\tau > t)} E \left[\int_{(t, T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} ds \middle| \mathcal{F}_t \right] \\ &\quad + 1_{(\tau > t)} E \left[X \exp \left\{ - \int_t^T r_u du - \int_t^T \hat{\eta}_u dW_u - \frac{1}{2} \int_t^T \hat{\eta}_u^2 du + \int_t^T \tilde{\eta}'_u du \right\} \middle| \mathcal{F}_t \right] \\ &= (1 - H_t) \left\{ E \left[\int_{(0, T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_0^s r_u du - \int_0^s \hat{\eta}_u dW_u - \frac{1}{2} \int_0^s \hat{\eta}_u^2 du + \int_0^s \tilde{\eta}'_u du \right\} ds \middle| \mathcal{F}_t \right] \right. \\ &\quad + E \left[X \exp \left\{ - \int_0^T r_u du - \int_0^T \hat{\eta}_u dW_u - \frac{1}{2} \int_0^T \hat{\eta}_u^2 du + \int_0^T \tilde{\eta}'_u du \right\} \middle| \mathcal{F}_t \right] \\ &\quad \left. - \int_{(0, t]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_0^s r_u du - \int_0^s \hat{\eta}_u dW_u - \frac{1}{2} \int_0^s \hat{\eta}_u^2 du + \int_0^s \tilde{\eta}'_u du \right\} ds \right\} \\ &\quad \exp \left\{ \int_0^t r_u du + \int_0^t \hat{\eta}_u dW_u + \frac{1}{2} \int_0^t \hat{\eta}_u^2 du - \int_0^t \tilde{\eta}'_u du \right\} \\ &= [I_1(t) - I_2(t)] I_3(t) (1 - H_t). \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= E \left[\int_{(0, T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_0^s r_u du - \int_0^s \hat{\eta}_u dW_u - \frac{1}{2} \int_0^s \hat{\eta}_u^2 du + \int_0^s \tilde{\eta}'_u du \right\} ds \middle| \mathcal{F}_t \right] \\ &\quad + E \left[X \exp \left\{ - \int_0^T r_u du - \int_0^T \hat{\eta}_u dW_u - \frac{1}{2} \int_0^T \hat{\eta}_u^2 du + \int_0^T \tilde{\eta}'_u du \right\} \middle| \mathcal{F}_t \right], \end{aligned}$$

$$I_2(t) = \int_{(0,t]} (b_s - c_{s-} \tilde{\eta}'_s) \exp \left\{ - \int_0^s r_u du - \int_0^s \hat{\eta}_u dW_u - \frac{1}{2} \int_0^s \hat{\eta}_u^2 du + \int_0^s \tilde{\eta}'_u du \right\} ds,$$

$$I_3(t) = \exp \left\{ \int_0^t r_u du + \int_0^t \hat{\eta}_u dW_u + \frac{1}{2} \int_0^t \hat{\eta}_u^2 du - \int_0^t \tilde{\eta}'_u du \right\}.$$

Obviously, I_1 is a $\mathcal{F}_t$ -martingale. By the martingale representation theorem, there exists a $\mathcal{F}_t$ -adapted process $(\sigma_t^W)_{0 \leq t \leq T}$, such that

$$I_1(t) = I_1(0) + \int_0^t \sigma_u^W dW_u.$$

With the finite variation processes I_2, H , we have

$$\begin{aligned} dI_1(t) &= \sigma_t^W dW_t, \\ dI_2(t) &= (b_t - c_{t-} \tilde{\eta}'_t) \exp \left\{ - \int_0^t r_u du - \int_0^t \hat{\eta}_u dW_u - \frac{1}{2} \int_0^t \hat{\eta}_u^2 du + \int_0^t \tilde{\eta}'_u du \right\} dt \\ &= (b_t - c_{t-} \tilde{\eta}'_t) I_3^{-1}(t) dt, \\ dI_3(t) &= I_3(t) (r_t dt + \hat{\eta}_t^2 dt + \hat{\eta}_t dW_t - \tilde{\eta}'_t dt), \end{aligned}$$

and

$$d \langle I_1, I_3 \rangle_t = \sigma_t^W I_3(t) \hat{\eta}_t dt.$$

By Itô formula and $dM_t = dH_t - 1_{(\tau > t)} \lambda_t dt = 0, \forall t \geq \tau$,

$$\begin{aligned} dV_t &= (dI_1(t) - dI_2(t)) I_3(t) (1 - H_t) + (I_1(t) - I_2(t)) (1 - H_t) dI_3(t) - (I_1(t) - I_2(t)) I_3(t) dH_t \\ &\quad + d \langle I_1, I_3 \rangle_t \\ &= (\sigma_t^W dW_t - (b_t - c_{t-} \tilde{\eta}'_t) I_3^{-1}(t) dt) I_3(t) (1 - H_t) + V_t (r_t dt + \hat{\eta}_t^2 dt + \hat{\eta}_t dW_t - \tilde{\eta}'_t dt) \\ &\quad + \sigma_t^W I_3(t) \hat{\eta}_t dt - (I_1(t) - I_2(t)) I_3(t) (1_{(\tau > t)} \lambda_t dt + dM_t) \\ &= V_t r_t dt + (I_3(t) \sigma_t^W (1 - H_t) + V_t \hat{\eta}_t) dW_t + \hat{\eta}_t (I_3(t) \sigma_t^W (1 - H_t) + V_t \hat{\eta}_t) dt + c_{t-} 1_{(\tau > t)} \lambda_t \tilde{\eta}_t dt \\ &\quad - c_{t-} 1_{(\tau > t)} \lambda_t dt - (V_t \lambda_t \tilde{\eta}_t dt - V_t \lambda_t dt) - b_t (1 - H_t) dt - V_t \lambda_t dt \\ &\quad - (I_1(t) - I_2(t)) I_3(t) dM_t \\ &= V_t r_t dt + (I_3(t) \sigma_t^W (1 - H_t) + V_t \hat{\eta}_t) dW_t + \hat{\eta}_t (I_3(t) \sigma_t^W (1 - H_t) + V_t \hat{\eta}_t) dt \\ &\quad + (c_{t-} - V_t) 1_{(\tau > t)} \lambda_t \tilde{\eta}_t dt - c_{t-} (dH_t - dM_t) - b_t (1 - H_t) dt - V_t dM_t \\ &= V_t r_t dt + \hat{\sigma}_t dW_t + \hat{\sigma}_t \hat{\eta}_t dt + \tilde{\sigma}_t 1_{(\tau > t)} \lambda_t \tilde{\eta}_t dt + \tilde{\sigma}_t dM_t - b_t (1 - H_t) dt - c_{t-} dH_t \\ &= V_t r_t dt + \hat{\sigma}_t dW_t + \hat{\sigma}_t \hat{\eta}_t dt + \tilde{\sigma}_t 1_{(\tau > t)} \lambda_t \tilde{\eta}_t dt + \tilde{\sigma}_t dM_t - dB_t. \end{aligned}$$

where $\hat{\sigma}_t = I_3(t) (1 - H_t) \sigma_t^W + V_t \hat{\eta}_t$ is a $\mathcal{G}_t$ -adapted process with satisfying $\hat{\sigma}_t = 0, \forall t \geq \tau$. Thus we've proved Proposition 4.1.

Remark The formula of V_t given in (21) is equivalent to

$$\begin{aligned} V_t = E \left[\int_{(t,T]} \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right. \right. \\ \left. \left. + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} dB_s | \mathcal{G}_t \right]. \end{aligned}$$

In fact, denote $Z(t, s) = \exp\{-\int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du\}$, by the formula (20), we have

$$\begin{aligned}
& E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} dB_s | \mathcal{G}_t \right] \\
&= E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} b_s 1_{(\tau > s)} ds | \mathcal{G}_t \right] \\
&\quad + E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} c_s - dH_s | \mathcal{G}_t \right] \\
&\quad + E \left[Z(t, T) X 1_{(\tau > T)} \exp \left\{ \int_t^T 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du + \int_t^T \ln(1 - \tilde{\eta}_u) dH_u \right\} | \mathcal{G}_t \right] \\
&= E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right\} b_s 1_{(\tau > s)} ds | \mathcal{G}_t \right] \\
&\quad + E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right\} (1 - \tilde{\eta}_s) c_s - (dM_s + 1_{(\tau > s)} \lambda_s ds) | \mathcal{G}_t \right] \\
&\quad + E \left[Z(t, T) X 1_{(\tau > T)} \exp \left\{ \int_t^T 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right\} | \mathcal{G}_t \right] \\
&= E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right\} (b_s - c_s - \tilde{\eta}'_s) 1_{(\tau > s)} ds | \mathcal{G}_t \right] \\
&\quad + E \left[Z(t, T) X 1_{(\tau > T)} \exp \left\{ \int_t^T 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right\} | \mathcal{G}_t \right].
\end{aligned}$$

The above equation holds because $E[\int_{(t,T]} Z(t, s) \exp\{\int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u \ln u\} (1 - \tilde{\eta}_s) c_s - \ln M_s | \mathcal{G}_t] = 0$.

Then by Corollary 5.1.1 and Proposition 5.1.2 in [18]

$$\begin{aligned}
& E \left[\int_{(t,T]} Z(t, s) \exp \left\{ \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} dB_s | \mathcal{G}_t \right] \\
&= 1_{(\tau > t)} E \left[\int_{(t,T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ -\int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} ds | \mathcal{F}_t \right] \\
&\quad + 1_{(\tau > t)} E \left[X \exp \left\{ -\int_t^T r_u du - \int_t^T \hat{\eta}_u dW_u - \frac{1}{2} \int_t^T \hat{\eta}_u^2 du + \int_t^T \tilde{\eta}'_u du \right\} | \mathcal{F}_t \right].
\end{aligned}$$

Corollary 4.2 In proposition 4.1, for any $t \in [0, T]$, instead the formula of V_t by

$$\begin{aligned}
V_t = E \left[\int_{(t,T]} \exp \left\{ -\int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s 1_{(\tau > u)} \tilde{\eta}_u \lambda_u du \right. \right. \\
\left. \left. + \int_t^s \ln(1 - \tilde{\eta}_u) dH_u \right\} dB_s | \mathcal{G}_t \right] \quad (23)
\end{aligned}$$

and $V_T = X 1_{(\tau > T)}$, then $\{V_t\}$ is still the RA-discount of B .

Noticing that (M_t) is a pure jump martingale with respect to $\mathbb{G}$, and (W_t) is a continuous martingale with respect to $\mathbb{F}$, therefore $(M_t), (W_t)$ are orthogonal. Suppose $0 \leq \tilde{\eta}_t < 1, \lambda_t > 0, \forall t \in [0, T]$, then $\lambda_t(\tilde{\eta}_t - 1) < 0$. If $\hat{\eta}_t, \tilde{\eta}_t, \lambda_t$ satisfy

$$\exp \left\{ \frac{1}{2} \int_0^T \hat{\eta}_t^2 dt \right\} < +\infty, \int_0^T \tilde{\eta}'_t dt > -\infty.$$

Define process $\theta = (\theta_t)_{0 \leq t \leq T}$ as

$$d\theta_t = \theta_t(-\hat{\eta}_t dW_t - \tilde{\eta}_t dM_t),$$

that is

$$\begin{aligned} \theta_t &= \varepsilon_t \left(-\int_0^t \hat{\eta}_s dW_s \right) \varepsilon_t \left(-\int_0^t \tilde{\eta}_s dM_s \right) \\ &= \exp \left\{ -\int_0^t \hat{\eta}_s dW_s - \frac{1}{2} \int_0^t \hat{\eta}_s^2 ds \right\} \exp \left\{ \int_0^{t \wedge \tau} \tilde{\eta}_u \lambda_u du + \int_{(0,t]} \ln(1 - \tilde{\eta}_u) dH_u \right\}. \end{aligned}$$

It is easy to see that θ is a $\mathcal{G}_t$ -martingale, and $E(\theta_T) = 1$.

Define

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{G}_t} = \theta_t,$$

then $\mathbb{P}^*$ is the equivalent measurement of $\mathbb{P}$. Let

$$\begin{aligned} W_t^* &= W_t + \int_0^t \hat{\eta}_s ds \\ M_t^* &= M_t + \int_0^{t \wedge \tau} \tilde{\eta}_s \lambda_s ds = H_t - \left(-\int_0^{t \wedge \tau} \tilde{\eta}_s' ds \right) \end{aligned} \quad (24)$$

then $(W_t^*), (M_t^*)$ are $\mathcal{G}_t$ -martingale under $\mathbb{P}^*$ and $(-\int_0^{t \wedge \tau} \tilde{\eta}_s' ds)$ is the compensator of $\mathcal{H}_t$ under $\mathbb{P}^*$.

Proposition 4.3 Suppose $0 \leq \tilde{\eta}_t < 1, \lambda_t > 0, \forall t \in [0, T]$ and $\hat{\eta}_t, \tilde{\eta}_t, \lambda_t$ satisfy

$$\exp \left\{ \frac{1}{2} \int_0^T \hat{\eta}_t^2 dt \right\} < +\infty, \int_0^T \tilde{\eta}_t' dt > -\infty$$

then $(V_t)_{0 \leq t \leq T}$ given by (21) satisfies the following equation under $\mathbb{P}^*$

$$V_t = E^* \left(\int_{(t,T]} \exp \left\{ -\int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right), \quad 0 \leq t \leq T \quad (25)$$

Proof By the property of conditional expectation,

$$\begin{aligned} E^* \left(\int_{(t,T]} \exp \left\{ -\int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right) &= E \left(\theta_T \int_{(t,T]} \exp \left\{ -\int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right) \theta_t^{-1} \\ &= \theta_t^{-1} \left\{ E \left(\theta_T \int_{(t,T]} \exp \left\{ -\int_t^s r_u du \right\} b_s 1_{(\tau > s)} ds | \mathcal{G}_t \right) \right. \\ &\quad + E \left(\theta_T \int_{(t,T]} \exp \left\{ -\int_t^s r_u du \right\} c_s - dH_s | \mathcal{G}_t \right) \\ &\quad \left. + E \left(\theta_T X 1_{(\tau > T)} \exp \left\{ -\int_t^T r_u du \right\} | \mathcal{G}_t \right) \right\}. \end{aligned}$$

Let $A_s = \int_{(t,s]} \exp \left\{ -\int_t^u r_v dv \right\} b_u 1_{(\tau > u)} du$, then (A_s) is a $\mathcal{G}_s$ -predictable process. By Itô formula

$$\theta_T A_T = \theta_t A_t + \int_{(t,T]} A_s d\theta_s + \int_{(t,T]} \theta_s dA_s,$$

where $\int_{(t,T]} A_u d\theta_u$ is a $\mathbb{G}$ -martingale and $A_t = 0$, thus following

$$\begin{aligned} &\theta_t^{-1} E(\theta_T A_T | \mathcal{G}_t) \\ &= \theta_t^{-1} E \left(\int_{(t,T]} \theta_s dA_s | \mathcal{G}_t \right) \end{aligned}$$

$$\begin{aligned}
&= \theta_t^{-1} E \left(\int_{(t,T]} \theta_s \exp \left\{ - \int_t^s r_u du \right\} b_s 1_{(\tau > s)} ds | \mathcal{G}_t \right) \\
&= E \left(\int_{(t,T]} \exp \left\{ - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du \right\} \exp \left\{ \int_t^{s \wedge \tau} \tilde{\eta}_u \lambda_u du + \int_{(t,s]} \ln(1 - \tilde{\eta}_u) dH_u \right\} \right. \\
&\quad \left. \exp \left\{ - \int_t^s r_u du \right\} b_s 1_{(\tau > s)} ds | \mathcal{G}_t \right) \\
&= 1_{(\tau > t)} E \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta} \lambda_u du \right\} \right. \\
&\quad \left. \exp \left\{ - \int_t^s \lambda_u du \right\} b_s ds | \mathcal{F}_t \right) \\
&= 1_{(\tau > t)} E \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} b_s ds | \mathcal{F}_t \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\theta_T \int_{(t,T]} \exp \left\{ - \int_t^s r_u du \right\} c_{s-} dH_s \\
&= \varepsilon_T \left(- \int_0^\cdot \hat{\eta}_s dW_s \right) \exp \left\{ \int_0^{T \wedge \tau} \tilde{\eta}_u \lambda_u du + \int_{(0,T]} \ln(1 - \tilde{\eta}_u) dH_u \right\} \exp \left\{ - \int_t^\tau r_u du \right\} c_{\tau-} 1_{(t < \tau \leq T)} \\
&= \varepsilon_T \left(- \int_0^\cdot \hat{\eta}_s dW_s \right) \exp \left\{ \int_0^\tau \tilde{\eta}_u \lambda_u du + \ln(1 - \tilde{\eta}_\tau) \right\} \exp \left\{ - \int_t^\tau r_u du \right\} c_{\tau-} 1_{(t < \tau \leq T)} \\
&= \varepsilon_T \left(- \int_0^\cdot \hat{\eta}_s dW_s \right) \int_{(t,T]} \exp \left\{ - \int_t^s r_u du + \int_0^s \tilde{\eta}_u \lambda_u du \right\} (1 - \tilde{\eta}_s) c_{s-} dH_s.
\end{aligned}$$

Similar to the first part proof, we have

$$\begin{aligned}
&E \left(\varepsilon_T \left(- \int_0^\cdot \hat{\eta}_s dW_s \right) \int_{(t,T]} \exp \left\{ - \int_t^s r_u du + \int_0^s \tilde{\eta}_u \lambda_u du \right\} (1 - \tilde{\eta}_s) c_{s-} dH_s | \mathcal{G}_t \right) \\
&= E \left(\int_{(t,T]} \exp \left\{ - \int_0^s \hat{\eta}_u dW_u - \frac{1}{2} \int_0^s \hat{\eta}_u^2 du \right\} \exp \left\{ - \int_t^s r_u du + \int_0^s \tilde{\eta}_u \lambda_u du \right\} (1 - \tilde{\eta}_s) c_{s-} dH_s | \mathcal{G}_t \right).
\end{aligned}$$

thus following

$$\begin{aligned}
&\theta_t^{-1} E \left(\theta_T \int_{(t,T]} \exp \left\{ - \int_t^s r_u dt \right\} c_{s-} dH_s | \mathcal{G}_t \right) \\
&= E \left(\int_{(t,T]} \exp \left\{ - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du - \int_t^s r_u dt + \int_t^s \tilde{\eta}_u \lambda_u du \right\} (1 - \tilde{\eta}_s) c_{s-} dH_s | \mathcal{G}_t \right) \\
&= 1_{(\tau > t)} E \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}_u \lambda_u du \right\} \right. \\
&\quad \left. \exp \left\{ - \int_t^s \lambda_u du \right\} (1 - \tilde{\eta}_s) \lambda_s c_{s-} ds | \mathcal{F}_t \right) \\
&= -1_{(\tau > t)} E \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} \tilde{\eta}'_s c_{s-} ds | \mathcal{F}_t \right).
\end{aligned}$$

We also can gain easily by Lemma 5.1.2 in [18]

$$\theta_t^{-1} E \left(\theta_T X 1_{(\tau > T)} \exp \left\{ - \int_t^T r_u du \right\} | \mathcal{G}_t \right)$$

$$= 1_{(\tau > t)} E \left(X \exp \left\{ - \int_t^T r_u du - \int_t^T \hat{\eta}_u dW_u - \frac{1}{2} \int_t^T \hat{\eta}_u^2 du + \int_t^T \tilde{\eta}'_u du \right\} | \mathcal{F}_t \right).$$

Thus,

$$\begin{aligned} & E^* \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right) \\ &= 1_{(\tau > t)} E \left[\int_{(t,T]} (b_s - c_s - \tilde{\eta}'_s) \exp \left\{ - \int_t^s r_u du - \int_t^s \hat{\eta}_u dW_u - \frac{1}{2} \int_t^s \hat{\eta}_u^2 du + \int_t^s \tilde{\eta}'_u du \right\} ds | \mathcal{F}_t \right] \\ &+ 1_{(\tau > t)} E \left[X \exp \left\{ - \int_t^T r_u du - \int_t^T \hat{\eta}_u dW_u - \frac{1}{2} \int_t^T \hat{\eta}_u^2 du + \int_t^T \tilde{\eta}'_u du \right\} | \mathcal{F}_t \right] \\ &= V(t). \end{aligned}$$

The proof of Proposition 4.3 is completed. From the proposition, we can conclude the RA-discount of B is no arbitrage.

Theorem 4.4 *Assume the conditions of Proposition 4.1 and Proposition 4.3 hold. Let $\{(V'(t), \hat{\sigma}'(t), \tilde{\sigma}'(t)) : 0 \leq t \leq T\}$ be any adapted solution of BSDE(22) such that $V'(t) = \hat{\sigma}'(t) = \tilde{\sigma}'(t) = 0, \forall t \geq \tau$. Then $V'(t)$ must be given by (21) or (25). In other words, the formula (21) or (25) gives the RA-discount value process of the security B.*

Proof Let $(V'(t), \hat{\sigma}'(t), \tilde{\sigma}'(t))$ be any adapted solution to (22), define

$$\bar{V}(t) = \left[V'(t) + \int_{(0,t]} \exp \left\{ - \int_t^s r_u du \right\} dB_s \right] \exp \left\{ - \int_0^t r_u du \right\},$$

then

$$\begin{aligned} d\bar{V}(t) &= \left[V'(t) + \int_{(0,t]} \exp \left\{ - \int_t^s r_u du \right\} dB_s \right] \exp \left\{ - \int_0^t r_u du \right\} (-r_t dt) \\ &+ \left[dV'(t) + dB_t + \int_{(0,t]} \exp \left\{ - \int_t^s r_u du \right\} dB_s r_t dt \right] \exp \left\{ - \int_0^t r_u du \right\} \\ &= \exp \left\{ - \int_0^t r_u du \right\} [\hat{\sigma}'(t) \hat{\eta}_t dt + \hat{\sigma}'(t) dW_t + 1_{(\tau > t)} \tilde{\sigma}'(t-) \tilde{\eta}_t dt + \tilde{\sigma}'(t-) dH_t] \\ &= \exp \left\{ - \int_0^t r_u du \right\} [\hat{\sigma}'(t) dW_t^* + \tilde{\sigma}'(t-) dM_t^*]. \end{aligned}$$

where W^*, M^* are defined as (24). Therefore, $\{\bar{V}(t), 0 \leq t \leq T\}$ is a $(\mathcal{G}_t)$ -martingale under $\mathbb{P}^*$.

On the other hand, let

$$V^0(t) = E^* \left(\int_{(t,T]} \exp \left\{ - \int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right),$$

and

$$\bar{V}^0(t) = V^0(t) \exp \left\{ - \int_0^t r_s ds \right\} + \int_{(0,t]} \exp \left\{ - \int_0^s r_u du \right\} dB_s,$$

$\forall t \in [0, T]$,

$$\bar{V}^0(t) = E^* \left(\int_{(0,T]} \exp \left\{ - \int_t^s r_u du \right\} dB_s | \mathcal{G}_t \right).$$

This implies that $(\bar{V}^0(t))$ is also a $(\mathcal{G}_t)$ -martingale under $\mathbb{P}^*$. Furthermore,

$$\bar{V}^0(T) = X 1_{(\tau > T)} \exp \left\{ - \int_0^T r_s ds \right\} + \int_{(0,T]} \exp \left\{ - \int_t^s r_u du \right\} dB_s = \bar{V}(T).$$

These yield

$$\bar{V}(t) = \bar{V}^0(t), \forall t \in [0, T].$$

Consequently,

$$V'(t) = V^0(t), \forall t \in [0, T].$$

i.e. $V'(t)$ is given by (25). From Proposition 4.3, $V'(t)$ is also given by (21). This theorem implies the nonnegative adapted solutions of BSDE(22) which satisfy $\hat{\sigma}_t = \tilde{\sigma}_t = 0, \forall t \geq \tau$ are unique. This unique process $(V(t))$ is called the RA-discount value of B.

5 Conclusions

In this paper, we used the concept of Risk-Adjust value (RA value) to develop the risky asset pricing theory, and proposed a new approach to price defaultable securities with continuous market risk at discrete time and continuous time under the primal probability measure by the backward stochastic differential equation (BSDE) theory. Then, we proved this RA value can be hedged completely with additional no arbitrage conditions. The RA value obeys the rules that the security price should be equal to the discounted value of its future cash flows with the so-called Risk-Adjust discount factors, which depend on the term structure of the risk free interest rate and risk premiums. To measure risk premiums, we use the market price process of continuous market risk and the market price process of default risk. These two prices can be estimated from the financial market data by methods in literatures, like [6, 13], etc. So, given the two market prices of risk (deterministic or stochastic), the hazard rate of default time under primal probability measure and the risk-free interest rate (if available), we can estimate the TR value of any defaultable security under primal probability measure. And furthermore, when there is no arbitrage opportunity in the security market, the tradable security with its RA value will not create any arbitrage opportunity.

In sum, the pricing models developed by this research have the following characters.

- All of the pricing formulas are given under the real probability measure, under which the parameters can be easily calibrated by market data.
- There are explicit economic interpretations in these price frameworks, and these RA values show the effect of the risk on the security values.
- The pricing framework satisfies the capital minimization rules, but doesn't depend on the assumptions of no arbitrage. Certainly, when the security market is indeed no arbitrage, the RA value is also an no arbitrage value. So there have more relax conditions for parameters than in no-arbitrage-models.
- In these model, the variability of securities are not the setting, but a part of the conclusion, which can be calculated together with the RA value. Furthermore, the variability here is not essential. The essential amount we need is the effect of one unit of risk on the security value. In other words, we give up the method that uses the variability to measure risk, and replace it by the effect the risk on the security value implied in the security price.
- Finally, we relax the conditions on parameters which may violate the uniqueness requirement of BSDE solutions, but ensure the existence of solutions of the BSDE.

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